A new class of pursuer strategies for the discrete-time lion and man problem

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Abstract

This paper addresses a discrete-time pursuit-evasion game, known as the lion and man problem. The pursuer is chasing the evader within the positive quadrant of a two-dimensional environment and wins the game when it reaches the evader position. A new family of pursuer strategies is proposed, which relies on the minimization of a user-defined function of the environment coordinates. The approach guarantees capture in finite time, no matter which is the strategy adopted by the evader. The degree of freedom associated to the choice of the function to be minimized enhances the flexibility of the pursuer strategy. Moreover, numerical simulations show the superiority of the proposed solution with respect to the most common pursuit strategies available in the literature.

Key words: Pursuit-evasion games; mobile robotics; game theory; autonomous agents; lion and man problem

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1 Introduction

Pursuit-evasion problems are extremely popular because they play a central role in a number of applications in several fields, including mobile robotics, search-and-rescue, surveillance, tracking, harvesting and many others. Although the topic has a rich history (see [18] for a beautiful overview), recent years have witnessed a steadily growing interest, motivated by the development of new technologies and the possibility of applying pursuit-evasion paradigms to many different contexts. The wide literature on the subject can be classified according to the assumptions made on the agents model, on the environment they move in and on the information patterns available to them. The interested reader is referred to the surveys [9,19] for a taxonomy of the existing approaches, and to papers [25,22,11,5,13,4,1,8,26,21,10] and references therein for a more extensive review of recent contributions.

A first major classification concerns whether to approach the problem in continuous or discrete time. In the former case, it is well known that an evader can indefinitely escape a single pursuer, even within very simple bounded environments like a circular arena [17]. A discrete-time formulation is the one in which the agents move one at a time, with a fixed bound on the distance they can travel at each move. One of the most popular versions of this game is the so called *lion and man problem*, originally proposed by David Gale [12]: a pursuer (the lion) and an evader (the man) are moving in the positive quadrant of a two-dimensional environment. The lion wins the game if it is able to reach the man in finite time. This basic pursuit-evasion setting is important for several reasons. First, it is known that there exist winning strategies for the lion, provided that it starts on the upper right side with respect to the man. This opens up the question about which is the most efficient pursuit strategy. Even more importantly, these lion's moves are employed as low-level procedures for addressing pursuit problems in more complex settings, like polygonal bounded environments [14,2], topological spaces [3], or searches with visibility limitations [20].

A celebrated solution of the lion and man problem has been proposed by Sgall [23]: the lion always keeps itself on the segment connecting a fixed point (called *center*) and the man's position. In fact, this is the lion's move

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commonly adopted in the above mentioned works. A similar approach has been adopted in [16], for the case of multiple pursuers in an n-dimensional environment. In [6], it has been shown that the time required for capture can be significantly reduced by suitably moving the center during the game. In all these approaches, capture is guaranteed by the fact that each lion's move increases by a finite quantity the so-called *cleared area*, i.e., the portion of the environment in which the man cannot enter, otherwise he will be caught in the next move.

In this paper, a new family of lion strategies is introduced. The main idea is to exploit a user-defined function of the environment coordinates, named *clearing function*, whose level curves are strictly decreasing towards the origin of the plane, where capture is expected to occur. The lion's move aims at minimizing the value of the clearing function, which corresponds to maximizing the cleared area. The shape of the level curves implicitly defines that of the cleared area, thus providing a remarkable degree of freedom in the pursuer strategy, which can be exploited in more complex or time-varying scenarios. The main contribution of the paper is to show that capture is guaranteed in finite time, irrespectively of the choice of the clearing function to be minimized. Moreover, the results of a campaign of numerical simulations demonstrate that the proposed approach outperforms the lion strategies proposed in previous works, in terms of time required by the lion to win the game.

The main advantage of the pursuer strategy based on the clearing function, with respect to the approaches adopted in [23,16,6], is that the lion's move is not dictated only by the *local* configuration of the game (namely, the relative position between the lion and the man). It is rather driven by a *global* view of the game, which is expressed by the values taken by the clearing function in each point of the environment. This not only allows one to remarkably reduce the capture time, but it also opens up the possibility to "pave" the environment with a clearing function whose level curves reflect the relative importance of different areas.

The paper is organized as follows. The lion and man problem is formulated in Section 2, while the new family of lion strategies is presented in Section 3. Section 4 reports some general properties of the lion's move which are exploited to devise an efficient algorithm for its computation. The main technical result is given in Section 5, where it is shown that, by playing the proposed strategies, the lion always wins the game, no matter which strategy is chosen by the man. Numerical analysis and comparisons with other lion strategies are reported in Section 6. Finally, Section 7 contains concluding remarks and mentions several future developments. A preliminary version of the approach proposed in the paper has been presented in [7].

2 Problem formulation

The notation adopted in the paper is standard. Let $\mathbb{R}^n_+$ denote the n-dimensional Euclidean space of non-negative numbers. A row vector with elements $v_1, \ldots, v_n$ is denoted by $V = [v_1, \ldots, v_n]$, while V' is the transpose of V. The phase of a vector V is denoted by $\angle V$. Given two vectors V and W, notation $V \succeq W$ ($V \succ W$) denotes the componentwise (strict) inequality. Denote by $\boldsymbol{\mathcal{C}}(C, r)$ the circle (including its interior) of center C and radius r, and by $\delta\boldsymbol{\mathcal{C}}(C, r)$ the corresponding circumference.

In this paper, we consider the version of the *lion and man problem* formulated by David Gale [12]. Two players move in the non-negative quadrant of the Cartesian plane. Time is assumed discrete, while space is continuous. At each round (hereafter called *time*) both players are allowed to travel a distance less or equal to a given radius r from their current position (w.l.o.g., $r = 1$ is assumed). Let us denote by $M_t \in \mathbb{R}^2_+$ and $L_t \in \mathbb{R}^2_+$ the man and lion position at time t, respectively. Hence, $\|M_{t+1} - M_t\| \leq 1$, $\|L_{t+1} - L_t\| \leq 1$. At the generic time t, the man moves first from M_t to M_{t+1}. Then, the lion observes the new position of the man and computes his new position L_{t+1} as a function of L_t and M_{t+1}. In this respect, the information pattern is that of a Stackelberg game, in which the man is the leader and the lion is the follower [24,15].

The game ends (lion wins) if the lion moves exactly to the man position. The man wins if he is able to escape indefinitely from the lion. It is assumed that the initial man coordinates are strictly smaller than the corresponding lion coordinates, i.e. $L_0 \succ M_0$, otherwise it is straightforward to observe that the man wins the game by moving straight upwards or to the right.

The proposed lion strategy aims at minimizing a cost function defined as follows.

Definition 1 *A function $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ is called a* clearing function (C-function) *if it satisfies the following properties:*

i) $f(0) = 0$ *and* $f(P) < \infty$, $\forall P \in \mathbb{R}^2_+$*;*
ii) $\forall P_2 \succeq P_1$, $P_2 \neq P_1$, *one has* $f(P_2) > f(P_1)$*;*
iii) let $F_\beta = \{P \in \mathbb{R}^2_+ : f(P) = \beta\}$ *be the* β*-level curve of* f*; for any fixed* $\beta > 0$*, there exists a strictly convex function* g_β *such that for any* $P = [x, y]' \in F_\beta$, $y = g_\beta(x)$ *and* $g'_\beta(x) \leq 0$, $g''_\beta(x) > 0$, $\forall x \geq 0$.

As it will become clear in the next section, the level curves F_β separate the portion of the non-negative quadrant in which the man can move (*contaminated region*) from the *cleared region* [19], which is the zone where the man cannot enter, otherwise he is captured in the next move. Notice that F_β collapses to the origin for $\beta = 0$.

Due to the symmetry of the game w.r.t. the coordinate axis, a natural choice is to deal with clearing functions which are symmetric w.r.t. the bisector of the non-negative quadrant. In the following, two examples of such functions are reported.

Quarter of circumference

Let us consider

$$f([x,y]') = x + y + \sqrt{2xy} \ . \tag{1}$$

For a given β, F_β is the lower left quarter of the circumference $\mathcal{C}([\beta,\, \beta]', \beta)$. In fact, from $(x-\beta)^2 + (y-\beta)^2 = \beta^2$, solving for β one has $\beta = x + y \pm \sqrt{2xy}$. From (1), we are interested in the level curve solution $\beta = x + y + \sqrt{2xy}$. The same level curve can be written as $y = g_\beta(x) = \beta - \sqrt{2x\beta - x^2}$, with $x \in [0, \beta]$, and it is easy to check that $g_\beta(x)$ is strictly convex for $x \in [0, \beta]$.

Branch of hyperbola

Let

$$f([x,y]') = \frac{x + y + \sqrt{(x+y)^2 + 4\, m\, x\, y}}{2} \tag{2}$$

with $m > 0$. It can be shown that, for fixed β, the level curve F_β is a branch of equilateral hyperbola centered in $\left[-\frac{\beta}{m},\, -\frac{\beta}{m}\right]'$. In fact, the equation defining $F_\beta = \{[x,y]' \colon f([x,y]') = \beta\}$ is

$$x + y + \frac{m}{\beta}xy - \beta = 0 \tag{3}$$

which can be rewritten as

$$\left(x + \frac{\beta}{m}\right)\left(y + \frac{\beta}{m}\right) - \frac{\beta^2}{m}\left(1 + \frac{1}{m}\right) = 0 \ .$$

Moreover, the same level curve can be written as $y = g_\beta(x) = \frac{\beta - x}{1 + \frac{m}{\beta}x}$, with $x \in [0, \beta]$, which is strictly convex.

In Fig. 1, the level curves F_β for different values of β related to the two C-functions introduced above are reported. For the branch of hyperbola C-function, the parameter m is equal to 2.

3 C-Function Lion Strategy

In this section, the proposed lion strategy, hereafter referred to as *Clearing Function Lion Strategy (C-FLS)*, is described. Such a strategy is based on the minimization of the C-function β level until capture occurs.

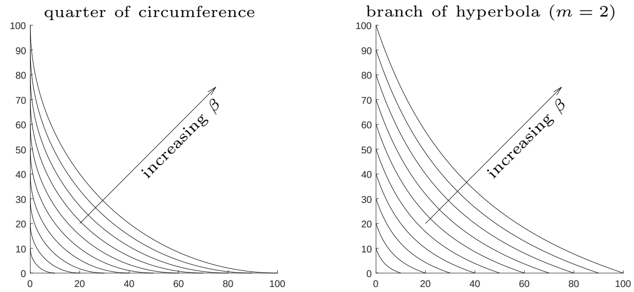

Fig. 1. Level curves F_β for different C-functions and values of β from 10 to 100 (with step 10).

Definition 2 *At any time t, let $\mathcal{R}_t$ denote the locus of points in the non-negative quadrant that are equidistant to M_t and L_t, i.e.,*

$$\mathcal{R}_t = \{P \in \mathbb{R}^2_+ \,:\, \|P - M_t\| = \|P - L_t\|\} \ . \tag{4}$$

The equality in (4) defines a portion of a line, known as *line of control*, which provides the Voronoi partition of the environment associated to M_t and L_t [13,26]. Let us write $\mathcal{R}_t$ as

$$\mathcal{R}_t = \{[x,y]' \in \mathbb{R}^2_+ \colon y = a_t x + b_t\} \ . \tag{5}$$

Clearly, if $L_t \succ M_t$, one has $a_t < 0$ and $b_t > 0$. As a consequence, since the environment is restricted to the non-negative quadrant, the line of control is indeed a closed segment and the Voronoi cell associated to the man is a triangle. It is worth remarking that if the man crosses the line of control and exits its Voronoi cell, the lion will catch him in the next move.

Let f denote the C-function chosen once and for all at the beginning of the game. At a given time t, let

$$\beta_t = \max_{P \in \mathcal{R}_t} f(P) \tag{6}$$

and

$$P_t = \arg\max_{P \in \mathcal{R}_t} f(P) = F_{\beta_t} \cap \mathcal{R}_t \ . \tag{7}$$

The uniqueness of P_t in (7) is guaranteed by the strict convexity of g_{β_t}.

At a time t, let $\mathcal{L}_{t+1}$ denote the set of feasible lion's moves, i.e.,

$$\mathcal{L}_{t+1} = \{L \colon L \succ M_{t+1} \text{ and } \|L - L_t\| \le 1\} \ . \tag{8}$$

Definition 3 (C-FLS) *Let M_{t+1} and L_t be given. The Clearing Function Lion Strategy (C-FLS) selects the position of the lion $L_{t+1} \in \mathcal{L}_{t+1}$ in order to minimize β_{t+1}, i.e.,*

$$L_{t+1} = \arg\min_{L \in \mathcal{L}_{t+1}} \max_{P \in \mathcal{R}_{t+1}} f(P) \ . \tag{9}$$

Once L_{t+1} has been chosen, $\mathcal{R}_{t+1}$ is set according to (4) and hence by (6) and (7) one has

$$\beta_{t+1} = \min_{L \in \mathcal{L}_{t+1}} \max_{P \in \mathcal{R}_{t+1}} f(P) \tag{10}$$

$$P_{t+1} = \arg \max_{P \in \mathcal{R}_{t+1}} f(P) = F_{\beta_{t+1}} \cap \mathcal{R}_{t+1} \ . \tag{11}$$

The rationale behind the C-FLS is to push the current β-level curve as close as possible to the origin, in order to minimize the contaminated region. In fact, if at a given time t the man crosses F_{β_t}, he also crosses the line of control $\mathcal{R}_t$ and therefore the lion will catch him. Therefore, the minimization of β at every time t aims at maximizing the cleared region defined by the chosen C-function. Hereafter, we refer to L_{t+1} in (9) as the optimal lion's move at time $t+1$. Fig. 2 shows an example of an optimal lion's move, along with the involved level curves.

Remark 1 *It is worth highlighting the difference between the approach proposed in this paper and that usually adopted in previous works like [23,16,6]. In those papers, the proof of capture is based on* invariants *related to the center of the game, guaranteeing that the cleared region steadily increases by a finite quantity. In particular, in [23,16] the pursuer keeps increasing its distance from the center (which is set at the beginning of the game and remains fixed), while in [6] the center is steadily pushed towards the origin in order to further reduce the contaminated region. In this work, the considered invariant is related to the maximum value of the C-function along the line of control resulting from the lion's move. In fact, in Section 5 it will be shown that by minimizing such a value, the lion will be able to force β_t below a certain threshold, depending only on the chosen C-function, under which capture occurs in the next move.*

From here on, if not otherwise specified, we assume that the lion plays the C-FLS. For a generic C-function f, the computation of L_{t+1} in (9) is not straightforward. In the next section, some properties of the C-FLS are derived in order to fully characterize the optimal lion's move and an algorithm for its computation is provided.

4 Computation of the optimal lion's move

4.1 Properties of the C-FLS

At a generic time t, let M_{t+1} and L_t be given. The aim is to compute the optimal lion's move L_{t+1} according to Definition 3. Hereafter, we assume $\|M_{t+1} - L_t\| > 1$, otherwise the lion catches the man at time $t+1$.

Let $\mathcal{A}$ be the arc of circumference $\delta\boldsymbol{\mathcal{C}}(L_t, 1)$ visible from M_{t+1}, i.e.,

$$\mathcal{A} = \{P \in \delta\boldsymbol{\mathcal{C}}(L_t, 1) \colon (M_{t+1} - P)'(L_t - P) \le 0\} \tag{12}$$

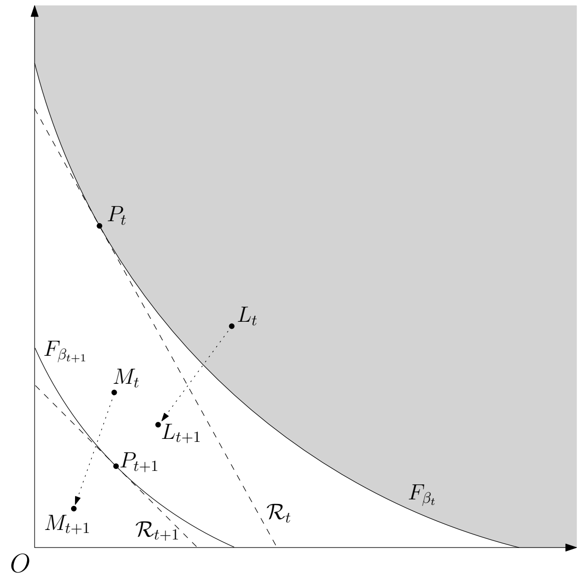

Fig. 2. Example of an optimal lion's move. The grey region denotes the cleared area at time t.

and let $\mathcal{Q} = \{Q \in \mathbb{R}^2_+ : Q \succeq M_{t+1}\}$ be the positive cone with vertex at M_{t+1}. Define $\mathcal{A}_+ = \mathcal{A} \cap \mathcal{Q}$ and set the interval $\Theta = [\underline{\theta}, \overline{\theta}]$ so that

$$\mathcal{A}_+ = \{L \in \mathcal{Q} \colon L = L_t + [\cos\theta, \sin\theta]', \ \theta \in \Theta\} \ . \tag{13}$$

In other words Θ denotes the angular interval spanning the arc $\mathcal{A}_+$ (Fig. 3 shows an example in which $\mathcal{A}_+ = \mathcal{A}$).

The following lemma defines a necessary condition for the lion's move to be optimal.

Lemma 1 *Let L_t and M_{t+1} be given. The optimal lion's move at time $t+1$ is such that $L_{t+1} \in \mathcal{A}_+$.*

Proof: Firstly, notice that L_{t+1} must belong to $\mathcal{Q}$, otherwise condition $L_{t+1} \in \mathcal{L}_{t+1}$ required by Definition 3 does not hold. We want to prove $L_{t+1} \in \mathcal{A}$. By contradiction, let us consider $\widehat{L}\colon \|\widehat{L} - L_t\| \le 1$ such that $\widehat{L} \notin \mathcal{A}$. Choose $\widetilde{L} \in \mathcal{A}$ such that $\widetilde{L}$ belongs to the segment with extremal points M_{t+1} and $\widehat{L}$. Notice that such a point always exists by definition of $\mathcal{A}$, see, e.g., Fig. 3. Let $\widehat{\mathcal{R}}$ and $\widetilde{\mathcal{R}}$ be the lines of control corresponding to $\widehat{L}$ and $\widetilde{L}$, respectively. Since M_{t+1}, $\widetilde{L}$ and $\widehat{L}$ are collinear, then $\widehat{\mathcal{R}}$ and $\widetilde{\mathcal{R}}$ are parallel. Because $\|M_{t+1} - \widetilde{L}\| < \|M_{t+1} - \widehat{L}\|$, then $\widetilde{\mathcal{R}}$ is closer to M_{t+1} than $\widehat{\mathcal{R}}$. Hence, if $\widetilde{P} = \arg\max_{P \in \widetilde{\mathcal{R}}} f(P)$, there exists $\widehat{P} \in \widehat{\mathcal{R}}$ such that $\widehat{P} \succ \widetilde{P}$. Therefore, by exploiting (6) and item ii) of Definition 1, one has

$$\beta_{t+1} = \max_{P \in \widehat{\mathcal{R}}} f(P) \ge f(\widehat{P}) > f(\widetilde{P}) = \max_{P \in \widetilde{\mathcal{R}}} f(P) \ .$$

Thus, by (9) it follows that $\widehat{L}$ cannot be the optimal lion's move. □

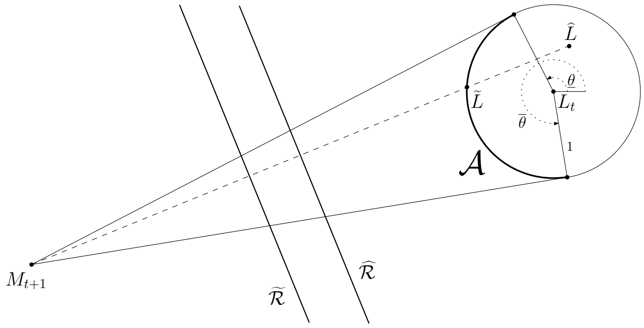

Fig. 3. Proof of Lemma 1. Being M_{t+1}, $\widetilde{L}$ and $\widehat{L}$ collinear, $\widehat{L}$ cannot be optimal.

The following theorem will be used to devise a bisection procedure for computing the optimal lion's move.

Theorem 1 *Let L_t, M_{t+1} be given. Denote by L_{t+1} the optimal lion's move according to* (9) *and let $\theta^* = \angle(L_{t+1} - L_t)$. Let $\widetilde{L} \in \mathcal{A}_+$ and $\widetilde{\mathcal{R}}$ be the corresponding line of control. Denote by $\widetilde{\theta} = \angle(\widetilde{L} - L_t)$, $\widetilde{P} = \arg\max_{P \in \widetilde{\mathcal{R}}} f(P)$ and $\widetilde{\varphi} = \angle(\widetilde{P} - L_t)$. Then,*

$$\text{if } \widetilde{\varphi} < \widetilde{\theta} \text{ then } \theta^* < \widetilde{\theta}\ , \tag{14a}$$

$$\text{if } \widetilde{\varphi} = \widetilde{\theta} \text{ then } \theta^* = \widetilde{\theta}\ , \tag{14b}$$

$$\text{if } \widetilde{\varphi} > \widetilde{\theta} \text{ then } \theta^* > \widetilde{\theta}\ . \tag{14c}$$

Proof: Consider the case $\widetilde{\varphi} < \widetilde{\theta}$. Notice that this implies $\widetilde{\theta} > \underline{\theta}$ (in fact, if $\widetilde{\theta} = \underline{\theta}$, one necessarily has $\widetilde{\varphi} > \widetilde{\theta}$, being $\widetilde{L} - L_t$ parallel to $\widetilde{\mathcal{R}}$). For any $\breve{L} \in \mathcal{A}_+$, let $\breve{\theta} = \angle(\breve{L} - L_t)$, $\breve{\mathcal{R}}$ be the associated line of control, $\breve{P} = \arg\max_{P \in \breve{\mathcal{R}}} f(P)$ and $\breve{\varphi} = \angle(\breve{P} - L_t)$. Being $\breve{\mathcal{R}}$ and $\breve{P}$ continuous functions of $\breve{L}$, it is always possible to choose $\breve{L} \in \mathcal{A}_+$ sufficiently close to $\widetilde{L}$, so that

$$\breve{\varphi} < \breve{\theta} < \widetilde{\theta} \tag{15}$$

where the first inequality follows from $\widetilde{\varphi} < \widetilde{\theta}$ and the continuity argument (an example is shown in Fig. 4).
To prove (14a), it is sufficient to show that any $\widehat{L}$ such that $\widehat{\theta} = \angle(\widehat{L} - L_t) > \breve{\theta}$ cannot be the optimal lion's move. Let $\widehat{\mathcal{R}}$ be the line of control associated to $\widehat{L}$ and define $Q = [x_Q, y_Q]' = \breve{\mathcal{R}} \cap \widehat{\mathcal{R}}$. Since $\|Q - M_{t+1}\| = \|Q - \breve{L}\| = \|Q - \widehat{L}\|$, Q is the center of a circumference passing through M_{t+1}, $\breve{L}$ and $\widehat{L}$. As $\breve{L}$ and $\widehat{L}$ lie also on $\delta\mathcal{C}(L_t, 1)$, the vector $Q - L_t$ bisects the angle formed by vectors $\breve{L} - L_t$ and $\widehat{L} - L_t$, i.e., $\angle(Q - L_t)$ must be equal to $\theta_Q = (\widehat{\theta} + \breve{\theta})/2$. Since $\widehat{\theta} > \breve{\theta}$, one has $\theta_Q > \breve{\theta}$ and then it is straightforward to conclude that Q lies on $\breve{\mathcal{R}}$ below $\breve{P}$, i.e., if $\breve{P} = [x_P,\, y_P]'$ one has $y_Q < y_P$.
Now, let $\breve{a}$ and $\widehat{a}$ be the angular coefficients of $\breve{\mathcal{R}}$ and $\widehat{\mathcal{R}}$, respectively. We want to show that $\widehat{a} < \breve{a}$. Define the vectors $\breve{V} = \breve{L} - M_{t+1}$ and $\widehat{V} = \widehat{L} - M_{t+1}$ and let $\breve{\phi}$ and $\widehat{\phi}$ their corresponding phases. Since $\widehat{L} \succeq M_{t+1}$ and $\widehat{\theta} > \breve{\theta}$, one has $\breve{\phi} > \widehat{\phi} \geq 0$. Being $\breve{\mathcal{R}}$ and $\widehat{\mathcal{R}}$ orthogonal to $\breve{V}$ and $\widehat{V}$, respectively, $\widehat{a} < \breve{a}$ immediately follows.
Since $y_Q < y_P$ and $\widehat{a} < \breve{a}$, there exists $\widehat{P} \in \widehat{\mathcal{R}}$ such that $\widehat{P} \succ \breve{P}$. By (6) and by item ii) in Definition 1, one has

$$\max_{P \in \widehat{\mathcal{R}}} f(P) \geq f(\widehat{P}) > f(\breve{P}) = \max_{P \in \breve{\mathcal{R}}} f(P)$$

and so $\widehat{L}$ cannot be optimal and then $\theta^* \leq \breve{\theta} < \widetilde{\theta}$, according to (15). To prove (14b) and (14c), a similar reasoning can be applied. A sketch of the three different cases is reported in Fig. 5.

□

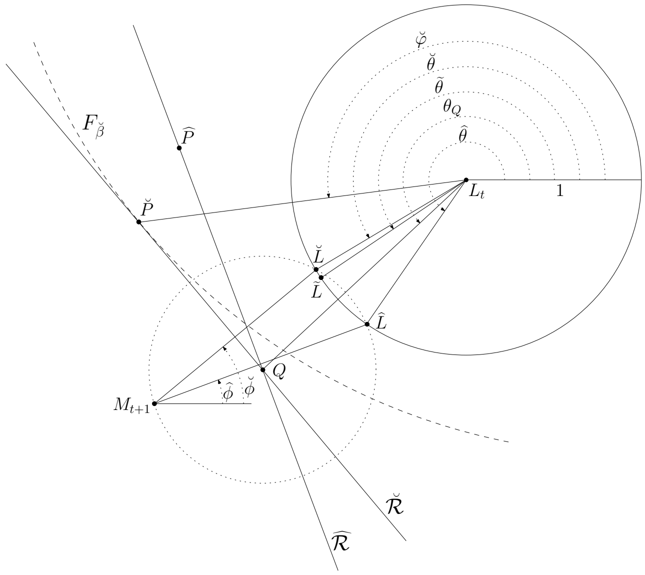

Fig. 4. Proof of Theorem 1. There exists $\widehat{P} \in \widehat{\mathcal{R}}$ such that $f(\widehat{P}) > f(\breve{P}) = \max_{P \in \breve{\mathcal{R}}} f(P)$, so $\widehat{L}$ is not optimal.

The next corollary gives a characterization of the optimal lion's move in terms of P_{t+1}.

Corollary 1 *Let L_t and M_{t+1} be given. Then, the optimal lion's move at time $t+1$ is such that*

$$L_{t+1} = L_t + \frac{P_{t+1} - L_t}{\|P_{t+1} - L_t\|} \tag{16}$$

where P_{t+1} is given by (11)*. Moreover,*

$$\|P_{t+1} - L_{t+1}\| = \|P_{t+1} - L_t\| - 1 \tag{17}$$

and

$$\|P_{t+1} - L_t\| - \|P_{t+1} - M_{t+1}\| = 1\ . \tag{18}$$

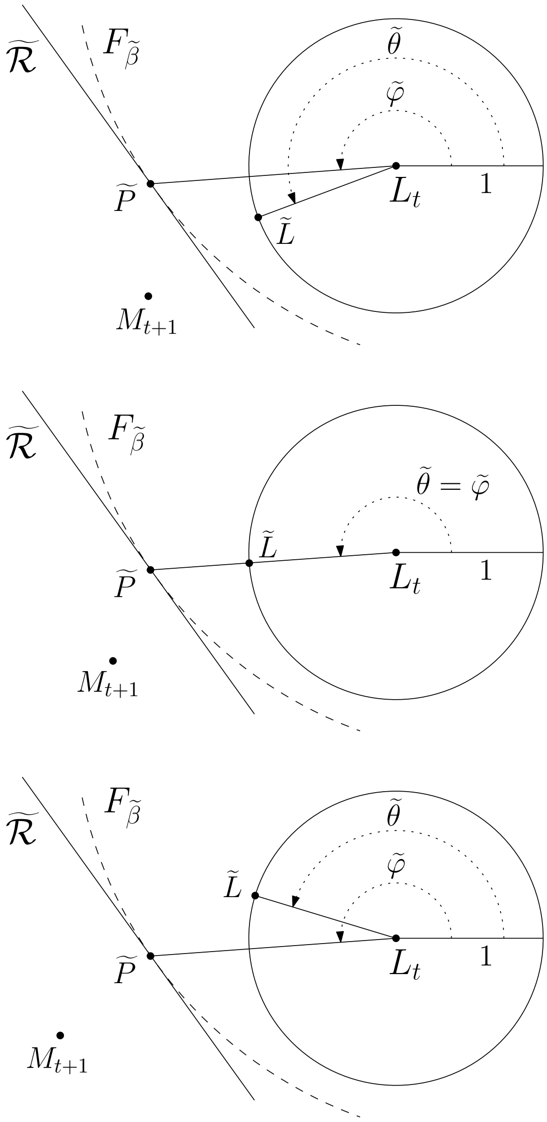

Fig. 5. Sketch of the three cases of Theorem 1. Top: $\widetilde{\varphi} < \widetilde{\theta}$. Middle: $\widetilde{\varphi} = \widetilde{\theta}$. Bottom: $\widetilde{\varphi} > \widetilde{\theta}$.

Proof: By adopting the same continuity argument as in Theorem 1, one can conclude that there always exists one $\widetilde{\theta}$ such that $\widetilde{\varphi} = \widetilde{\theta}$ (notice that (14a) or (14c) cannot hold for all $\widetilde{\theta} \in \Theta$). Hence, one has

$$\theta^* = \angle(L_{t+1} - L_t) = \widetilde{\varphi} = \angle(P_{t+1} - L_t) \ ,$$

i.e., L_t, L_{t+1} and P_{t+1} are collinear. Then, (16) follows directly. By (16), one easily gets (17). Moreover, since $P_{t+1} \in \mathcal{R}_{t+1}$, one has $\|P_{t+1} - L_{t+1}\| = \|P_{t+1} - M_{t+1}\|$ and then (18) holds. □

Notice that, by Corollary 1 and by the uniqueness of P_{t+1}, the optimal lion's move L_{t+1} turns out to be unique.

4.2 C-FLS algorithm

A procedure for the computation of the optimal C-FLS lion's move is reported in Algorithm 1. In the procedure, ε denotes a given tolerance, $\widetilde{P}$ is the current candidate for the solution of (11), and $\widetilde{\varphi}$ is the phase of $\widetilde{P} - L_t$. By exploiting Theorem 1, a bisection on the interval $\Theta = [\underline{\theta}, \overline{\theta}]$ is performed. The main task is the computation of $\widetilde{P} = \arg\max_{P \in \widetilde{\mathcal{R}}} f(P)$, which is carried out by the function *compute_tangent_point*. In fact, once the line of control $\widetilde{\mathcal{R}} = \{[x, y]' \in \mathbb{R}^2_+ : y = ax + b\}$ has been obtained, the computation of $\widetilde{P}$ boils down to finding the level curve of $f(\cdot)$ tangent to $\widetilde{\mathcal{R}}$. Clearly, this depends on the specific C-function selected. However, an analytical solution for $\widetilde{P}$ is often available. If the C-function is the quarter of circumference defined in (1), the point $\widetilde{P}$ is given by

$$\widetilde{P} = \left[\frac{(1+a)\widetilde{\beta} - ab}{a^2 + 1}, \ a\frac{(1+a)\widetilde{\beta} - ab}{a^2+1} + b \right]'$$

where $\widetilde{\beta} = \max_{P \in \widetilde{\mathcal{R}}} f(P) = \dfrac{b(a - 1 - \sqrt{1 + a^2})}{2a}$. Similarly, for the branch of hyperbola in (2), one has

$$\widetilde{P} = \left[-\frac{bm + (a+1)\widetilde{\beta}}{2am}, \ \frac{bm - (a+1)\widetilde{\beta}}{2m} \right]'$$

where $\widetilde{\beta}$ is the largest root of $\widetilde{\beta}^2 \left[(1+a)^2 + 4am\right] + \widetilde{\beta}\left[2mb(1-a)\right] + b^2 m^2 = 0$.

It is worth stressing that, once the C-function has been chosen, a single iteration of the main loop of Algorithm 1 requires only few computations which can be easily expressed in analytic form. Therefore, the computational complexity of the algorithm depends only on the number of iterations required to achieve the desired tolerance. Since by (13) one has $\overline{\theta} - \underline{\theta} < \pi$, the number of iterations needed by the algorithm is upper bounded by $\lceil \log_2(\pi/\varepsilon) \rceil$.

5 Finite-time capture result

In this section, it will be shown that the C-FLS guarantees that the lion is able to capture the man in finite time, for every choice of the C-function satisfying Definition 1.

At a given time t, let M_t and L_t be given. Let us define

$$\mathcal{Q}(P_t) = \{[x, y]' \in \mathbb{R}^2_+ : y \leq a_t x + b_t\} \qquad (19)$$

Algorithm 1 Algorithm for the computation of the optimal lion's move.

```
Data: L_t, M_{t+1}
Result: L_{t+1}
Set θ̲, θ̄ according to (13)
while (θ̄ − θ̲) > ε do
    θ̃ = (θ̄ + θ̲)/2
    L̃ = L_t + [cos θ̃, sin θ̃]'
    R̃ = line_of_control(L̃, M_{t+1})
    P̃ = compute_tangent_point(R̃, f(·))
    φ̃ = ∠(P̃ − L_t)
    if φ̃ < θ̃ then
        θ̄ = θ̃
    else
        θ̲ = θ̃
    end if
end while
L_{t+1} = L̃
```

where a_t and b_t are the coefficients of $\mathcal{R}_t$ as in (5). Hence, $\mathcal{Q}(P_t)$ is the triangle bounded by the line of control $\mathcal{R}_t$ and the coordinate axes.

Before the derivation of the main result, let us introduce two sufficient conditions for the lion to capture the man in one move.

Proposition 1 *Given L_t and M_t, let $R_x = [x_R,\, 0]'$ and $R_y = [0,\, y_R]'$ denote the intersections of $\mathcal{R}_t$ with the coordinate axes. Let $l = \max\{x_R,\, y_R\}$. If $l \le \frac{1}{\sqrt{2}}$, then the game ends in one move.*

Proof: By the definition of $\mathcal{Q}(P_t)$ in (19), if the man moves to a point $M_{t+1} \notin \mathcal{Q}(P_t)$, i.e., he crosses the line of control, the lion catches him in one move. Let $\mathcal{S}$ be the minimum square in $\mathbb{R}^2_+$ containing $\mathcal{Q}(P_t)$, see, e.g., Fig. 6. Clearly, the side of $\mathcal{S}$ is equal to $l = \max\{x_R,\, y_R\}$. Since $M_t \in \mathcal{Q}(P_t)$, by the definition of line of control in (4), one has $L_t \in \mathcal{S}$. Since $l \le \frac{1}{\sqrt{2}}$, for any $L_t \in \mathcal{S}$ one has $\boldsymbol{\mathcal{C}}(L_t, 1) \supseteq \mathcal{Q}(P_t)$. Therefore, the lion is able to catch the man in one move for any $M_{t+1} \in \mathcal{Q}(P_t)$. □

Proposition 2 *Let*

$$\beta_{min} = \min\{f([1/\sqrt{2},\, 0]'), f([0,\, 1/\sqrt{2}]')\} \; . \qquad (20)$$

If, at some step t, $\beta_t \le \beta_{min}$, then the game ends in one move.

Proof: The proof is a direct consequence of Proposition 1 and of the properties of the C-functions. □

Let us now introduce a function which will be instrumental to prove that capture occurs in finite time. At a given time t, let

$$J_t = f(P_t) + \alpha\|P_t - L_t\| = f(P_t) + \alpha\|P_t - M_t\| \quad (21)$$

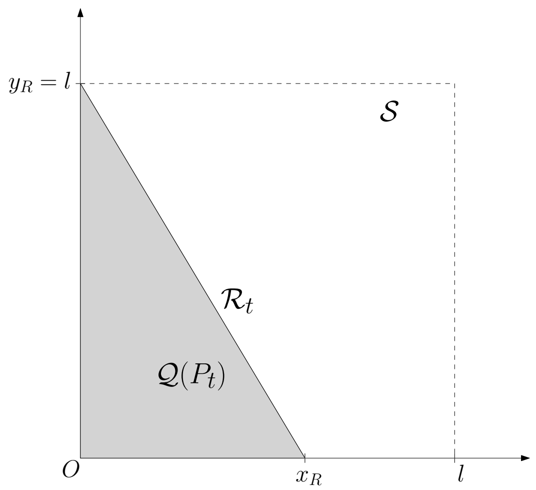

Fig. 6. Sketch of the proof of Proposition 1. If $M_t \in \mathcal{Q}(P_t)$ and $l \le 1/\sqrt{2}$, then the game ends in one move.

for a fixed $\alpha > 0$. The aim is to prove that, for a suitable α, J_t goes to zero in a finite number of game moves. In fact, this implies that also $f(P_t)$ must go to zero, thus guaranteeing that the lion captures the man, according to Proposition 2. By (18) one has

$$\begin{aligned}\|P_{t+1} - M_{t+1}\| &= \|P_{t+1} - L_t\| - 1 \\ &\le \|P_{t+1} - P_t\| + \|P_t - L_t\| - 1\end{aligned}$$

and then

$$\begin{aligned}J_{t+1} &= f(P_{t+1}) + \alpha\|P_{t+1} - M_{t+1}\| \\ &\le f(P_{t+1}) + \alpha(\|P_{t+1} - P_t\| + \|P_t - L_t\| - 1) \; . \end{aligned} \quad (22)$$

Let us define $\Delta J_t = J_t - J_{t+1}$. Then, by (21), (22) and the definition of P_t, one has

$$\Delta J_t \ge f(P_t) - f(P_{t+1}) + \alpha(1 - \|P_{t+1} - P_t\|) \; . \quad (23)$$

In order to prove that J_t goes to zero in finite time, some lemmas are introduced. The first one states that the unique point $P \in \mathcal{Q}(P_t)$ such that $f(P) = \beta_t$ is $P = P_t$.

Lemma 2 *At a given time t, let P_t be given. Then, $f(P) < f(P_t)$, $\forall P \in \mathcal{Q}(P_t) \setminus \{P_t\}$.*

Proof: For all $P \in \mathcal{R}_t$, $P \neq P_t$, by (7) one has $f(P) < f(P_t)$. Moreover, $\forall P \in \mathcal{Q}(P_t) \setminus \mathcal{R}_t$, there exists $\overline{P} \in \mathcal{R}_t$ such that $P \prec \overline{P}$ and hence, by item ii) in Definition 1, $f(P) < f(\overline{P})$. Since, by (7), one has $f(\overline{P}) \le f(P_t)$, the result follows. □

The next lemma shows that, if the lion plays the C-FLS, β_t is a non-increasing function of t.

Lemma 3 *Let M_t and P_t be given. Then, for any man's move M_{t+1}, one has $P_{t+1} \in \mathcal{Q}(P_t)$. Moreover, $f(P_{t+1}) \le f(P_t)$.*

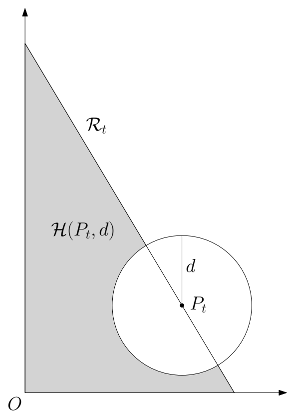

Fig. 7. The grey region denotes the set $\mathcal{H}(P_t, d)$.

Proof: By contradiction, assume $P_{t+1} \notin \mathcal{Q}(P_t)$ and hence $\|P_{t+1} - M_t\| > \|P_{t+1} - L_t\|$. One has

$$\begin{aligned}\|P_{t+1} - M_{t+1}\| &\geq \|P_{t+1} - M_t\| - \|M_{t+1} - M_t\| \\ &\geq \|P_{t+1} - M_t\| - 1 > \|P_{t+1} - L_t\| - 1 \\ &= \|P_{t+1} - L_{t+1}\|\end{aligned}$$

where the last equality is due to (17). Hence, P_{t+1} is not equidistant from M_{t+1} and L_{t+1}, i.e., $P_{t+1} \notin \mathcal{R}_{t+1}$ which leads to a contradiction. Being $P_{t+1} \in \mathcal{Q}(P_t)$, by Lemma 2, $f(P_{t+1}) \leq f(P_t)$ follows directly. □

For a given $d > 0$, let us define the following quantities which are instrumental in the derivation of the main result:

$$\mathcal{H}(P_t, d) = \{P \in \mathcal{Q}(P_t) \colon \|P - P_t\| \geq d\} \tag{24}$$

$$\Delta(P_t, d) = \min_{P_{t+1} \in \mathcal{H}(P_t, d)} f(P_t) - f(P_{t+1}) \tag{25}$$

$$\Delta_d = \min_{P_t : \beta_{min} \leq f(P_t) \leq \beta_0} \Delta(P_t, d) \tag{26}$$

where β_{min} is defined as in (20) and β_0 depends on the game initial conditions, according to (6).

The set $\mathcal{H}(P_t, d)$ denotes the set of points in $\mathcal{Q}(P_t)$ whose distance from P_t is greater or equal to d. An example of $\mathcal{H}(P_t, d)$ is shown in Fig. 7. The value $\Delta(P_t, d)$ represents the minimum "gap" between $\beta_t = f(P_t)$ and $\beta_{t+1} = f(P_{t+1})$ for $P_{t+1} \in \mathcal{H}(P_t, d)$, i.e., when the distance between P_{t+1} and P_t is at least d. Finally, Δ_d is the minimum value of $\Delta(P_t, d)$ for all the possible choices of P_t such that $\beta_{min} \leq f(P_t) \leq \beta_0$ (notice that, according to Lemma 3 and Proposition 2, the interval $[\beta_{min}, \beta_0]$ contains all the feasible values of β_t from the beginning of the game to its end).

Lemma 4 *Let P_t and $d > 0$ be given such that $\mathcal{H}(P_t, d) \neq \emptyset$. Then,*

i) $\Delta(P_t, d) > 0$ and $\Delta_d > 0$;

ii) $\Delta(P_t, \widehat{d}) \leq \Delta(P_t, d)$ and $\Delta_{\widehat{d}} \leq \Delta_d$, $\forall \widehat{d} \in (0, d)$.

Proof: i) Let $d > 0$ be fixed. It follows that $\mathcal{H}(P_t, d) \subseteq \mathcal{Q}(P_t) \setminus \{P_t\}$ and by Lemma 2, $f(P) < f(P_t)$ for all $P \in \mathcal{H}(P_t, d)$. Since $\mathcal{H}(P_t, d)$ is a compact set, one has $\Delta(P_t, d) > 0$. Moreover, $\Delta_d > 0$ holds because the minimum in (26) is evaluated on a compact set.
ii) From (24), one has $\mathcal{H}(P_t, \widehat{d}) \supseteq \mathcal{H}(P_t, d)$. So, by (25), one gets $\Delta(P_t, \widehat{d}) \leq \Delta(P_t, d)$ for all P_t and therefore $\Delta_{\widehat{d}} \leq \Delta_d$. □

Let

$$\overline{d} = \max \left\{ \max_{\substack{P_a, P_b \in \mathbb{R}^2_+ : \\ f(P_a), f(P_b) \leq \beta_0}} \|P_a - P_b\| \ , \ 1 \right\} . \tag{27}$$

be the maximum distance between two points P_a and P_b for which the corresponding β is not greater than β_0. If such a distance is less than 1, $\overline{d}$ is set to 1.

Notice that, by Lemma 3, $f(P_t) \geq f(P_{t+1})$ and since P_t and P_{t+1} are such that $f(P_t), f(P_{t+1}) \leq \beta_0$, by (27) one has

$$\|P_{t+1} - P_t\| \leq \overline{d}, \ \forall t . \tag{28}$$

The next lemma states that the function J_t in (21) strictly decreases at each move.

Lemma 5 *Let $\alpha = \dfrac{\Delta_{1/2}}{\overline{d} - 1/2}$. Then, $\Delta J_t \geq \alpha/2 > 0$ for all t.*

Proof: By Lemma 4-i), $\Delta_{1/2} > 0$ and since $\overline{d} \geq 1$ one has $\alpha > 0$. Let $d_t = \|P_{t+1} - P_t\|$. We consider three possible cases.
First, let us assume $0 \leq d_t \leq 1/2$. Since $f(P_t) \geq f(P_{t+1})$, by (23) one has $\Delta J_t \geq \alpha(1 - \|P_{t+1} - P_t\|) \geq \alpha/2$.
Now, let us consider $1/2 < d_t \leq 1$. Since $d_t \leq 1$, by (23) one gets

$$\begin{aligned}\Delta J_t &\geq f(P_t) - f(P_{t+1}) \geq \Delta(P_t, d_t) \geq \Delta(P_t, 1/2) \\ &\geq \Delta_{1/2} = (\overline{d} - 1/2)\,\alpha \geq \alpha/2\end{aligned}$$

where the second inequality is due to the fact that $P_{t+1} \in \mathcal{H}(P_t, d_t)$, the third inequality holds by Lemma 4-ii) and the last inequality comes from $\overline{d} \geq 1$.
Finally, let us address the case $d_t > 1$. By (28), we may restrict our analysis to $1 < d_t \leq \overline{d}$. Hence,

$$\begin{aligned}\Delta J_t &\geq \Delta(P_t, 1) + \alpha(1 - d_t) \geq \Delta_1 - \alpha\,(\overline{d} - 1) \\ &\geq \Delta_{1/2} - \alpha\,(\overline{d} - 1) = (\overline{d} - 1/2)\,\alpha - \alpha\,(\overline{d} - 1) = \alpha/2\end{aligned}$$

where the third inequality holds by Lemma 4-ii). This concludes the proof. □

The role of the function J_t is clarified by the proof of Lemma 5. In fact, there are moves in which the decrease of $f(P_t)$ may be negligible (or even null). However, in such cases also $\|P_{t+1} - P_t\|$ must be small, and hence (23) guarantees a significant decrease of J_t.

We are finally ready to state the main result of this section, which guarantees finite time capture independently on the strategy adopted by the man.

Theorem 2 *Let $L_0 \succ M_0$ and let the lion play the C-FLS for a fixed C-function f. Then, there exists a finite time t for which $L_t = M_t$.*

Proof: Let us consider the function J_t in (21). By Lemma 5, at each step such a function decreases at least by a finite quantity $\alpha/2 = \frac{1}{2}\frac{\Delta_{1/2}}{\bar{d}-1/2} > 0$. So, there exists a finite $\bar{t}$ such that $J_{\bar{t}} < \beta_{min}$ which implies $f(P_{\bar{t}}) = \beta_{\bar{t}} < \beta_{min}$. Then, by Proposition 2 the game ends at the next move and the theorem is proved. □

The capture result in Theorem 2 is based on the exact computation of the lion's move in (9). Nevertheless, it can be easily seen that the result still holds if the lion's move is computed according to the bisection procedure in Algorithm 1. In fact, J_t in (21) is affected by an error which depends continuously on the tolerance ε in Algorithm 1. Hence, one can choose ε sufficiently small to guarantee that ΔJ_t is always larger than a fixed positive quantity, by following an argument similar to that in the proof of Lemma 5. Then, the proof of Theorem 2 still holds.

6 Comparison with other strategies

In this section, the proposed lion strategy is compared to other strategies available in the literature. In particular, we consider the lion strategy proposed in [23], hereafter referred to as *Lion Strategy 1 (LS_1)*, which is based on a fixed center, and the one presented in [6], denoted as *Lion Strategy 2 (LS_2)*, in which the center is updated at each move (see Remark 1). Concerning the C-FLS, the two C-functions described in Section 2 have been adopted. The strategy based on (1) is referred to as $C\text{–}FLS_{circ}$, while that using the C-function (2) is named $C\text{–}FLS_{hyper}$.

In order to simulate the considered pursuit strategies, one has to define also the evasion strategy adopted by the man. It is worth remarking that the optimal man strategy (i.e., the worst-case man behaviour for the lion) has not been devised yet for any of the considered approaches. Therefore, several alternative man strategies are used in the simulations. At any time t, such strategies set the man position M_{t+1} so that $\|M_{t+1} - M_t\| = 1$. They are defined as follows.

MS_1: This man strategy has been proposed in [23]: M_{t+1} is set such that $(M_{t+1} - M_t)$ is orthogonal to $(L_t - M_t)$. Between the two candidate points, the one with the greatest product of the two coordinates is chosen. The choice of the orthogonal direction is motivated by a trade-off between going towards the lion (which is not a good choice!) and moving in the opposite direction, which results in approaching the region close to the origin, where capture occurs (see Proposition 1). In [23] it is also shown that this man strategy is optimal for some specific initial lion and man positions.

MS_2: This strategy works like MS_1 with the only difference that among the two candidate man's moves, the one with smaller product of the coordinates is chosen.

MS_{circ}: This strategy has been explicitly designed to play against the C-FLS when the chosen C-function is the quarter of circumference in (1). As long as $\|M_t - P_t\| > 1$, the man moves towards P_t, i.e., $M_{t+1} = M_t + (P_t - M_t)/\|P_t - M_t\|$. When $\|M_t - P_t\| \leq 1$, the man moves to the point in $\mathcal{Q}(P_t) \setminus \mathcal{R}_t$ which maximizes $f(M_{t+1})$. Notice that, whenever $\|M_t - P_t\| > 1$, β_t cannot decrease, irrespectively of the lion strategy. In fact, if the lion plays C-FLS (with the same C-function) one has $\beta_{t+1} = \beta_t = \beta_0$, while if the lion plays a different strategy, β_t may even increase.

MS_{hyper}: This is the same as MS_{circ}, but choosing the branch of hyperbola defined in (2) as C-function. By performing several games and by comparing the resulting performance, the tuning parameter m has been set to $m = 2$.

Example 1 *First, in order to illustrate the different behaviors of the considered approaches, a single game has been played with different man and lion strategies. The initial conditions are set to $M_0 = [1, 4]'$ and $L_0 = [2, 6]'$. For each lion algorithm, the best man strategy (i.e., the strategy which allows the man to survive for the longer number of moves) among the considered ones is adopted. In Fig. 8, the paths traveled by the players for each lion strategy are reported along with the game length. It is apparent that the two lion strategies based on C-functions outperform the approaches LS_1 and LS_2. Notice that, as expected, the man strategies MS_{circ} and MS_{hyper} are those allowing the man to survive longer against the corresponding C-function lion strategies $C{-}FLS_{circ}$ and $C{-}FLS_{hyper}$. When the distance between the two players becomes small, a chattering phenomenon may be observed; the reason is due to the fact that, to delay capture as much as possible, the man moves back and forth orthogonally to $(L_t - M_t)$, causing the zigzag paths.*

Example 2 *A simulation campaign consisting of $K =$ 10000 games has been performed with the following parameters. Let $M_0 = [x_m, y_m]'$ and $L_0 = [x_l, y_l]'$ denote the man and lion initial positions, respectively. Such positions have been randomly generated with uniform distribution such that $L_0 \succ M_0$; in particular, the initial positions have been generated so that $x_m, y_m \in (0, 10)$ and $(x_l - x_m), (y_l - y_m) \in (0, 10)$.*

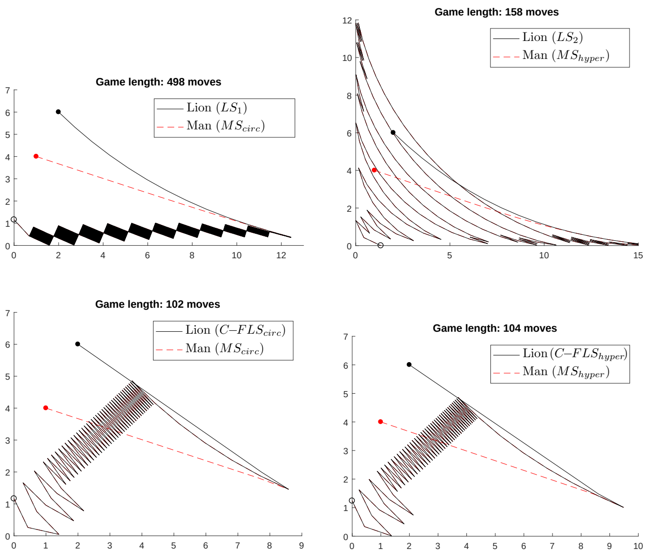

Fig. 8. Example 1. Paths traveled by the lion (solid black) and by the man (dashed red) for different strategies. Dots denote initial positions while circles denote the capture points.

In Table 1, the average game length over all the simulated games are reported for each combination of lion and man strategies. The maximum game length for each lion strategy is emphasized in boldface. It is evident that the lion algorithms based on C-functions require, on average, a number of moves much smaller than that needed by LS_1 *and* LS_2*. Regarding the man strategy, one has that the methods* MS_{circ} *and* MS_{hyper} *are those which allow the man to survive longer. This suggest that also for the man it is convenient to play a game based on the optimization of a C-function.*

Table 1
Example 2. Average game length for different strategies

		Man Strategy			
		MS_1	MS_2	MS_{circ}	MS_{hyper}
Lion Strategy	LS_1	3558.49	3557.79	5043.60	**5198.51**
	LS_2	594.73	1058.61	1011.05	**1865.07**
	$C\text{–}FLS_{circ}$	255.31	42.78	**660.55**	145.96
	$C\text{–}FLS_{hyper}$	253.14	133.68	**763.93**	693.66

For a given game k*, let us denote by* $N^a(k)$ *the number of moves needed by the lion to catch the man by using strategy* a*, for the best man strategy among the considered ones. Let us define the following performance indexes to compare the proposed new lion strategies with* LS_1 *and* LS_2*.*

$$R^{LS_1} = \frac{1}{K}\sum_{k=1}^{K}\frac{N^a(k)}{N^{LS_1}(k)} \ ,$$

$$R^{LS_2} = \frac{1}{K}\sum_{k=1}^{K}\frac{N^a(k)}{N^{LS_2}(k)} \ .$$

Table 2
Example 2. Indexes R^{LS_1} and R^{LS_2} for different lion strategies

		R^{LS_1}	R^{LS_2}
Lion Strategy	LS_1	1	3.467
	LS_2	0.324	1
	$C\text{–}FLS_{circ}$	0.201	0.638
	$C\text{–}FLS_{hyper}$	0.211	0.676

In Table 2, numerical values of R^{LS_1} *and* R^{LS_2} *are reported. Notice that,* $C\text{–}FLS_{circ}$ *allows the lion to win the game in a number of moves which is, on average, about 64% of those needed by* LS_2 *and about 20% w.r.t. the commonly used method proposed in [23].*

It should be stressed that, considering the performance $N^a(k)$ *of the different lion strategies, among the 10000 played games,* $C\text{–}FLS_{circ}$ *took less moves than the other strategies 7738 times, while* $C\text{–}FLS_{hyper}$ *796 times. In the remaining 1466 games, the two strategies based on C-functions required the same number of moves to win. For all the played games, methods* LS_1 *and* LS_2 *always needed a number of moves greater than the two strategies based on the C-functions.*

7 Conclusions

The new approach to the lion and man problem presented in this paper has two main advantages with respect to the existing techniques. First, it provides a general framework, within which the pursuer can choose the preferred C-function, which corresponds to setting the shape of the cleared region. The second main benefit is the apparent superiority in terms of time required to achieve capture, testified by the extensive simulation campaign on randomly generated pursuit problems.

The contribution of the paper paves the way to a number of subsequent developments. From a theoretical point of view, the optimal strategy is still an open problem, on both lion and man side. Understanding the man behaviour which guarantees a longer survival might be beneficial to devise a meaningful criterion for choosing the most appropriate C-function. Another objective is to prove that the C-function approach can be successfully applied to closed environments (e.g., convex/nonconvex polygons), regions containing obstacles or specific zones to be preserved. Moreover, it is possible to modify the C-function during the game, to adapt it to changing conditions. The application of the proposed lion strategies to the more complex scenarios described above is the subject of ongoing research.

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