Sets computing the symmetric tensor rank

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Abstract. Let $\nu_d : \mathbb{P}^r \to \mathbb{P}^N$, $N := \binom{r+d}{r} - 1$, denote the degree d Veronese embedding of $\mathbb{P}^r$. For any $P \in \mathbb{P}^N$, the symmetric tensor rank $sr(P)$ is the minimal cardinality of a set $S \subset \nu_d(\mathbb{P}^r)$ spanning P. Let $\mathcal{S}(P)$ be the set of all $A \subset \mathbb{P}^r$ such that $\nu_d(A)$ computes $sr(P)$. Here we classify all $P \in \mathbb{P}^n$ such that $sr(P) < 3d/2$ and $sr(P)$ is computed by at least two subsets of $\nu_d(\mathbb{P}^r)$. For such tensors $P \in \mathbb{P}^N$, we prove that $\mathcal{S}(P)$ has no isolated points.

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1. Introduction

Let $\nu_d : \mathbb{P}^r \to \mathbb{P}^N$, $N := \binom{r+d}{r} - 1$, denote the degree d Veronese embedding of $\mathbb{P}^r$. Set $X_{r,d} := \nu_d(\mathbb{P}^r)$. For any $P \in \mathbb{P}^N$, the *symmetric rank* or *symmetric tensor rank* or, just, the *rank* $sr(P)$ of P is the minimal cardinality of a finite set $S \subset X_{r,d}$ such that $P \in \langle S \rangle$, where $\langle \ \ \rangle$ denote the linear span.

For any $P \in \mathbb{P}^N$, let $\mathcal{S}(P)$ denote the set of all finite subsets $A \subset \mathbb{P}^r$ such that $\nu_d(A)$ computes $sr(P)$, i.e. the set of all $A \subset \mathbb{P}^r$ such that $P \in \langle \nu_d(A) \rangle$ and $\sharp(A) = sr(P)$. Notice that if $A \in \mathcal{S}(P)$, then $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$.

The study of the sets $\mathcal{S}(P)$ has a natural role in the theory of symmetric tensors. Indeed, if we interpret points $P \in \mathbb{P}^n$ as symmetric tensors, then $\mathcal{S}(P)$ is the set of all the representations of P as a sum of rank 1 tensors. For many applications, it is crucial to have some information about the structure of $\mathcal{S}(P)$. We do not recall the impressive literature on the subject (but see [15], for a good references' repository). The interest in the theory is growing, since applications of tensors are actually increasing in Algebraic Statistics, and then in Biology, Chemistry and also Linguistics (see e.g. [15] and [16]). Let us mention one relevant aspect, from our point of view. If we are looking for one specific decomposition of P as a sum of tensors of rank 1, and we find some

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decomposition, how to ensure that the found decomposition is the expected one? Of course, if $\mathcal{S}(P)$ is a singleton, the answer is obvious. In a recent paper ([8]) Buczyński, Ginensky and Landsberg proved that $\sharp(\mathcal{S}(P)) = 1$ when the rank is small, i.e. $sr(P) \leq (d+1)/2$. This important uniqueness theorem (which holds more generally for 0-dimensional schemes, see [7] Proposition 2.3) turns out to be sharp, even if $r = 1$. For larger values of the rank, one can determine the uniqueness of the decomposition, when an element $A \in \mathcal{S}(P)$ satisfies some geometric properties (e.g. when no 3 points of A are collinear, see [2], Theorem 2 or when A is in *general uniform position*, see [4]).

In this paper, we describe more closely the set $\mathcal{S}(P)$, for tensors whose rank sits in the range $sr(P) < 3/2$.

In particular, we show that for each P with $\sharp(\mathcal{S}(P)) > 1$, the set $\mathcal{S}(P)$ has no isolated points.

This result has a consequence. Assume we are given $Q \in \mathbb{P}^n$ with $sr(Q) < 3d/2$, and we find $A \in \mathcal{S}(Q)$ which is isolated in $\mathcal{S}(Q)$. Then we can conclude that A is the unique element of $\mathcal{S}(Q)$ (in other words, Q is *identifiable*). This means that, in the specified range, given one decomposition $A \in \mathcal{S}(P)$, one can conclude that A is unique, just by performing an analysis $\mathcal{S}(P)$ in a neighbourhood of A. This sounds to be much easier than looking for other points of $\mathcal{S}(P)$ in the whole space.

Our precise statement is:

Theorem 1. *Assume $r \geq 2$. Fix a positive integer $t < 3d/2$. Fix $P \in \mathbb{P}^N$ such that $sr(P) = t$ and the symmetric rank of P is computed by at least two different sets $A, B \subset \mathbb{P}^r$. Then $sr(P)$ is computed by an infinite family of subsets of $\mathbb{P}^r$, and this family has no isolated points.*

We notice that the notion of "isolated points" requires an algebraic structure of the set $\mathcal{S}(P)$. As well-known (and checked in Section 2), the set $\mathcal{S}(P)$ is constructible in the sense of Algebraic Geometry ([14], Ex. II.3.18 and Ex. II.3.19). This makes more precise the expression "no isolated point" above (see Remark 2 in Section 2 for the details).

We also prove that the bound $t < 3d/2$, in the statement of Theorem 1, is sharp. Indeed, Example 1 provides one tensor P with $sr(P) = 3d/2$ (so d is even), and $\sharp(\mathcal{S}(P)) = 2$.

In the proof, it is not difficult to see that if there are at least two elements in $\sharp(\mathcal{S}(P)) = 2$, when $sr(P) < 3d/2$, then the shape of the Hilbert functions of A and B shows that both sets have a large intersection with either a line, or a conic of $\mathbb{P}^r$ (we will refer to [2] and [13], for this part of the theory). Then, we perform a (maybe tedious, but necessary) analysis of the behaviour of sets of points, with a big intersection with either a line or a conic.

We also provide a deeper description of $\mathcal{S}(P)$, still in the range $sr(P) < 3/2$ and assuming that $\mathcal{S}(P)$ is not a singleton (hence it is infinite). Indeed, we have the following:

Theorem 2. *Assume $r \geq 2$ and $d \geq 3$. Fix a positive integer $t < 3d/2$. Fix $P \in \mathbb{P}^N$ such that $sr(P) = t$. Then, the set $\mathcal{S}(P)$ is not a single point if and only if P may be described in one of the following way:*

(a) *for any* $A \in \mathcal{S}(P)$*, there is a line* $D \subset \mathbb{P}^r$ *such that* $\sharp(A \cap D) \geq \lceil (d+2)/2 \rceil$*; set* $F := A \setminus A \cap D$*; the set* $\langle \nu_d(A \cap D) \rangle \cap \langle \{P\} \cup \nu_d(F) \rangle$*, is formed by a unique point* P_D *and* $\mathcal{S}(P_D)$ *is infinite; for each* $E \in \mathcal{S}(P_D)$ *we have* $E \cap F = \emptyset$ *and* $E \cup F \in \mathcal{S}(P)$.

(b) *for any* $A \in \mathcal{S}(P)$*, there is a smooth conic* $T \subset \mathbb{P}^m$ *such that* $\sharp(A \cap T) \geq d+1$*; set* $F := A \setminus A \cap T$*; the set* $\langle \nu_d(A \cap T) \rangle \cap \langle \{P\} \cup F \rangle$*, is formed by a unique point* P_T *and* $\mathcal{S}(P_T)$ *is infinite; for each* $E \in \mathcal{S}(P_T)$ *we have* $E \cap F = \emptyset$*; every element of* $\mathcal{S}(P)$ *is of the form* $E' \cup F$ *for some* $E' \subset T$ *computing* $\mathcal{S}(P_T)$ *with respect to the rational normal curve* $\nu_d(T)$.

(c) d *is odd; for any* $A \in \mathcal{S}(P)$*, there is a reducible conic* $T = L_1 \cup L_2 \subset \mathbb{P}^m$*,* $L_1 \neq L_2$*, such that* $\sharp(A \cap L_1) = \sharp(A \cap L_2) = (d+1)/2$ *and* $L_1 \cap L_2 \notin A$.

Let us mention that if L is a linear subspace of dimension m in $\mathbb{P}^r$, then the Veronese embedding ν_d, restricted to L, can be identified with a d-th Veronese embedding of $\mathbb{P}^m$. Thus, if Q is a point of the linear span $\langle \nu_d(L) \rangle$, then we can consider the rank of Q, either with respect to $X_{r,d}$, or with respect to $X_{m,d}$. Fortunately, in our cases where this ambiguity could arise, by a result contained in [16] (which corresponds essentially to the symmetric case of [9], Proposition 2.2) the two ranks are equal, and every decomposition $A \in \mathcal{S}(Q)$, with respect to $X_{r,d}$, is contained in $X_{m,d}$. Indeed, we have:

Remark 1. Take P_D (resp. P_T) as in case (a) (resp. (b)) of Theorem 2. By [18], Proposition 3.1, or [17], subsection 3.2, $sr(P_D)$ (resp. $sr(P_T)$) is equal to its symmetric rank with respect to the rational normal curve $\nu_d(D)$ (resp. $\nu_d(T)$). By [16], Exercise 3.2.2.2, each element of $\mathcal{S}(P_D)$ (resp. $\mathcal{S}(P_T)$) is contained in D (resp. T).

Several algorithms are available, to get an element of $\mathcal{S}(P_D)$ or $\mathcal{S}(P_T)$ ([11], [17], [5]).

Finally, we wish to thank J. Landsberg, who pointed out to us the importance of studying the existence of isolated points $A \in \mathcal{S}(P)$, when $\mathcal{S}(P)$ is not a singleton. We also thank the anonymous referee, for several useful suggestions on a preliminary version of the paper.

2. Preliminaries

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

Recall, from the introduction, than $\nu_d : \mathbb{P}^r \to \mathbb{P}^N$, $N := \binom{r+d}{r} - 1$ denotes the degree d Veronese embedding of $\mathbb{P}^r$. Call $X_{r,d}$ the image of this map.

For any closed subscheme $W \subseteq \mathbb{P}^r$, let $\langle W \rangle$ denote the linear span of W. If W sits in some hyperplane, $\langle W \rangle$ is the intersection of all the hyperplanes of $\mathbb{P}^r$ containing W.

For any integer $m > 0$ and any integral, positive-dimensional subvariety $T \subset \mathbb{P}^r$, we let $\Sigma_m(T)$ denote the embedded m-th secant variety of X, i.e. the closure in $\mathbb{P}^r$ of the union of all $(m-1)$-dimensional linear subspaces spanned by m points of T. We take the closure with respect to the Zariski topology.

Notice that, over the complex number field, the closure in the euclidean topology gives the same set.

For any integer $k > 0$, let $\mathrm{Hilb}^k(\mathbb{P}^r)^0$ denote the set of all finite (0–dimensional) reduced subsets of $\mathbb{P}^r$, with cardinality k. $\mathrm{Hilb}^k(\mathbb{P}^r)^0$ is a smooth and quasi-projective variety of dimension rk.

Remark 2. We observe that the set $\mathcal{S}(P)$, defined in the introduction, is always constructible.

Indeed, let $G := G(k-1, r)$ denote the Grassmannian of all $(k-1)$-dimensional linear subspaces of $\mathbb{P}^r$. For any point $P \in \mathbb{P}^r$, set $G(k-1,r)(P) := \{V \in G(k-1,r) : P \in V\}$ and $G(k-1,r)(P)_+ := \{V \in G(k-1,r)(P) : P$ is spanned by k points of $V \cap X$. Notice that, by definition, $G(k-1,r)(P)_+ = \emptyset$ for all $k < sr(P)$ and $G(sr(P)-1,r)(P)_+ \neq \emptyset$. Now, put $\mathcal{J} := \{(S,V) \in \mathrm{Hilb}^{sr(P)}(\mathbb{P}^r)^0 \times G(sr(P)-1,k)(P)_+ : P \in \langle \nu_d(S)\rangle\}$. This set $\mathcal{J}$ is locally closed. If π_1 denotes the projection onto the first factor, then $\mathcal{S}(P)$ is exactly the image $\pi_1(\mathcal{J})$. Hence, a theorem of Chevalley guarantees that $\mathcal{S}(P)$ is a constructible set ([14], Ex. II.3.18 and Ex. II.3.19).

We are interested in isolated points of $\mathcal{S}(P)$. Notice that Z is an isolated point for $\mathcal{S}(P)$ when Z is an irreducible component of the closure of $\mathcal{S}(P)$. Thus, the notion of *isolated points* for $\mathcal{S}(P)$ are equal both if we use the Zariski or the Euclidean topology on $\mathcal{S}(P)$.

Remark 3. Let X be any projective scheme and D any effective Cartier divisor of X. For any closed subscheme Z of X, we denote with $\mathrm{Res}_D(Z)$ the residual scheme of Z with respect to D. i.e. the closed subscheme of X with ideal sheaf $\mathcal{I}_Z : \mathcal{I}_D$ (where $\mathcal{I}_Z, \mathcal{I}_D$ are the ideal sheaves of Z and D, respectively).

We have $\deg(Z) = \deg(Z \cap D) + \deg(\mathrm{Res}_D(Z))$. If Z is a finite reduced set, then $\mathrm{Res}_D(Z) = Z \setminus Z \cap D$. For every $L \in \mathrm{Pic}(X)$ we have the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D) \to \mathcal{I}_Z \otimes L \to \mathcal{I}_{Z\cap D, D} \otimes (L|D) \to 0 \qquad (1)$$

From (1) we get

$$h^i(X, \mathcal{I}_Z \otimes L) \leq h^i(X, \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D)) + h^i(D, \mathcal{I}_{Z\cap D, D} \otimes (L|D))$$

for every integer $i \geq 0$.

3. The proofs

We will make an extensive use of the following two results.

Lemma 1. *Let $A, B \in \mathbb{P}^r$ be two zero-dimensional schemes such that $A \neq B$. Assume the existence of $P \in \langle \nu_d(A)\rangle \cap \langle \nu_d(B)\rangle$ such that $P \notin \langle \nu_d(A')\rangle$ for any $A' \subsetneq A$ and $P \notin \langle \nu_d(B')\rangle$ for any $B' \subsetneq B$. Then $h^1(\mathbb{P}^r, \mathcal{I}_{A\cup B}(d)) > 0$.*

Proof. See [2], Lemma 1. □

The following lemma was proved (with D a hyperplane) in [3], Lemma 8. The same proof works for an arbitrary hypersurface D of $\mathbb{P}^r$.

Lemma 2. *Fix positive integers r, d, t such that $t \le d$ and finite sets $A, B \subset \mathbb{P}^r$. Assume the existence of a hypersurface $D \subset \mathbb{P}^r$ of degree t, such that $h^1(\mathcal{I}_{(A\cup B)\setminus(A\cup B)\cap D}(d-t)) = 0$. Set $F := A \cap B \setminus (D \cap A \cap B)$.*

Then $\nu_d(F)$ is linearly independent. Moreover $\langle\nu_d(A)\rangle \cap \langle\nu_d(B)\rangle$ is the linear span of the two supplementary subspaces $\langle\nu_d(F)\rangle$ and $\langle\nu_d(A)\rangle\cap\langle\nu_d(B)\rangle$.

Assume there is $P \in \langle\nu_d(A)\rangle \cap \langle\nu_d(B)\rangle$ such that $P \notin \langle\nu_d(A')\rangle$ for any $A' \subsetneq A$, and $P \notin \langle\nu_d(B')\rangle$ for any $B' \subsetneq B$. Then $A = (A \cap D) \sqcup F$, $B = (B \cap D) \sqcup F$, which implies that $A \setminus A \cap D = B \setminus B \cap D$, and these sets are equal to F.

Next, we need to point out first the case of the Veronese embeddings $X_{1,d}$ of $\mathbb{P}^1$. This (already non–trivial) case anticipates some features of the behaviour of the sets $\mathcal{S}(P)$, in higher dimension.

Lemma 3. *Assume $r = 1$ and hence $N = d$. Fix $P \in \mathbb{P}^d$ such that $sr(P)$ is computed by at least two different subsets of $X_{1,d}$. Then $\dim(\mathcal{S}(P)) > 0$ and $\mathcal{S}(P)$ has no isolated points.*

Proof. Let t be the border rank of P, i.e. the minimal integer such that P sits in the secant variety $\Sigma_t(X_{1,d})$. The dimension of secant varieties of irreducible curve is well known ([1], Remark 1.6), and it turns out that $t \le \lfloor (d+2)/2 \rfloor$. Take A, B computing $sr(P)$ and such that $A \ne B$. Lemma 1 gives $h^1(\mathcal{I}_{A\cup B}(d)) > 0$. Since any set of at most $d+1$ points is separated by divisors of degree d, we see that $\sharp(A \cup B) \ge d + 2$. Hence $\sharp(A) = \sharp(B) \ge t$ and equality holds only if $t = (d + 2)/2$ and $A \cap B = \emptyset$.

(i) First assume $t = (d + 2)/2$, so that, as we observed above, t is also the symmetric rank of P. In this case, by [1], Remark 1.6, a standard dimensional count proves that $\Sigma_t(X_{1,d}) = \mathbb{P}^d$. Moreover, $(\mathcal{S}(Q))$ can be described as the fiber of a natural proper map of varieties. Namely, let $G(t-1, d)$ denotes the Grassmannian of $(t-1)$-dimensional linear subspaces of $\mathbb{P}^d$. Let $\mathbb{I} := \{(O, V) \in \mathbb{P}^d \times G(t-1, d) : O \in V\}$ denote the incidence correspondence, and π_1, π_2 denote the morphisms induced from the projections to the two factors. Since $X_{1,d}$ is a rational normal curve, of degree d, notice that $\dim(\langle W\rangle) = t - 1$ for *every* $W \in \mathrm{Hilb}^t(X_{1,d})$. Thus, the map $Z \mapsto \langle Z\rangle$ defines a proper morphism $\phi : \mathrm{Hilb}^t(X_{1,d}) \to G(t-1, d)$. Set $\Phi := \pi_2^{-1}(\phi(\mathrm{Hilb}^t(X_{1,d})))$. By construction, $\mathcal{S}(P)$ corresponds to the fiber of the map $\pi_{1|\Phi} : \Phi \to \mathbb{P}^d$ over P. Φ (the *abstract secant variety*) is an integral variety of dimension $\dim(\Phi) = d+1$ ([1]). Since ψ is proper and Φ is integral, every fiber of $\pi_{1|\Phi}$ has dimension at least 1 and no isolated points ([14], Ex. II.3.22 (d)). Thus, the claim holds, in this case.

(ii) Now assume $d \ge 2t - 1$. Hence $t < sr(P)$. A theorem of Sylvester (see [11], or [17], Theorem 4.1) proves that, in this case, $sr(P) = d + 2 - t$. Moreover, by [17] §4, there is a unique zero-dimensional scheme $Z \subset \mathbb{P}^1$ such that $\deg(Z) = t$ and $P \in \langle\nu_d(Z)\rangle$. As $t < sr(P)$, this subscheme Z cannot be reduced.

Fix any $A \in \mathcal{S}(P)$. Since $h^1(\mathcal{I}_{A\cup Z}(d)) > 0$ (Lemma 1) and $\deg(A) + \deg(Z) = d+2$, we have $Z \cap A = \emptyset$. Fix any $E \subset A$ such that $d - \sharp(E) = 2t-2$. Let $Y_E \subset \mathbb{P}^{2t-2}$ be the image of $X_{1,d}$ under the projection π_E from the linear subspace $\langle \nu_d(E)\rangle$. Notice that Y_E is again a rational normal curve, of degree $2t-2$, so that it coincides, up to a projectivity, with $X_{1,2t-2}$.

We have $Z \cap E = \emptyset$. Moreover $\deg(Z) + \sharp(E) \le d+1$, so that, by the properties of the rational normal curve mentioned above, the set $\nu_d(Z)\cup\nu_d(E)$ is linearly independent. It follows $\langle \nu_d(Z)\rangle \cap \langle \nu_d(E)\rangle = \emptyset$. Hence π_E is a morphism at each point of $\langle \nu_d(Z)\rangle$ and maps it isomorphically onto a $(t-1)$-dimensional linear subspace of $\mathbb{P}^{2t-2}$. As $\deg(A) \le d+1$, for the same reason we also have $\langle \nu_d(A \setminus E)\rangle \cap \langle \nu_d(E)\rangle = \emptyset$. It follows that the symmetric rank of $\pi_E(P)$ (with respect to Y_E) is exactly t, and $\pi_E(\nu_d(A \setminus E))$ is one of the elements of the set $\mathcal{S}(\pi_E(P))$. Moreover, for any $U \in \mathcal{S}(\pi_E(P))$ the set $U \cup E$ computes $sr(P)$. We saw above that $\pi_E(\nu_d(A\setminus E))$ is not an isolated element of $\mathcal{S}(\pi_E(P))$. Thus A is not an isolated element of $\mathcal{S}(P)$. □

Now, we are ready to prove our first main result.

Proof of Theorem 1. Since $A \ne B$, Lemma 1 gives $h^1(\mathcal{I}_{A\cup B}(d)) > 0$. Then, since $\sharp(A\cup B) \le 2t < 3d$, one of the following cases occurs ([13], Th. 3.8):

- (i) there is a line $D \subset \mathbb{P}^r$ such that $\sharp(D \cap (A\cup B)) \ge d+2$;
- (ii) there is a conic $T \subset \mathbb{P}^r$ such that $\sharp(T \cap (A\cup B)) \ge 2d+2$.

We will proof the statement, by showing that Lemma 3 implies that we can move the points of $A\cap D$ (in case (i)), or $A \cap T$ (in case (ii)), in a continuous family, whose elements, together with $A \setminus (A\cap D)$, determine a non trivial family of sets in $\mathcal{S}(P)$, which generalizes A.

(a) In this step, we assume the existence of a line $D \subset \mathbb{P}^r$ such that $\sharp(D \cap (A\cup B)) \ge d+2$.

Set $F := A\setminus(A\cap D)$. Let $H \subset \mathbb{P}^r$ be a general hyperplane containing D. Since $A\cup B$ is finite and H is general, we have have $(A\cup B)\cap H = (A\cup B)\cap D$.

First assume $h^1(\mathcal{I}_{(A\cup B)\setminus(A\cup B)\cap D}(d-1)) = 0$. Lemma 2 gives $A \setminus (A\cap D) = B\setminus(B\cap D)$. Hence $\sharp(A\cap D) = \sharp(B\cap D)$ and $A\cap D \ne B\cap D$, since $A \ne B$. The Grassmann's formula shows that $\langle \nu_d(A)\rangle \cap \langle \nu_d(B)\rangle$ is the linear span of its (supplementary) subspaces $\langle \nu_d(A\setminus(A\cap D))\rangle$ and $\langle \nu_d(A\cap D)\rangle\cap\langle \nu_d(B\cap D)\rangle$. This means that one can find a point $P_D \in \langle \nu_d(A\cap D)\rangle \cap \langle \nu_d(B\cap D)\rangle$ such that $P \in \langle \{P_D\}\cup\nu_d(A\setminus A\cap D)\rangle = \langle \{P_D\}\cup\nu_d(F)\rangle$. We notice that $\nu_d(A\cap D)$ and $\nu_d(B\cap D)$ are two different subsets of the rational normal curve $\nu_d(D)$, and they computes the rank of P_D, with respect to $\nu_d(D) = X$ (which can be identified with $X_{1,d}$, see the Introduction). Indeed, if P_D belongs to the span of a subset Z of $\nu_d(D)$, with cardinality smaller than $A\cap D$, then P would belong to the span of the subset $\nu_d(F)\cup Z$, of cardinality smaller than $sr(P)$, a contradiction. By Lemma 3, $A\cap D$ is not an isolated point of $\mathcal{S}(P_D)$.

Claim 1: Fix any $E \in \mathcal{S}(P_D)$. Then $sr(P) = \sharp(F) + sr(P_D)$ and $E\cup F \in \mathcal{S}(P)$.

Proof of Claim 1: By [16], Exercise 3.2.2.2, (see also Remark 1), every element of $\mathcal{S}(P_D)$ is contained in D and in particular it is disjoint from F. Since $P_D \in \langle \nu_d(E)\rangle$ and $P \in \langle \{P_D\} \cup \nu_d(F)\rangle$, we have $P \in \langle \nu_d(E \cup F)\rangle$. Hence, to prove Claim 1 it is sufficient to prove $\sharp(E \cup F) \le sr(P)$. Since $F \cap D = \emptyset$, we have $\sharp(E \cup F) = sr(P) + sr(P_D) - \sharp(A \cap D)$. Since $P_D \in \langle \nu_d(A \cap D)\rangle$, we have $\sharp(A \cap D) \ge sr(P_D)$ by the definition of $sr(P_D)$, concluding the proof of Claim 1.

Claim 1 implies that A is not an isolated point of $\mathcal{S}(P)$. Namely, let Δ be an integral affine curve and $o \in \Delta$ such that there is $\{\alpha_\lambda\}_{\lambda \in \Delta} \subseteq \mathcal{S}(P_D)$ with $\alpha_o = A \cap D$ and $\alpha_\lambda \subset D$ for all $\lambda \in \Delta$ (Lemma 3). By Claim 1, we have $F \cup \alpha_\lambda \in \mathcal{S}(P)$ for all $\lambda \in \Delta$.

Now assume $h^1(\mathcal{I}_{(A \cup B) \setminus (A \cup B) \cap D}(d-1)) > 0$. Since $\sharp((A \cup B) \setminus (A \cup B) \cap D) \le 2d-2 \le 2d-1$, again there is a line $L \subset \mathbb{P}^m$ such that $\sharp(L \cap ((A \cup B) \setminus (A \cup B) \cap D)) \ge d+1$. Let $H_2 \subset \mathbb{P}^m$ be a general quadric hypersurface containing $D \cup L$ (it exists, because if $L \cap D = \emptyset$, then $r \ge 3$). Since $L \cup D$ is the base locus of the linear system $|\mathcal{I}_{L \cup D}(2)|$, $A \cup B$ is finite and H_2 is general in $|\mathcal{I}_{L \cup D}(2)|$, we have $H_2 \cap (A \cup B) = (L \cup D) \cap (A \cup B)$. By Lemma 2, $A \setminus (A \cap (D \cup L)) = B \setminus (B \cap (D \cup L))$. Since $\sharp((A \cup B) \setminus (A \cup B) \cap H_2) \le 3d - 2d - 3 \le d-1$, we have $h^1(\mathcal{I}_{(A \cup B) \setminus (A \cup B) \cap H_2}(d-2)) = 0$. Lemma 3 gives $A \setminus (A \cap (D \cup L)) = B \setminus (B \cap (D \cup L))$. Notice that either $\sharp(A \cap L) \ge (d+2)/2$, or $\sharp(B \cap L) \ge (d+2)/2$, since $\sharp((A \cup B) \cap (D \cup L)) \ge 2d+3$ and $\sharp(A \cap (D \cup L)) = \sharp(B \cap (D \cup L))$.

Assume $x := \sharp(A \cap L) \ge (d+2)/2$. Since $P \in \langle \nu_d(A)\rangle$ and $P \notin \langle \nu_d(A')\rangle$ for any $A' \subsetneq A$, the set $\langle \{P\} \cup \nu_d(A \setminus A \cap L)\rangle \cap \langle \nu_d(A \cap L)\rangle$ is a single point. Call $P_{L,A}$ this point. Since A computes $sr(P)$, we see that $A \cap L$ computes the rank of $P_{L,A}$, with respect to the rational normal curve $\nu_d(L)$. Since $2x+1 > d$, as explained in the proof of Lemma 3, $A \cap L$ is not an isolated point of $\mathcal{S}(P_{L,A})$ (w.r.t. $\nu_d(L)$). On the other hand, as in Claim 1, adding $A \setminus (A \cap L)$ to a sets in $\mathcal{S}(P_{L,A})$ we obtain sets in $\mathcal{S}(P)$. As above, this implies that A is not an isolated point of $\mathcal{S}(P)$.

In the same way we conclude if $\sharp(B \cap D) \ge (d+2)/2$.

(b) Here we assume the non-existence of a line $D \subset \mathbb{P}^m$ such that $\sharp(D \cap (A \cup B)) \ge d+2$. Hence there is a conic $T \subset \mathbb{P}^m$ such that $\sharp(T \cap (A \cup B)) \ge 2d+2$.

Since A computes $sr(P)$, the set $\langle \{P\} \cup \nu_d(A \setminus A \cap T)\rangle \cap \langle \nu_d(A \cap T)\rangle$ is a single point. Call this point P_T. Let H_2 be a general element of $|\mathcal{I}_T(2)|$. Since $\mathcal{I}_T(2)$ is spanned outside T and $A \cup B$ is finite, we have $H_2 \cap (A \cup B) = T \cap (A \cup B)$. Since $\sharp(A \cup B) - \sharp((A \cup B) \cap T) \le d-2 \le d-1$, we have $h^1(\mathcal{I}_{A \cup B \setminus (A \cup B) \cap H_2}(d-2)) = 0$. Lemma 3 gives $A \setminus A \cap T = B \setminus B \cap T$.

First assume that T is a smooth conic. Hence $\nu_d(T)$ is a rational normal curve of degree $2d$. In this case, the conclusion follows by repeating the proof of the case $h^1(\mathcal{I}_{(A \cup B) \setminus (A \cup B) \cap D}(d-1)) = 0$ of step (a), including Claim 1, with $\nu_d(T)$ instead of $\nu_d(D)$, and applying Lemma 3 for the integer $2d$.

Now assume that T is singular. Since $A \cup B$ is reduced, we may find T as above which is not a double line, say $T = L_1 \cup L_2$ with $L_1 \neq L_2$. Since

$\sharp((A\cup B)\cap T)\ge 2d+2$ and $\sharp((A\cup B)\cap R)\le d+1$ for every line R, we have $\sharp((A\cup B)\cap L_1)=\sharp((A\cup B)\cap L_2)=d+1$ and $L_1\cap L_2\notin(A\cup B)$. If either $\sharp(A\cap L_i)=\ge (d+2)/2$ or $\sharp(B\cap L_i)\ge (d+1)/2$ for some i, we may repeat the proof of the case $h^1(\mathcal{I}_{(A\cup B)\setminus(A\cup B)\cap D}(d-1))>0$ taking $L_1\cup L_2$ instead of $L\cup D$.

Thus, it remains to consider the case where d is odd and $\sharp(A\cap L_i)=\sharp(B\cap L_i)=(d+1)/2$ for all i. Set $\{O\}:=L_1\cap L_2$. Since $\langle\nu_d(L_1)\rangle\cap\langle\nu_d(L_2)\rangle=\{\nu_d(O)\}$ and $P\notin\langle\nu_d(L_i)\rangle$, $i=1,2$, the linear space $\langle\nu_d(L_i)\rangle\cap\langle\{\nu_d(P_T)\}\cup\nu_d(L_{2-i})\rangle$ is a line $D_i\subset\langle\nu_d(L_i)\rangle$ passing through $\nu_d(O)$. The set $\langle\nu_d(A\cap L_i)\rangle\cap D_i$ is a point $P_{A,i}\in D_i\setminus\{\nu_d(O)\}$. Notice that $\langle D_1\cup D_2\rangle$ is a plane and $P_T\in\langle D_1\cup D_2\rangle\setminus(D_1\cup D_2)$. Hence for each $U_1\in D_1\setminus\{\nu_d(O)\}$ there is a unique $U_2\in D_2\setminus\{O\}$ such that $P_T\in\langle\{U_1,U_2\}\rangle$. By construction, $P_{L_i,A}$ has symmetric tensor rank $sr_{L_i}(P_{L_i,A})=(d+1)/2$ with respect to the rational normal curve $\nu_d(L_i)$ ([17], Theorem 4.1 or [5], §3) (we also have $sr(P)=(d+1)/2$, by [18], Proposition 3.1). The non-empty open subset $\langle\nu_d(L_i)\rangle\setminus\Sigma_{(d-1)/2}(\nu_d(L_i))$ of $\langle\nu_d(L_i)\rangle$ is the set of all $Q\in\langle\nu_d(L_i)\rangle$ whose symmetric rank with respect to $v_d(L_i)$ is exactly $sr_{L_i}(Q)=(d+1)/2$. Since $h^1(\mathbb{P}^1,\mathcal{I}_E(d))=0$ for every set $E\subset\mathbb{P}^1$ such that $\sharp(E)\le d+1$, for every $Q\in\langle\nu_d(L_i)\rangle\setminus\Sigma_{(d-1)/2}(\nu_d(L_i))$ there is a unique $A_{i,Q}\subset L_i$ such that $\nu_d(A_{i,Q})$ computes $sr_{L_i}(P)$. Set $\mathcal{U}_i:=\langle\nu_d(L_i)\rangle\setminus\Sigma_{(d-1)/2}(\nu_d(L_i))\cap D_i$. For each $Q_1\in D_1\cap\nu_d(L_i))\setminus\Sigma_{(d-1)/2}(\nu_d(L_i)$, call Q_2 the only point of $D_2\setminus\{O\}$ such that $P\in\langle\{Q_1,Q_2\}\rangle$. By moving $Q_1\in D_1$, we find an integral one-dimensional variety $\Delta:=\{F\cup A_{L_1,Q_1}\cup A_{L_2,Q_2}\}\subseteq\mathcal{S}(P)$ with $A\in\Delta$. Hence A is not an isolated point of $\mathcal{S}(P)$. □

The following example shows that the bound $sr(P)<3d/2$ in the statement of Theorem 1 is sharp, for large d.

Example 1. Fix an even integer $d\ge 6$. Assume $m\ge 2$. Here we construct $P\in\mathbb{P}^n$ such that $sr(P)=3d/2$ and its symmetric rank is computed by exactly two subsets of $X_{m,d}$

Fix a 2-dimensional linear subspace $M\subseteq\mathbb{P}^r$ and a smooth plane cubic $C\subset M$. Since $h^1(M,\mathcal{I}_C(d))=h^1(M,\mathcal{O}_M(d-3))=0$, we have $\deg(\nu_d(C))=3d$, $\dim(\langle\nu_d(C)\rangle)=3d-1$ and $\nu_d(C)$ is a linearly normal elliptic curve of $\langle\nu_d(C)\rangle$. Since no non-degenerate curve is defective ([1], Remark 1.6), we have $\Sigma_{3d/2}(\nu_d(C))=\langle\nu_d(C)\rangle$ and $\Sigma_{3d/2}(\nu_d(C))\setminus\Sigma_{(3d-2)/2}(\nu_d(C))$ is a non-empty open subset of the secant variety $\Sigma_{3d/2}(\nu_d(C))$. Fix a general $P\in\Sigma_{3d/2}(\nu_d(C))$. Since $\nu_d(C)$ is not a rational normal curve, by [10], Theorem 3.1 and [10], Proposition 5.2, there are exactly 2 (reduced) subsets of $\nu_d(C)$, of cardinality $3d/2$, which compute the symmetric rank of P. Thus, to settle the example, it is sufficient to prove that any $B\subset\mathbb{P}^m$ such that $\nu_d(B)$ computes $sr(P)$, is a subset of C. Obviously $\sharp(B)\le 3d/2$.

Assume $B\nsubseteq C$. Let H_3 be a general cubic hypersurface containing C (hence $H_3=C$ if $r=2$). Set $B':=B\setminus B\cap C$. Since B is finite and H_3 is general, we have $B\cap H_3=B\cap C$. Since $A\subset C$, we have $B'=(A\cup B)\setminus(A\cup B)\cap C$. Lemma 1 gives $h^1(\mathcal{I}_{A\cup B}(d))>0$. Hence $h^1(M,\mathcal{I}_{A\cup B}(d))>0$. Remark 3 gives that either $h^1(C,\mathcal{I}_{(A\cup B)\cap C}(d))>0$ or $h^1(\mathcal{I}_{B'}(d-3))>0$.

(a) First assume $h^1(\mathcal{I}_{B'}(d-3)) > 0$. Since $d \ge 3$ and $\sharp(B') \le 2d-1$, there is a line $D \subset M$ such that $\sharp(D \cap B') \ge d-1$ (see [5], Lemma 34, or [13], Th. 3.8). Since $\nu_d(B)$ is linearly independent, we have $\sharp(D \cap B) \le d+1$.

Assume $\sharp(D \cap (A \cup B)) \le d+1$. Hence $h^1(D, \mathcal{I}_{(A\cup B)\cap D}(d)) = 0$. Remark 3 gives $h^1(M, \mathcal{I}_{(A\cup B)\setminus(A\cup B)\cap D}(d-1)) > 0$. Set $F := (A\cup B) \setminus ((A \cup B) \cap D)$. We easily compute $\sharp(F) < 3(d-1)$. By [13], Theorem 3.8, we get that either there is a line D_1 such that $\sharp(F \cap D_1) \ge d+1$ or there is a conic D_2 such that $\sharp(D_2 \cap F) \ge 2d$. As $P \in \Sigma_{3d/2}(\nu_d(C))$ is general, then also A is general in C (hence reduced). Thus, no 3 of its points are collinear and no 6 of its points are contained in a conic. Hence if D_1 exists, we get $\sharp(B) \ge 2d-2$, while if D_2 exists, we get $\sharp(B) \ge d-1+(2d-5) = 3d-6$; both lead to a contradiction, because $d \ge 6$ and $\sharp(B) = 3d/2$.

Now assume $\sharp(D \cap (A\cup B)) \ge d+2$. Let $H \subset \mathbb{P}^m$ be a general hyperplane containing D. Since $A \cup B$ is finite and H is general, we have $H \cap (A \cup B) = D \cap (A \cup B)$. If $h^1(\mathcal{I}_{(A\cup B)\setminus(A\cup B)\cap H}(d-1)) = 0$, then Lemma 2 gives $B \setminus B \cap D = A \setminus A \cap D$. Hence $\sharp(A \cap D) = \sharp(B \cap D)$. Since $\sharp(A \cap D) \le 2$, we get $d \le 2$, a contradiction. Now assume $h^1(\mathcal{I}_{(A\cup B)\setminus(A\cup B)\cap H}(d-1)) > 0$. Since $\sharp((A\cup B) \setminus (A\cup B) \cap H) \le 2d-2$, there is a line $L \subset \mathbb{P}^m$ such that $\sharp(L \cap (A \cup B) \setminus ((A\cup B) \cap D)) \ge d+1$. Let $H_2 \subset \mathbb{P}^m$ be a general quadric hypersurface containing $L \cup D$. As usual, since $A \cup B$ is finite, $L \cup D$ is the base locus of the linear system $|\mathcal{I}_{L\cup D}(2)|$ and H_2 is general in $|\mathcal{I}_{L\cup D}(2)|$, we have $H_2 \cap (A\cup B) = (L\cup D) \cap (A \cup B)$. Since $\sharp((A\cup B)\setminus(A\cup B)\cap H_2) \le d-3$, we have $h^1(\mathcal{I}_{(A\cup B)\setminus(A\cup B)\cap H_2}(d-2)) = 0$. Hence Lemma 2 gives $A \setminus A \cap H = B \setminus B \cap H$. Hence $\sharp((A \cap (L \cup D)) = \sharp(B \cap (L\cup D))$. This is absurd, because $d \ge 4$ while, by generality, no 6 points of A are on a conic.

(b) Assume $h^1(C, \mathcal{I}_{(A\cup B)\cap C}(d)) > 0$ and $h^1(\mathcal{I}_{B'}(d-3)) = 0$. Since C is a smooth elliptic curve and $\deg(\mathcal{O}_C(d)) = 3d$, either $\deg((A\cup B)\cap C) \ge 3d+1$ or $\deg((A \cup B) \cap C) = 3d$ and $\mathcal{O}_C((A\cup B)\cap C) \cong \mathcal{O}_C(d)$. Hence $\sharp(B \cap C) \ge (3d-1)/2$. Therefore $\sharp(B') \le 2$. Taking $D := C$ in Lemma 2 we get $B' = \emptyset$, because $A \subset C$.

Next, we prove Theorem 2, a more precise description of the positive dimensional components of $\mathcal{S}(P)$, when $sr(P) < 3d/2$.

Proof of Theorem 2. Fix $A \in \mathcal{S}(P)$. and assume the existence of $B \in \mathcal{S}(P)$ such that $B \ne A$. At the beginning of the proof of Theorem 1 we showed that either:

(i) there is a line $D \subset \mathbb{P}^r$ such that $\sharp(D \cap (A \cup B)) \ge d+2$;
(ii) there is a conic $T \subset \mathbb{P}^r$ such that $\sharp(T \cap (A \cup B)) \ge 2d+2$.

(i) Here we assume the existence of a line $D \subset \mathbb{P}^r$ such that $\sharp((A\cup B) \cap D) \ge d+2$. We proved in step (a) of the proof of Theorem 1 that $\sharp(A \cap D) = \sharp(B \cap D)$. Hence $\sharp(A \cap D) \ge \lceil (d+2)/2 \rceil$. Set $F := A \setminus A \cap D$. Since $P \in \langle \nu_d(A) \rangle$ and $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$, the set $\langle \nu_d(A \cap D) \rangle \cap \langle \{P\} \cup \nu_d(F) \rangle$ is a single point. Let P_D denote this point. Lemma 3 and [16], Exercise 3.2.2.2, give that $\mathcal{S}(P_D)$ is infinite and each element of it is contained in D. Thus, to prove that we are in case (a) of the statement, it is sufficient to prove that

$E\cup F\in\mathcal{S}(P)$ for any $E\in\mathcal{S}(P_D)$. This assertion is just Claim 1 of the proof of Theorem 1.

(ii) Now assume the non-existence of a line D as above. Then, there is a (reduced) conic $T\subset\mathbb{P}^r$ such that $\sharp(T\cap(A\cup B))\geq 2d+2$ and $A\setminus A\cap T=B\setminus B\cap T$. Hence $\sharp(A\cap T)=\sharp(B\cap T)\geq d+1$. We consider separately the cases in which T is smooth or T is singular.

(ii.1) Assume T is smooth. Set $F:=A\setminus A\cap T$. As in step (i), we see that $\langle\nu_d(A\cap D)\rangle\cap\langle\{P\}\cup\nu_d(F)\rangle$ is a single point, P_T. Moreover, we see that $\sharp(A\cap T)=sr(P_T)$ and $\mathcal{S}(P_T)$ is infinite, since $\{F\cup E\}_{E\in\mathcal{S}(P_T)}\subseteq\mathcal{S}(P)$. To conclude that we are in case (b), we need to prove that every element of $\mathcal{S}(P)$ is of the form $F\cup E$, $E\in\mathcal{S}(P_T)$. Fix any $B\in\mathcal{S}(P)$ such that $B\neq A$. Since $\sharp(A\cup B)<3d$ and $h^1(\mathcal{I}_{A\cup B}(d))>0$, either there is a line D_1 such that $\sharp((A\cup B)\cap D_1)\geq d+2$, or there is a reduced conic $T_2\neq T$ such that $\sharp((A\cup B)\cap T_2)\geq 2d+2$ ([13], Theorem 3.8).

Assume the existence of the line D_1. If $h^1(\mathcal{I}_{(A\cup B)\setminus((A\cup B)\cap D_1)}(d-1))=0$, then Lemma 2 gives $A\setminus A\cap D_1=B\setminus B\cap D_1$. Since $\sharp(A)=sr(P)=\sharp(B)$, we get $\sharp(A\cap D_1)=\sharp(B\cap D_1)\geq(d+2)/2$, which contradicts the fact that we are not in case (i). Therefore $h^1(\mathcal{I}_{(A\cup B)\setminus(A\cup B)\cap D_1}(d-1))>0$. Hence there is a line D_2 such that $\sharp(D_2\cap((A\cup B)\setminus(A\cup B)\cap D_1))\geq d+1$. Let H_2 be a general quadric hypersurface containing $D_1\cup D_2$ (it exists, because if $D_1\cap D_2=\emptyset$, then $m\geq 3$). Since $\sharp((A\cup B)\setminus(A\cup B)\cap H_2)\leq(3d-1)-2d-3\leq d-1$, we have $h^1(\mathcal{I}_{(A\cup B)\setminus(A\cup B)\cap H_2}(d-2))=0$. Hence Lemma 2 implies $A\setminus A\cap H_2=B\setminus B\cap H_2$. Since H_2 be a general quadric hypersurface containing $D_1\cup D_2$, we have $A\cap H_2=A\cap(D_1\cup D_2)$ and $B\cap H_2=B\cap(D_1\cup D_2)$. Since $T\cap(D_1\cup D_2)\leq 4$, we get $2d+3\leq\sharp((A\cup B)\cap(D_1\cup D_2))\leq 8$, contradicting the assumption $d\geq 3$.

Assume the existence of the conic T_2 and assume $T\neq T_2$. In step (ii) of the proof of Theorem 1, we proved that $A\setminus T_2\cap A=B\setminus T_2\cap B$. Since $\sharp(A)=sr(P)=\sharp(B)$, we get $\sharp(A\cap T_2)=\sharp(B\cap T_2)$. Since $\sharp(T\cap T_2)\leq 4$ and $\sharp(A\setminus A\cap T)\leq(3d-1)/2-d-1$, we have $\sharp(A\cap T_2)\leq(3d-1)/2-d+3=(d+5)/2$. Hence $\sharp(A\cap T_2)=\sharp(B\cap T_2)\geq 2d+2-(d+5)/2=(3d-1)/2$. Since $\sharp(A\cap T_2)+\sharp(B\cap T_2)\geq\sharp((A\cup B)\cap T_2)\geq 2d+2$ we get $d=3$ and $A\subset T$. Hence $\sharp(B\cap T_2)\geq 4$ so that $B\subset T_2$. Thus $A\subset T$ and $B\subset T_2$ and moreover $A\setminus A\cap T_2=B\setminus B\cap T_2=\emptyset$. It follows that $A=T\cap T_2$. Since $A\subset T$ and T is a smooth conic, we have $P\in\langle\nu_3(T)\rangle$ and the symmetric rank of P, with respect to the rational normal curve $\nu_3(T)\subset\mathbb{P}^6$, is 4. It follows that $\mathcal{S}(P)$ is infinite. By [16], Exercise 3.2.2.2, we have $B\subset\nu_3(T)$ for all $B\in\mathcal{S}(P)$. Hence (b) holds, in this case.

Finally, assume that T_2 exists and $T=T_2$. I.e. assume $\sharp(T\cap(A\cup B))\geq 2d+2$. In step (ii) of the proof of Theorem 1, we proved that $A\setminus T\cap A=B\setminus T\cap B$ and that $B\cap T$ computes $sr(P_T)$. Hence $B\in\{F\cup E\}_{E\in\mathcal{S}(P_T)}$.

(ii.2) Here we assume the existence of a *reducible* conic T such that $\sharp(A\cap T)\geq d+1$. Write $T=L_1\cup L_2$ with $\sharp(A\cap L_1)\geq\sharp(A\cap L_2)$. If $\sharp(A\cap L_1)\geq(d+2)/2$, then, by step (i), we are in case (a). If $\sharp(A\cap L_1)<(d+2)/2$, then we get $\sharp(A\cap L_1)=\sharp(A\cap L_2)=(d+1)/2$ and $L_1\cap L_2\notin A$. We also get that

d is odd. It remains simply prove that $\mathcal{S}(P) \neq \{A\}$. Indeed, we proved that $\mathcal{S}(P)$ is infinite in the second part of step (ii) of the proof of Theorem 1.

The proof of the statement is completed. $\square$

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