# RANK 2 ARITHMETICALLY COHEN-MACAULAY BUNDLES ON A GENERAL QUINTIC SURFACE

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ABSTRACT. Rank 2 arithmetically Cohen-Macaulay vector bundles on a general quintic hypersurface of the three-dimensional projective space are classified.

## 1. INTRODUCTION

Let  $Y \subset \mathbb{P}^m$  be a smooth *n*-dimensional projective variety, embedded by  $\mathcal{O}_Y(1)$ , with n > 0. Assume that the coordinate ring  $R_Y$  is Cohen-Macaulay, and take a vector bundle  $\mathcal{E}$  on Y. Then the bundle  $\mathcal{E}$  is called *arithmetically Cohen-Macaulay* (ACM) if it has no intermediate cohomology, i.e. if:

 $\mathrm{H}^{p}(Y,\mathcal{E}(t)) = 0, \quad \text{for } p \neq 0, n \text{ and for all } t \in \mathbb{Z}.$ 

It is natural to ask whether it is possible to classify ACM bundles on a fixed variety Y. By the results of Horrocks [Hor64] and Knörrer in [Knö87] the answer is affirmative when Y is a projective space or a smooth quadric hypersurface. On the other hand, very few varieties are of *finite Cohen-Macaulay type*. Namely, by the result of [BGS87] and [EH88], the set of isomorphism classes of ACM indecomposable bundles over Y is infinite (up to twist by  $\mathcal{O}_Y(t)$ ) unless Y is: a rational normal curve, a projective space, a smooth quadric, a Veronese surface in  $\mathbb{P}^4$ , or a the rational normal scroll S(1,2) in  $\mathbb{P}^4$ .

Considerable efforts have thus been driven toward a classification of ACM bundles, at least of low rank (say of rank 2), on some special classes of varieties. Several techniques are available to approach the problem. To mention a few: derived categories (see for instance the paper [AO91]); quivers and matrix factorization (see [Eis80], [Knö87], [Yos90]); liaison theory (see [CH04], [CDH05]); computer-aided algebra (e.g. Schreyer's appendix to [Bea00], see also [Fae07]). One relevant instance of available classifications is the case of *prime* Fano threefolds, taken up in [Mad02], [AC00], [AF06], [Fae05], [BF08].

<sup>2000</sup> Mathematics Subject Classification. Primary 14J60. Secondary 13C14, 14F05, 14D20, 14J29.

 $Key\ words\ and\ phrases.$  Cohen-Macaulay modules. ACM bundles. Intermediate cohomology. Quintic surface.

Even more attention has been paid to the case of hypersurfaces in  $\mathbb{P}^m$  of degree d, which we denote by  $Y_d$ . Particularly pertinent to our study are the papers [Mad00], [CM00], [CM04], [CM05], [MKRR07a], [MKRR07b]. If we summarize the results obtained in the literature, we get that no rank 2 ACM indecomposable bundle exists on the general  $Y_d$  for  $m \ge 5$ ,  $d \ge 3$ , and for m = 4,  $d \ge 6$ . Moreover, the geometry of these bundles has been studied in great detail in case m = 4,  $d \le 4$ , see for instance the papers [Dru00], [IM00a], [IM00b].

Let us now consider the question for surfaces in  $\mathbb{P}^3$  of degree d. For d = 1, 2, we have seen that the issue is settled by Horrocks and Knörrer. For d = 3, 4 the classification follows from [Mad00], [Fae08]. However, the problems remains wide open for  $d \ge 5$ , except the specific case of bundles admitting a resolution by linear forms, and even this case can be managed only with the aid of the computer, see [Bea00].

This paper is devoted to the classification of ACM bundles of rank 2 on a general quintic surface X. After choosing an *initial* twist for the bundle  $\mathcal{E}$ , we completely classify the pairs of integers which are the Chern classes of an indecomposable rank 2 ACM bundle over a general quintic surface. More precisely, fix a twist the bundle  $\mathcal{E}$  so that  $\mathrm{H}^{0}(X, \mathcal{E}(-1)) = 0$ ,  $\mathrm{H}^{0}(X, \mathcal{E}) \neq 0$  (then  $\mathcal{E}$  is called *initialized*). Our main result is the following:

**Theorem.** On the general quintic surface  $X \subset \mathbb{P}^3$  there exist initialized indecomposable ACM bundles  $\mathcal{E}$  of rank 2 with the following invariants:

$$(1.1) \quad \frac{c_1(\mathcal{E}) \| -2 | -1 | 0 0 0 | 1 1 1 | 2 2 2 2 2 | 3 | 4}{c_2(\mathcal{E}) \| 1 | 2 | 3 4 5 | 6 8 10 | 11 12 13 14 | 20 | 30}$$

Moreover, these are the only possible Chern classes for such  $\mathcal{E}$ . The bundle  $\mathcal{E}$  is stable for  $c_1(\mathcal{E}) > 0$  and the moduli space of semistable sheaves  $M_X(2; c_1(\mathcal{E}), c_2(\mathcal{E}))$  is smooth of the expected dimension at a general point.

Observe that these are also the possible Chern classes of initialized indecomposable rank 2 ACM bundles on a general quintic threefold in  $\mathbb{P}^4$ . We do not know however, for general threefold hypersurfaces of degree 5, if bundles with all the listed Chern classes actually exist. See [CM05] for a discussion.

Our first method is to degenerate the surface  $Y_d$  to a reducible surface  $Y_{d-1} \cup H$ , where  $H \subset \mathbb{P}^3$  is a plane. This allows to take care of the cases with  $c_1 \leq 2$  in (1.1) by an inductive argument. Since this fails for higher  $c_1(\mathcal{E})$ , we introduce a second degeneration. The idea is to prove the existence on a quintic surface X of the bundles with the desired property (i.e. with  $\mathrm{H}^1(X, \mathcal{E}(t)) = 0$  for all  $t \in \mathbb{Z}$ ) by deforming a rank 2 bundle  $\mathcal{F}$  with some nonvanishing intermediate cohomology. More precisely, we take a general deformation of a bundle  $\mathcal{F}$  whose cohomology is as close as possible to the ACM condition, namely only  $\mathrm{h}^1(X, \mathcal{F}) = 2$  (for  $c_1(\mathcal{F}) = 3$ ) or  $\mathrm{h}^1(X, \mathcal{F}) = \mathrm{h}^1(X, \mathcal{F}(-1)) = 1$  (for  $c_1(\mathcal{F}) = 4$ ).

In the next section we fix some notation and review some basic results. In Section 3 we introduce our inductive method. In Section 4 we write down some general bounds for  $c_i(\mathcal{E})$ . Section 5 summarizes the classification of ACM bundle of rank 2 on surfaces of degree  $\leq 4$ , while Section 6 contains the proof of our result in the case  $c_1(\mathcal{E}) \geq 3$ .

#### 2. NOTATION AND BASIC RESULTS

We will work on an algebraically closed field  $\mathbf{k}$  of characteristic zero. The letter R will denote the coordinate ring of  $\mathbb{P}^3$ , i.e. the polynomial ring in 4 variables, and R(t) will be its graded piece of degree t. Given a subscheme  $Z \subset \mathbb{P}^3$ ,  $I_Z$  will denote the ideal of Z in R. We will write  $R_Z$  for the coordinate  $\mathbf{k}$ -algebra  $R/I_Z$ , and  $R_Z(t)$  for its homogeneous degree t piece.

A subscheme  $Z \subset \mathbb{P}^3$  is called *arithmetically Gorenstein* (aG) if  $R_Z$  is a Gorenstein ring. If the subscheme Z is a *complete intersection* (ci), of hypersurfaces of degree  $d_1, d_2, d_3$ , then its *type* will be the triple of integers  $(d_1, d_2, d_3)$ . We will consider the *difference Hilbert function* attached to Z:

$$\Delta h(R_Z, t) = \dim_{\mathbf{k}} R_Z(t) - \dim_{\mathbf{k}} R_Z(t-1).$$

We will consider mainly aG subschemes of codimension three. A description of the resolution of their ideals follows from the Buchsbaum–Eisenbud structure theorem (see [BE77]).

As a matter of terminology, any claim about the general element of a given family will mean that there exists a Zariski closed subset of the relevant parameter space such that the claim holds in the complement of this set. The parameter space will often be implicit, for example we will choose the general hypersurface of degree d in  $\mathbb{P}^3$  in an open subset of  $|\mathcal{O}_{\mathbb{P}^3}(d)|$ .

Let  $Y_d$  be a surface of degree d in  $\mathbb{P}^3$ , given by a homogeneous polynomial  $F_d$  in  $R(d) = \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d))$ . We will assume that  $Y_d$  general enough, to be smooth, and for  $d \geq 4$ , to have Picard number one. The canonical line bundle of  $Y_d$  is then  $\omega_{Y_d} \simeq \mathcal{O}_{Y_d}(d-4)$ . We will identify  $\mathrm{Pic}(Y_d)$  with  $\mathbb{Z} = \langle H \rangle$ , where H is the restriction to  $Y_d$  of the hyperplane class  $c_1(\mathcal{O}_{\mathbb{P}^3}(1))$ . For any sheaf  $\mathcal{E}$  on  $Y_d$ , the first Chern class  $c_1(\mathcal{E})$  can then be identified with an integer. The second Chern class  $c_2(\mathcal{E})$  of  $\mathcal{E}$  will be a multiple of the class of a point in  $Y_d$  - and thus it will also be denoted by an integer. We write  $\mathcal{E}(t)$  for  $\mathcal{E} \otimes \mathcal{O}_{Y_d}(1)^{\otimes t}$ .

Given a subscheme  $Z \subset Y_d \subset \mathbb{P}^3$ , the symbol  $\mathcal{I}_{Z,Y_d}$ , sometimes simplified to  $\mathcal{I}_Z$ , will denote the ideal sheaf of Z in  $Y_d$ . We have the exact sequence:

$$(2.1) 0 \to \mathcal{I}_Z \to \mathcal{O}_{Y_d} \to \mathcal{O}_Z \to 0.$$

Given a coherent sheaf  $\mathcal{F}$  on a projective variety Y, we will write  $h^k(Y, \mathcal{F})$ for the dimension of the cohomology group  $H^k(Y, \mathcal{F})$ . We will often omit the dependence on Y. The Euler characteristic  $\chi(\mathcal{F})$  is defined as  $\sum_{k=0,...,\dim(Y)} (-1)^k h^k(Y, \mathcal{F})$ . We will denote the moduli space of semistable sheaves on Y of rank r, with Chern classes  $c_1$ ,  $c_2$  by  $M_Y(r; c_1, c_2)$ . We refer to [HL97] for an account on this notion. **Definition 2.1.** For a sheaf  $\mathcal{E}$  on the surface  $Y_d$ , define the *initial twist* as the integer n such that  $\mathrm{H}^0(Y_d, \mathcal{E}(n)) \neq 0$  and  $\mathrm{H}^0(Y_d, \mathcal{E}(n-1)) = 0$ . The sheaf  $\mathcal{E}$  is called *initialized* if its initial twist is zero, i.e. if  $\mathrm{h}^0(Y_d, \mathcal{E}) > \mathrm{h}^0(Y_d, \mathcal{E}(-1)) = 0$ .

In some paper on the same subject (e.g. [Mad98]), bundles satisfying the previous property are called *normalized*. We prefer to switch to a new terminology, for in some classical paper on bundles (e.g. [Har77]), the word *normalized* has been used with a different meaning.

**Remark 2.2.** If  $\mathcal{E}$  is a bundle on  $Y_d$ , of rank r, by Riemann-Roch we have:

(2.2) 
$$c_1(\mathcal{E}(t)) = c_1(\mathcal{E}) + r t_2$$

(2.3) 
$$c_2(\mathcal{E}(t)) = c_2(\mathcal{E}) + c_1(\mathcal{E}) \left( \operatorname{rk}(\mathcal{E}) - 1 \right) dt + \binom{r}{2} dt^2,$$

(2.4) 
$$\chi(\mathcal{E}) = -c_2(\mathcal{E}) + \frac{d}{6} (3 c_1(\mathcal{E}) (c_1(\mathcal{E}) + 4 - d) + r (11 - 6 d + d^2)).$$

The following theorem summarizes well-known results about the Serre correspondence between (aG) zero-dimensional *locally complete intersection* (*lci* for short) subschemes  $Z \subset Y_d$  and (ACM) rank 2 bundles on  $Y_d$ . We refer to [HL97, Theorem 5.1.1] for a proof.

**Theorem 2.3.** Let  $Z \subset Y_d$  be a zero-dimensional lci subscheme of  $Y_d$ , and let c be an integer. Then the following are equivalent:

i) There exist a rank 2 vector bundle  $\mathcal{E}$  with  $c_1(\mathcal{E}) = c$  and an extension:

(2.5) 
$$0 \to \mathcal{O}_{Y_d}(-c) \to \mathcal{E}^* \to \mathcal{I}_Z \to 0.$$

ii) The pair  $(\mathcal{O}_{Y_d}(d+c-4), Z)$  has the Cayley-Bacharach property i.e. for any section  $s \in \mathrm{H}^0(Y_d, \mathcal{O}_{Y_d}(d+c-4))$ , and for any  $Z' \subset Z$  with  $\mathrm{len}(Z') = \mathrm{len}(Z) - 1$ , we have  $s_{|Z} = 0 \Leftrightarrow s_{|Z'} = 0$ .

**Remark 2.4.** The exact sequence (2.5) amounts to the Koszul complex associated to the section  $s_Z \in \mathrm{H}^0(Y_d, \mathcal{E})$  vanishing along Z. We say that  $\mathcal{E}$  is *associated* to Z. So, a general global section of  $\mathcal{E}$  vanishes on a subscheme of length len(Z). Notice that dualizing (2.5) we obtain the exact sequence:

(2.6) 
$$0 \to \mathcal{O}_{Y_d} \xrightarrow{s} \mathcal{E} \to \mathcal{I}_Z(c) \to 0.$$

In particular, when  $\mathcal{E}$  is initialized, we have:

(2.7) 
$$h^0(Y_d, \mathcal{I}_Z(c-t)) = 0 \quad \forall t > 0.$$

Equivalently,  $\mathcal{E}$  is initialized if and only if Z lies in no surfaces of degree e for  $e \leq c$ .

The following theorem is essentially well-known, so we only sketch a bit of the proof. For general reference on aG subschemes and difference Hilbert functions we refer to the book [IK99].

**Theorem 2.5.** In the previous setting, the following statements are equivalent:

- a) The scheme Z is aG;
- b) For all t, we have  $\dim_{\mathbf{k}} R_Z(t) + \dim_{\mathbf{k}} R_Z(c_1(\mathcal{E}) + d 4 t) = \operatorname{len}(Z)$ ;
- c) For all t, we have  $\Delta h(R_Z, t) = \Delta h(R_Z, c_1(\mathcal{E}) + d 3 t);$
- d) The bundle  $\mathcal{E}$  is ACM.

*Proof.* The equivalence of (a) and (b) is proved in [DGO85]. To see (b)  $\Leftrightarrow$  (c), just notice that:

$$\Delta h(R_Z, t) - \Delta h(R_Z, s-t) = \dim_{\mathbf{k}} R_Z(t) + \dim_{\mathbf{k}} R_Z(s-1-t) - -(\dim_{\mathbf{k}} R_Z(t-1) + \dim_{\mathbf{k}} R_Z(s-t)),$$

so, setting  $s = c_1(\mathcal{E}) + d - 4$ , we get that (c) holds if and only if  $\dim_{\mathbf{k}} R_Z(t) + \dim_{\mathbf{k}} R_Z(c_1(\mathcal{E}) + d - 4 - t)$  is constant in t. But this constant equals  $\operatorname{len}(Z)$  for  $\dim_{\mathbf{k}} R_Z(t) = \operatorname{len}(Z)$ , for big enough t. To see (b)  $\Leftrightarrow$  (d), we twist by  $\mathcal{O}_{Y_d}(t)$  the sequence (2.5) and take global sections. The cokernel of the transpose of the induced map  $\operatorname{H}^2(Y_d, \mathcal{I}_Z(t)) \simeq \operatorname{H}^2(Y_d, \mathcal{O}_{Y_d}(t)) \to \operatorname{H}^2(Y_d, \mathcal{E}^*(t))$  agrees, by Serre duality, with  $\operatorname{H}^0(Y_d, \mathcal{I}_Z(d + c_1(\mathcal{E}) - 4 - t))$ . Then we compute  $\operatorname{h}^1(Y_d, \mathcal{I}_Z(t)) = \operatorname{len}(Z) - \operatorname{dim}_{\mathbf{k}} R_Z(t)$ , we see that  $\operatorname{h}^1(Y_d, \mathcal{E}^*(t))$  is zero for all t if and only if condition (b) is satisfied.  $\Box$ 

**Remark 2.6.** Of course we have  $H^0(Y_d, \mathcal{O}_Z(t)) = R_Z(t) = 0$  for all negative t, so by Theorem 2.5, part (c) we get the following equalities:

(2.8)  $\Delta h(R_Z, t) = 0, \qquad \text{for } t < 0 \text{ and for } t > c_1(\mathcal{E}) + d - 3;$  $\dim_{\mathbf{k}} R_Z(t) = \operatorname{len}(Z), \qquad \text{for } t \ge c_1(\mathcal{E}) + d - 3.$ 

Similarly one obtains  $\Delta h(R_Z, c_1(\mathcal{E}) + d - 3) = \Delta h(R_Z, 0) = 1$ . In turn, this implies  $\dim_{\mathbf{k}} R_Z(c_1(\mathcal{E}) + d - 4) = d - 1$ . We get:

(2.9) 
$$h^{1}(Y_{d}, \mathcal{I}_{Z}(c_{1}(\mathcal{E}) + d - 4)) = 1.$$

Furthermore, we have:

(2.10) 
$$\ln(Z) = \sum_{t=0}^{c_1+d-3} \Delta h(R_Z, t).$$

Following [Bea00, page 18], given an aG subscheme Z of  $\mathbb{P}^3$ , we will call *index* of Z the integer  $i_Z$  such that  $\Delta h(R_Z, t) = \Delta h(R_Z, i_Z + 1 - t)$ , for all t. It is the largest integer j such that  $\dim_{\mathbf{k}} R_Z(j) < \operatorname{len}(Z)$ . If Z is the zero locus of a global section of a bundle  $\mathcal{E}$  on  $Y_d$  with  $c_1(\mathcal{E}) = c$ , we have:

$$i_Z = c + d - 4$$

**Remark 2.7.** The bundle  $\mathcal{E}^*$  provides an element of the extension group  $\operatorname{Ext}^1(\mathcal{I}_Z, \mathcal{O}_{Y_d}(-c_1))$ . By Serre duality we have:

(2.11) 
$$\operatorname{Ext}^{1}(\mathcal{I}_{Z}, \mathcal{O}_{Y_{d}}(-c_{1}))^{*} \simeq \operatorname{H}^{1}(Y_{d}, \mathcal{I}_{Z}(d+c_{1}(\mathcal{E})-4)).$$

Hence by (2.9) the group  $\operatorname{Ext}^1(\mathcal{I}_Z, \mathcal{O}_{Y_d}(-c_1))$  has dimension 1. Then, to the aG subscheme  $Z \subset X$  we associate a pair  $[\mathcal{E}_Z, s_Z]$ , where  $\mathcal{E}_Z$  fits in the

extension (2.5) and is uniquely determined, and  $s_Z$  is a global section of  $\mathcal{E}_Z$  (determined up to a nonzero scalar), and  $Z = \{s_Z = 0\}$ .

**Remark 2.8.** A rank 2 bundle  $\mathcal{E}$  on  $Y_d$  is decomposable into a direct sum of line bundles if and only if, for some integer a, there is a global section  $s \in \mathrm{H}^0(Y_d, \mathcal{E}(a))$  such that the cokernel of the induced map  $s : \mathcal{O}_{Y_d} \to \mathcal{E}(a)$ is isomorphic to  $\mathcal{O}_{Y_d}(c_1(\mathcal{E}(a)))$ , i.e. the 0-locus is empty. From the point of view of subschemes,  $Z \subset Y_d$  is a complete intersection of  $Y_d$  and two more surfaces if and only if its associated rank 2 bundle decomposes.

Note that, if an aG subscheme Z is contained in a plane H, it determines an rank 2 ACM bundle  $\mathcal{E}$  defined on H, so  $\mathcal{E}$  splits by Horrocks' criterion and Z is complete intersection in  $\mathbb{P}^3$ .

The following result was proved by Beauville in [Bea00], and is analogous to Buchsbaum-Eisenbud structure theorem. It allows to relate rank 2 ACM bundles over the hypersurface  $Y_d$  to skew-symmetric matrices whose determinant has degree 2d defined on  $\mathbb{P}^3$ . Recall that, to a skew-symmetric matrix fit is associated a polynomial Pf(f), the *Pfaffian* of f, with  $det(f) = Pf(f)^2$ .

**Theorem 2.9.** Let  $Y_d \subset \mathbb{P}^3$  be a smooth surface defined by the homogeneous form  $F_d$ ,  $\mathcal{E}$  be a rank 2 ACM bundle on  $Y_d$  with  $c_1(\mathcal{E}) = c_1 H$ . Let  $\iota : Y_d \hookrightarrow \mathbb{P}^3$ be the inclusion. Then the sheafified minimal graded free resolution of the sheaf  $\iota_*(\mathcal{E})$  takes the form:

$$0 \to \mathsf{P}(\mathcal{E})^*(c_1 - d) \xrightarrow{\mathsf{f}(\mathcal{E})} \mathsf{P}(\mathcal{E}) \xrightarrow{\mathsf{p}(\mathcal{E})} \iota_*(\mathcal{E}) \to 0,$$

where  $\mathsf{P}(\mathcal{E}) = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{3}}(a_{i}), f(\mathcal{E})$  is skew-symmetric, and  $\mathrm{Pf}(\mathsf{f}(\mathcal{E})) = F_{d}$ .

Conversely, given  $\mathsf{P} = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{3}}(a_{i})$  and a general skew-symmetric matrix  $\mathsf{f} : \mathsf{P}^{*}(c_{1} - d) \to \mathsf{P}$ , with  $\mathrm{Pf}(\mathsf{f}) = F_{d}$ , the sheaf  $\mathcal{E} = \mathrm{coker}(\mathsf{f})$  is a rank 2 ACM bundle, defined on the surface  $Y_{d}$  given by  $F_{d}$ , with  $c_{1}(\mathcal{E}) = c_{1} H$ .

## 3. A degeneration Lemma

For any variety Y, we denote with  $\operatorname{Hilb}_m(Y)$  the Hilbert scheme parametrizing closed subschemes of Y of finite length m. Recall that the Hilbert scheme parametrizing closed subschemes of length m of any smooth surface, is a smooth irreducible projective variety of dimension 2m (see for instance [HL97, pag. 104]).

Denote by  $\mathcal{G}(m, i)$  the subset of the Hilbert schemes  $\operatorname{Hilb}_m(\mathbb{P}^3)$  parametrizing length m subschemes of  $\mathbb{P}^3$  consisting of aG schemes of index i. These varieties are well understood in view of Buchsbaum-Eisenbud's structure theorem [BE77]. In particular, the number and the dimension of the irreducible components of this locally closed subset of  $\operatorname{Hilb}_m(\mathbb{P}^3)$ , are wellknown, see [Die96] and [IK99]. In particular a component of  $\mathcal{G}(m, i)$  is given once we fix the Hilbert function h of  $R_Z$ . We denote such a component by  $\mathcal{G}_{h}(m, i)$ . Write  $\mathcal{G}(m, i, d)$  for the incidence variety consisting of pairs (Z, Y), with  $Y \in |\mathcal{O}_{\mathbb{P}^3}(d)|, Z \in \mathcal{G}(m, i)$  and  $Z \subset Y$ . **Remark 3.1.** For any triple of integers (m, i, d), we have the incidence diagram:



The fibre  $p_{m,i,d}^{-1}(Y_d)$  consists of the family of aG subschemes  $Z \subset Y_d$ , of length m and index i, while  $q_d^{-1}(Z)$  is isomorphic to  $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^3, \mathcal{I}_{Z,\mathbb{P}^3}(d)))$ . In view of Theorem 2.5, we conclude that the ACM bundle  $\mathcal{E}$  having Chern classes  $c_1 = c_1(\mathcal{E})$  and  $c_2 = c_2(\mathcal{E})$  is defined on the general hypersurface of degree d if, setting  $m = c_2$ ,  $i = c_1 + d - 4$ , the map  $p_{m,i,d}$  in the diagram (3.1) is dominant. Computing dimensions in the same diagram, we get that  $p = p_{m,i,d}$  is dominant if and only if, for some Hilbert function h,  $p^{-1}(Y_d)$ has a component F where the following holds:

$$\binom{d+3}{3} + \dim(F) = \dim(\mathcal{G}_{\mathrm{h}}(m,i)) + \mathrm{h}^{0}(Y_{d},\mathcal{I}_{Z,Y_{d}}(d)).$$

Summing up we have the criterion:

the map 
$$p = p_{m,i,d} \iff$$
 there exists  $Y_d \in |\mathcal{O}_{\mathbb{P}^3}(d)|$  such that  $p^{-1}(Y_d)$  has a component of dimension  $\dim(\mathcal{G}_{\mathrm{h}}(m,i)) - \dim_{\mathbf{k}} R_Z(d)$ 

The following lemma introduces an inductive method on the degree d, to prove that the map  $p_{m,i,d}$  is dominant. Unfortunately the numerical assumption  $c_1 \leq 3$  limits its range of applicability.

**Lemma 3.2.** Fix integers d, m and fix  $c_1 \leq 3$ . Set  $i = c_1 + d - 4$ . Suppose that the map  $p = p_{m,i,d}$  of diagram (3.1) is dominant for general subschemes  $Z \subset \mathbb{P}^3$  contained in general fibres of p. Then  $p' = p_{m,i,d+1}$  is also dominant.

*Proof.* We would like to provide a surface of degree d + 1 satisfying the criterion introduced above. We start by taking a *general* surface  $Y_d$  of degree d, which we may assume to lie in the image of p by the hypothesis.

In view of the criterion of Remark (3.1), we are allowed to choose a component G of the quasi-projective scheme  $\mathcal{G}(m, i)$ , and a component F of  $p^{-1}(Y_d)$  such that:

$$\dim(F) = \dim(G) - \dim_{\mathbf{k}} R_Z(d).$$

Now, take the reducible hypersurface  $\overline{Y}_{d+1}$  defined as the union of the general hypersurface  $Y_d$  given above and a general plane in  $\mathbb{P}^3$ . The condition  $c_1 \leq 3$  entails  $\dim_{\mathbf{k}} R_Z(d) = \dim_{\mathbf{k}} R_Z(d+1)$ , indeed  $\dim_{\mathbf{k}} R_Z(t) = \operatorname{len}(Z) = m$  for any  $t \geq c_1 + d - 3$ .

Notice that, up to restriction to an open dense subset of the domain, the map  $q_{d+1}$  is equivalent to a  $\mathbb{P}^k$ -bundle onto the given component G, where  $k = h^0(\mathcal{I}_Z(d+1)) - 1$ , for a general element  $Z \in G$ . Indeed, the Hilbert function is constant on the components of  $\mathcal{G}(m, i)$ . Therefore the total space  $q_{d+1}^{-1}(G)$  is irreducible, and we consider the restriction of p' to this space. Observe also that the component F can be seen as a component of  $(p')^{-1}(\overline{Y}_{d+1})$  as well, since a general plane H will not meet the subscheme Z, so that a subscheme in the neighborhood of Z lies in  $Y_d$  if and only if it lies in  $\overline{Y}_{d+1}$ .

Then we are in position to apply [Har77, 3.22.b page 95]. This yields, for a general fibre of p' around  $\overline{Y}_{d+1}$ :

(3.2)  
$$\dim(p')^{-1}(Y_{d+1}) \leq \dim p^{-1}(Y_d) =$$
$$= \dim(\mathcal{G}(m,i)) - \dim_{\mathbf{k}} R_Z(d) =$$
$$= \dim(\mathcal{G}(m,i)) - \dim_{\mathbf{k}} R_Z(d+1)$$

We conclude that the dimension of the image of p' equals at least the dimension of  $|\mathcal{O}_{\mathbb{P}^3}(d+1)|$ . Then equality must hold and p' is dominant.  $\Box$ 

#### 4. General bounds for Chern classes of rank 2 ACM bundle

In the following proposition, we determine the range of  $c_1(\mathcal{E})$ , for an indecomposable rank 2 ACM bundle  $\mathcal{E}$  on a degree d surface  $Y_d \subset \mathbb{P}^3$ . The value of  $c_2(\mathcal{E})$  is uniquely determined only in the two maximal and minimal alternatives for  $c_1(\mathcal{E})$ . Otherwise,  $c_2(\mathcal{E})$  can be bounded form above and from below, see Proposition 4.2.

**Proposition 4.1.** Let  $Y_d$  as above,  $d \ge 4$  and let  $\mathcal{E}$  be a initialized indecomposable rank 2 ACM bundle over  $Y_d$ . Then any nonzero section of  $\mathcal{E}$ vanishes over a zero-dimensional, locally complete intersection subscheme  $Z \subset Y_d$  with len $(Z) = c_2(\mathcal{E})$  and we have:

$$(4.1) 3-d \le c_1(\mathcal{E}) \le d-1.$$

Moreover, we have the implications:

(4.2) 
$$c_1(\mathcal{E}) = 3 - d \iff c_2(\mathcal{E}) = 1,$$

(4.3) 
$$c_1(\mathcal{E}) = 4 - d \iff c_2(\mathcal{E}) = 2,$$

(4.4) 
$$c_1(\mathcal{E}) = d - 2 \Longrightarrow c_2(\mathcal{E}) = \frac{d(d-1)(d-2)}{3},$$

(4.5) 
$$c_1(\mathcal{E}) = d - 1 \Longrightarrow c_2(\mathcal{E}) = \frac{d(d-1)(2d-1)}{6}.$$

*Proof.* The inequalities (4.1) are similar of the main theorem of [Mad98], where the same bounds are proved for hypersurfaces with Picard group  $\mathbb{Z}$ .

Fix a global section  $s \in \mathrm{H}^{0}(Y_{d}, \mathcal{E})$ . We have  $\mathrm{Im}(s) \neq \mathcal{O}_{Y_{d}}$  by Remark 2.8. i.e. Z is not empty. If Z contains a divisor  $A \in |\mathcal{O}_{Y_{d}}(a)|$ , with  $a \geq 1$ , then  $\mathcal{I}_{Z} \subset \mathcal{O}_{Y_{d}}(-a)$ , so that  $\mathrm{h}^{0}(Y_{d}, \mathcal{E}(-a)) \neq 0$ , contradicting the hypothesis that  $\mathcal{E}$  is initialized. Thus s vanishes in codimension 2, so  $\mathrm{len}(Z) = c_{2}(\mathcal{E}) \geq 1$ .

Since  $\dim_{\mathbf{k}} R_Z(t) = 0$  for  $t \leq -1$ , applying property (b) of Theorem 2.5 to  $\mathcal{E}$ , with t = -1, one gets  $\dim_{\mathbf{k}} R_Z(c_1(\mathcal{E}) + d - 3) = \operatorname{len}(Z) > 0$ , hence

 $c_1(\mathcal{E}) \geq 3-d$ . Furthermore, when  $c_1(\mathcal{E}) = 3-d$  (resp.  $c_1(\mathcal{E}) = 4-d$ ), again by property (b) of Theorem 2.5 for t = 0, we get  $\text{len}(Z) = \dim_{\mathbf{k}} R_Z(0) = 1$ (resp.  $\text{len}(Z) = 2 \dim_{\mathbf{k}} R_Z(0) = 2$ ). So (4.2) and (4.3) follow. On the other hand, if  $c_1(\mathcal{E}) \geq d-1$ , we have:

$$\mathrm{h}^{2}(Y_{d},\mathcal{E}(-2)) = \mathrm{h}^{0}(Y_{d},\mathcal{E}^{*}\otimes\omega_{Y_{d}}) = \mathrm{h}^{0}(Y_{d},\mathcal{E}(d-c_{1}(\mathcal{E})-2)) = 0,$$

for  $\mathcal{E}$  is initialized. Since  $\mathcal{E}$  is ACM, and by duality  $h^2(Y_d, \mathcal{E}(-1)) \leq h^2(Y_d, \mathcal{E}(-2))$ , this implies that:

$$\chi(\mathcal{E}(-2)) = \chi(\mathcal{E}(-1)) = 0.$$

Solving the above equations in  $c_1(\mathcal{E})$  and  $c_2(\mathcal{E})$ , using Formula (2.4) one easily obtains (4.5) and (4.4).

In Proposition 4.1, (4.5) and (4.4) are called, respectively, the *maximal* and *submaximal* cases.

**Proposition 4.2.** Let  $Y_d$  and  $\mathcal{E}$  be as above,  $d \ge 4$ , set  $c_1 = c_1(\mathcal{E})$  and  $c_2 = c_2(\mathcal{E})$ . Then we have the following bounds on  $c_2$ .

**Lower bound:** We distinguish the two cases:

$$(4.6) c_1 \le 0 \Longrightarrow c_2 \ge c_1 + d - 2.$$

(4.7) 
$$c_1 \ge 1 \Longrightarrow c_2 \ge \frac{c_1}{6} (c_1 + 1) (3d - c_1 - 2).$$

**Upper bound:** According to the parity of  $c_1 + d - 3$  we have:

(4.8) 
$$c_1 + d - 3 = 2\ell \Longrightarrow c_2 \le \frac{(\ell+1)(\ell+2)}{6}(2\ell+3),$$

(4.9) 
$$c_1 + d - 3 = 2\ell - 1 \Longrightarrow \begin{cases} c_2 \le \frac{(\ell+1)(\ell+2)}{3}\ell, \\ c_2 \text{ is even.} \end{cases}$$

*Proof.* Take a nonzero global section of  $\mathcal{E}$  and consider its vanishing locus Z. We have  $\operatorname{len}(Z) = c_2$ . Recall that the difference Hilbert function  $\Delta h(R_Z, t)$  is concave and symmetric by Theorem 2.5, part (c).

In case  $c_1 \leq 0$ , the lower bound (4.6) amounts to  $\Delta h(R_Z, t)$  being greater or equal than 1 for all t between 0 and  $i_Z+1 = c_1+d-3$ . Equality is achieved for  $\Delta h(R_Z, t)$  constantly equal to 1, for  $0 \leq t \leq c_1+d-3$ . Such a subscheme has length  $c_1 + d - 2$  and is contained in a line, for dim<sub>k</sub>  $R_Z(1) = 2$ , hence  $H^0(\mathcal{I}_Z(1)) = 2$  by sequence (2.1).

Now, if  $c_1 \ge 1$ , by [IK99, Theorem 5.25], we can divide the summation interval  $0 \le t \le c_1 + d - 3$  into three subintervals:

- i) increasing:  $t \in I_1 := [0, c_1 1];$
- ii) concave:  $t \in I_2 := [c_1, d 3];$
- iii) decreasing:  $t \in I_3 := [d 2, c_1 + d 3];$

Notice that:

$$t \in I_1 \iff c_1 + d - 3 - t \in I_3.$$

So, using property (c) of Theorem 2.5, we obtain:

$$\sum_{t \in I_1} \Delta \mathbf{h}(R_Z, t) = \sum_{t \in I_3} \Delta \mathbf{h}(R_Z, t).$$

Putting this into (2.10), we get  $\operatorname{len}(Z) = 2 \sum_{t \in I_1} \Delta h(R_Z, t) + \sum_{t \in I_2} \Delta h(R_Z, t)$ . Formula (2.7) shows that  $h^0(Y_d, \mathcal{I}_Z(c_1 - 1 - t)) = 0$ , for all  $t \ge 0$ , so  $\Delta h(R_Z, t)$  agrees with  $\Delta h(R, t) = \binom{t+2}{2}$  for all t in  $I_1$  and  $\sum_{t \in I_1} = \binom{c_1+2}{3}$ . In the interval  $I_2$ , see (ii), the function  $\Delta h(R_Z, t)$  takes value at least

In the interval  $I_2$ , see (ii), the function  $\Delta h(R_Z, t)$  takes value at least  $\binom{c_1+1}{2}$  since  $\Delta h(R_Z)$  is concave (see [IK99]). So  $\sum_{t \in I_2} \Delta h(R_Z, t)$  is bounded below by this value, multiplied by the length of the interval (i.e.  $c_1 + d - 2$ ). Summing this to the above value for the intervals  $I_1$  and  $I_3$  we obtain:

$$(4.10) \quad c_2(\mathcal{E}) = \operatorname{len}(Z) = 2 \sum_{t \in I_1} \Delta \operatorname{h}(R_Z, t) + \sum_{t \in I_2} \Delta \operatorname{h}(R_Z, t) \ge \geq \frac{c_1}{3} (c_1 + 1) (c_1 + 2) + \frac{c_1}{2} (c_1 + 1) (d - c_1 - 2) = \frac{c_1}{6} (c_1 + 1) (3 d - c_1 - 2).$$

Now let us prove the upper bounds. For (4.8), observe that  $\Delta h(R_Z, t)$  is bounded above by  $\binom{t+2}{2}$ . Using again the symmetry of the difference Hilbert function and formula (2.10) we get:

$$c_{2}(\mathcal{E}) = \operatorname{len}(Z) = \Delta \operatorname{h}(R_{Z}, \ell) + \left(\sum_{t=0}^{\ell-1} \Delta \operatorname{h}(R_{Z}, t) + \sum_{t=\ell+1}^{2\ell} \Delta \operatorname{h}(R_{Z}, t)\right) =$$
$$= \Delta \operatorname{h}(R_{Z}, \ell) + 2\sum_{t=0}^{\ell-1} \Delta \operatorname{h}(R_{Z}, t) \le \binom{\ell+2}{2} + 2\binom{\ell+2}{3} = \frac{(\ell+1)(\ell+2)}{6} (2\ell+3)$$

The case (4.9) is proved analogously; one just needs to observe that the summation interval for  $\Delta$  h is symmetric, so len(Z) is even.

**Remark 4.3.** On any  $Y_d$ ,  $d \ge 2$  there exist initialized indecomposable rank 2 ACM bundles with minimal first Chern class  $c_1(\mathcal{E}) = 3 - d$ ,  $c_2(\mathcal{E}) = 1$ . Namely, let  $\iota : Y_d \hookrightarrow \mathbb{P}^3$  be the inclusion, take a point  $y \in Y_d$  and observe that  $Z = \{y\}$  is an aG subscheme of  $Y_d$ , and the pair  $(\mathcal{O}_{Y_d}(-1), Z)$  has the Cayley-Bacharach property. We obtain a bundle  $\mathcal{E}_y$  (indecomposable for y is not  $c_i$ ), and the exact sequences:

$$(4.12) 0 \to \mathcal{O}_{Y_d}(d-3) \to \mathcal{E}_y^* \to \mathcal{I}_y \to 0,$$

(4.13) 
$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^3}(3-2\,d) & \mathcal{O}_{\mathbb{P}^3} \\ \oplus & \oplus & \oplus \\ \mathcal{O}_{\mathbb{P}^3}(1-d)^3 & \mathcal{O}_{\mathbb{P}^3}(2-d)^3 \end{array} \rightarrow \iota_*(\mathcal{E}_y) \rightarrow 0.$$

Similarly, a length-2 subscheme  $Z_2 \subset Y_d$ ,  $Z_2$  is aG and the pair  $(\mathcal{O}_{Y_d}, Z_2)$  has the Cayley-Bacharach property. So, on  $Y_d$  it is defined an rank 2 ACM

bundle  $\mathcal{E}_{Z_2}$  with  $c_1(\mathcal{E}_{Z_2}) = 4 - d$ ,  $c_2(\mathcal{E}_{Z_2}) = 2$ . The bundle  $\mathcal{E}_{Z_2}$  is indecomposable for  $d \geq 3$  and we have the exact sequences:

These resolutions are obtained via a standard mapping cone construction from the obvious resolutions of the ideal sheaves  $\mathcal{I}_y$ ,  $\mathcal{I}_{Z_2}$ . The fact that the previous bundles are initialized follows immediately from their resolutions.

**Proposition 4.4.** Let  $\mathcal{E}$  be an initialized indecomposable rank 2 ACM bundle on  $Y_d$ ,  $d \ge 4$ , and suppose  $c_1(\mathcal{E}) = 5 - d$ . Then we have:  $c_2(\mathcal{E}) \in \{3, 4, 5\}$ . A general section of  $\mathcal{E}$  vanishes, respectively, along a length 3 ci subscheme of type (1, 1, 3), a length 4 ci subscheme of type (1, 2, 2), and a length 5 subscheme in  $Y_d$  contained in 5 independent quadrics.

All the three possibilities take place on any smooth surface  $Y_d$ .

*Proof.* Let Z be a nonzero section of  $\mathcal{E}$ . Since  $\mathcal{E}$  is ACM, property (c) of Theorem 2.5 gives  $\Delta h(R_Z, t) = \Delta h(R_Z, 2-t)$ , so  $\Delta h(R_Z, 2) = 1$ . Furthermore, we have  $1 \leq \Delta h(R_Z, 1) \leq 3$ , hence  $c_2(\mathcal{E}) = \text{len}(Z)$  runs between 3 and 5.

If  $c_2(\mathcal{E}) = 3$  (i.e. if  $\leq \Delta h(R_Z, 1) = 1$ ), then  $h^0(\mathcal{I}_Z(1)) = 2$ . This means that Z is contained in a line and therefore it is ci. Similarly,  $c_2(\mathcal{E}) = 4$ implies  $h^0(\mathcal{I}_Z(1)) = 1$ , so Z is contained in a plane and thus it is complete intersection. In case  $c_2(\mathcal{E}) = 5$ , we have  $h^0(\mathcal{I}_Z(1)) = 0$  and  $h^0(\mathcal{I}_Z(2)) = 5$ , which means 5 independent quadrics.

Clearly, any hypersurface  $Y_d$  contains 3 collinear points. Analogously, 4 general coplanar points in  $Y_d$  are aG. Finally, 5 general points in  $Y_d$  are also aG, indeed it suffices that they are non coplanar, for in this case  $\Delta h(R_Z, t)$  takes the form 1, 3, 1.

**Proposition 4.5.** Let  $\mathcal{E}$  be an initialized indecomposable rank 2 ACM bundle on  $Y_d$ ,  $d \geq 5$ , and suppose  $c_1(\mathcal{E}) = 6 - d$ . Let Z be the vanishing locus of a nonzero section of  $\mathcal{E}$ . Then we have:  $c_2(\mathcal{E}) \in \{4, 6, 8\}$ . If  $c_2(\mathcal{E}) \in \{4, 6\}$ , then Z is a ci subscheme of type (1, 1, 4), (1, 2, 3). If  $c_2(\mathcal{E}) = 8$ , then Z is contained in three independent quadrics (and no hyperplane).

All the above possibilities take place over a general hypersurface  $Y_d$ .

*Proof.* By Theorem 2.5, in our hypothesis the function  $\Delta h(R_Z)$  takes one of the three possible forms: 1, 1, 1, 1 or 1, 2, 2, 1 or 1, 3, 3, 1. In the first two cases the aG subscheme Z is contained in a plane and thus it is a complete intersection, see Remark 2.8. In the third case, i.e. if  $c_2(\mathcal{E}) = 8$ , then looking at the function  $\Delta h(R_Z)$ , one sees that Z is contained in three independent quadrics and no hyperplane.

For the second statement, of course any  $Y_d$  contains 4 collinear points. Moreover, intersecting  $Y_d$  with a conic C contained in a general plane, one gets 2d points on C. Choosing Z to be the union of 6 of them, one can find a cubic intersecting C at Z, so we find the subscheme  $Z \subset Y_d$  of type (1,2,3). To deal with the case  $c_2(\mathcal{E}) = 8$ , we can use Lemma 3.2, indeed Zis contained in no hyperplane and  $c_1 \leq 3$ . It suffices to prove that a general cubic surface  $Y_3$  contains the subscheme  $Z_8$  in question. This is clear: just take an elliptic quartic contained in  $Y_3$  and intersect with a general quadric to obtain  $Z_8$ .

**Remark 4.6.** We can say something more precise about the geometry of the subschemes  $Z \subset Y_d$  which are 0-loci of sections of an initialized rank 2 ACM bundle  $\mathcal{E}$  on  $Y_d$ ,  $d \ge 4$ , with  $c_1(\mathcal{E}) = 6 - d$  and  $c_2(\mathcal{E}) = 8$ .

Namely, assume that Z is a set of 8 points in very uniform position, on an irreducible quadric Q. Then, up to generalization, either Z is a ci subscheme of type (2, 2, 2), or it lies on a rational cubic curve  $C \subset Q$ .

The first case holds when three independent quadrics containing Z intersect in a finite set of points. The second one takes place when all the quadrics through Z contain a common curve.

Observe that both cases occur a general  $Y_d$ . Indeed, we proved above that the second one occurs. But since the second case is a degeneration of the first one (see [IK99, Theorem 5.25]), then also the first case takes place by diagram (3.1).

#### 5. RANK 2 ACM BUNDLES ON SURFACES OF DEGREE UP TO 4

Let us summarize here the well-known classification of rank 2 ACM bundles on a general surface  $Y_d \subset \mathbb{P}^3$  of degree d, for all  $d \leq 4$ .

For d = 2,  $Y_2$  is a smooth quadric  $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ , and any ACM bundle splits as a direct sum of twists of  $\mathcal{O}_Q(0,0)$ ,  $\mathcal{O}_Q(1,0)$  and  $\mathcal{O}_Q(0,1)$ . For d = 3 the result is well-known by [Fae08]. The case d = 4 follows immediately from the classification of rank 2 ACM bundles on a smooth quartic threefold in  $\mathbb{P}^4$ , achieved by Madonna in [Mad00]. We reproduce here these results and sketch an easy proof for the case d = 4. In the following theorem, in the column *Difference Hilbert Function* column, the  $t^{\text{th}}$  place represents the integer  $\Delta h(R_Z, t)$  (only non-zero values are displayed). The column *Resolution* illustrates (an instance of) the generators  $\mathsf{P}(\mathcal{E})$  in the sheafified minimal graded free resolution of  $\iota_*(\mathcal{E})$ , in the framework of Theorem 2.9. Note that, in order to write down the resolution, it suffices to show the generators  $\mathsf{P}(\mathcal{E})$ , for the syzygies are given by  $\mathsf{P}^*(c_1 - d)$ .

**Theorem 5.1.** Let  $\mathcal{E}$  be an initialized, indecomposable, rank 2 ACM bundle over a general surface  $Y_d$  of degree  $d \in \{3,4\}$  in  $\mathbb{P}^3$ , with  $c_1(\mathcal{E}) = c_1 H$ , let s be a general section of  $\mathcal{E}$  and set  $Z = \{s = 0\}$ . Then  $\mathcal{E}$  and Z are among the types summarized by the following table.

(5.1)							
degree	Chern	Diff. Hilbert Function	Resolution				
3	$ \begin{array}{c c} c_1(\mathcal{E}) & c_2(\mathcal{E}) \\ \hline 0 & 1 \\ \hline 1 & 2 \\ \hline 2 & 5 \\ \hline \end{array} $	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	$ \begin{array}{c} \begin{array}{c} P(\mathcal{E}) \\ \hline \\ \hline \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^3 \\ \hline \\ \mathcal{O}_{\mathbb{P}^3}^3 \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \\ \hline \\ \mathcal{O}_{\mathbb{P}^3}^6 \end{array} \end{array} $				
4	$ \begin{array}{c c} \hline c_1(\mathcal{E}) & c_2(\mathcal{E}) \\ \hline \hline -1 & 1 \\ 0 & 2 \\ \hline 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ \hline 2 & 8 \\ \hline 2 & 8 \\ \hline 3 & 14 \\ \hline \end{array} $	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\frac{\begin{array}{c} \begin{array}{c} P(\mathcal{E}) \\ \hline \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-2)^3 \\ \hline \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \\ \hline \mathcal{O}_{\mathbb{P}^3}^3 \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \\ \mathcal{O}_{\mathbb{P}^3}^2 \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^2 \\ \hline \mathcal{O}_{\mathbb{P}^3}^2 \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^5 \\ \hline \mathcal{O}_{\mathbb{P}^3}^4 \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^2 \\ \hline \mathcal{O}_{\mathbb{P}^3}^4 \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^2 \\ \hline \mathcal{O}_{\mathbb{P}^3}^8 \end{array} $				

Moreover, any class in the table is non-empty.

*Proof.* All the possibilities for the  $c_1$  are listed in the table, by the inequalities (4.1) of Proposition 4.1. The list of possible  $c_2$  follows then by (4.2), (4.3), (4.4), (4.5) of the same proposition, and by Proposition 4.2. The statement about the difference Hilbert Function follows directly from Theorem 2.5.

Remark 4.3 and Proposition 4.4 prove the existence for  $c_1 < 2$ . The remaining cases are a consequence of the main theorem of [Mad00]. Indeed, in the paper, it is proved that indecomposable ACM bundles with  $c_1 = 2$ ,  $c_2 = 8$  and  $c_1 = 3$ ,  $c_2 = 14$  exist on any smooth quartic hypersurface  $\tilde{Y}_4 \subset \mathbb{P}^4$ . Then just take as  $Y_4$  a hyperplane section of  $\tilde{Y}_4$ .

Alternatively, in order to prove existence for  $c_1 = 2$ ,  $c_2 = 8$  one just needs to prove that a general quartic can be obtained as the Pfaffian of a skew-symmetric matrix  $f : \mathcal{O}_{\mathbb{P}^3}^4(-2) \to \mathcal{O}_{\mathbb{P}^3}^4$ . Given a general matrix f, set  $\iota(\mathcal{E}_0) = \operatorname{coker}(f), Y_4^0 = \operatorname{Pf}(f)$ , where  $\iota : Y_4^0 \hookrightarrow \mathbb{P}^3$  is the natural embedding. Observe that  $\mathcal{E}_0$  is a stable bundle. By a parameter count, our claim is equivalent to the moduli space  $M_{Y_4^0}(2; 2, 8)$  being smooth of the expected dimension 10 at the point  $[\mathcal{E}_0]$ . But since  $Y_4^0$  is a smooth K3 surface, this is indeed the case, see for instance [HL97, Part II, Chapter 6]. The same approach has been used by Beauville to prove existence in the case  $c_1 = 3$ ,  $c_2 = 14$ , see [Bea00, Lemma 7.7].

To compute the resolutions, whenever  $\Delta h(R_Z, t)$  starts with 1, s, with  $s \leq 2$ , the scheme Z is contained in a plane, hence it is ci. The resolution of the ideal sheaf  $\mathcal{I}_Z$  follows immediately. By a standard mapping cone construction, it is easy to obtain the resolution of  $\mathcal{E}$  as well. One settles similarly the case  $c_2(\mathcal{E}) = 5$ . For the case  $c_2(\mathcal{E}) = 14$ , we see that the matrix  $f(\mathcal{E})$  of Theorem 2.9 can only have linear entries, working as in [Fae08,

Theorem 4.1]. Finally, in case  $c_2(\mathcal{E}) = 8$ , we have two cases, according to whether Z is contained or not in a twisted cubic curve: in both cases the resolution of  $\mathcal{E}$  follows easily from that of  $\mathcal{I}_Z$ . On the other hand, the matrix  $f(\mathcal{E})$  cannot have bigger size (say a square matrix of order 8), for in this case  $Y_4$  would be a determinantal quartic, contradicting generality. 

## 6. RANK 2 ACM BUNDLES ON THE QUINTIC SURFACE

From now on, we will denote by X a general quintic surface in  $\mathbb{P}^3$ , embedded in  $\mathbb{P}^3$  by  $\iota$ , defined by a homogeneous polynomial F of degree d = 5. We will write  $\mathcal{E}$  for an *initialized indecomposable rank* 2 *ACM bundle* on *X*. We have the following result, whose proof amounts to writing the conditions of Proposition 4.2.

**Proposition 6.1.** Let  $\mathcal{E}$  and X be as above. Then the Chern classes of  $\mathcal{E}$ , and the difference Hilbert function of the zero locus of a nonzero section Zof  $\mathcal{E}$  fall into one of the types summarized by the following table.

	Chern			Difference Hilbert Function								
· · · · · · · · · · · · · · · · · · ·	$c_1(\mathcal{E})$	$c_2(\mathcal{E})$		0	1	2	3	4	5	6		
	-2	1		1								
	-1	2		1	1							
	0	3		1	1	1						
	0	4		1	2	1						
	0	5		1	3	1						
5.1)	1	4		1	1	1	1					
,	1	6		1	2	2	1					
	1	8		1	3	3	1					
	2	11		1	3	3	3	1				
	2	12		1	3	4	3	1				
	2	13		1	3	5	3	1				
	2	14		1	3	6	3	1				
	3	20		1	3	6	6	3	1			
	4	30		1	3	6	10	6	3	1		

(6

In the Difference Hilbert Function column, the  $t^{th}$  place represents the integer  $\Delta h(R_Z, t)$  (only non-zero values are displayed).

The rest of the paper is devoted to a detailed analysis of the above cases. We will prove that all of these alternatives take place over the general hypersurface X.

Remark 6.2. In fact, we only need to prove the existence of bundles for the cases where  $c_1 \geq 2$ . Indeed, the cases  $c_1 = -2$  and  $c_1 = -1$  follow by Remark 4.3. The cases  $c_1 = 0, 1$  are covered by Proposition 4.4 and Proposition 4.5.

6.1. The case  $c_1 = 2$ . Here we assume  $c_1(\mathcal{E}) = 2$  and prove the existence of rank 2 ACM bundles whose second Chern class appears in Table (6.1).

**Proposition 6.3.** On a general quintic surface X there exists an indecomposable rank 2 ACM bundle  $\mathcal{E}$  with Chern classes  $c_1 = 2$  and  $c_2$ , for any  $c_2 \in \{11, \ldots, 14\}$ .

*Proof.* Going back to Diagram (3.1) of Remark 3.1, we need to prove that the map  $p_{m,i,d}$  is dominant, for any triple (m, i, d) = (m, 3, 5), with  $m = 11, \ldots, 14$ . In other words, we need to prove the existence, on a general quintic surface, of an aG set of points with index 3 and degree  $m = 11, \ldots, 14$ .

We will use Lemma 3.2, and start working on a surface of degree 4, indeed we are assuming  $c_1 = 2$ . So take a general quartic hypersurface  $Y_4$  in  $\mathbb{P}^3$ and let  $Z_m$  be an aG subscheme  $Z_m \subset Y_4$  with  $i_{Z_m} = 3$  and  $\operatorname{len}(Z_m) = m$ . We have thus an exact sequence:

(6.2) 
$$0 \to \mathcal{O}_{Y_4}(-3) \to \mathcal{E}_m^* \to \mathcal{I}_{Z_m} \to 0,$$

where, by Theorem 2.5, the sheaf  $\mathcal{E}_m$  is a rank 2 ACM stable bundle, with  $c_1(\mathcal{E}_m) = 3$  and  $c_2(\mathcal{E}_m) = m$ . One sees easily that  $h^0(Y_4, \mathcal{E}_m^*(2)) = 14 - m$ . Thus,  $\mathcal{E}_m$  is initialized if and only if m = 14. On the other hand, looking at Table (6.1), we assume  $m \ge 11$ . So we distinguish the two cases  $c_2(\mathcal{E}_m) = m = 14$ , or  $c_2(\mathcal{E}_m) = m \in \{11, 12, 13\}$ .

**Case 1.** On a general surface X of degree 5 there exists an initialized stable rank 2 ACM bundle  $\mathcal{E}_m$  with  $c_1(\mathcal{E}_m) = 2$  and  $c_2(\mathcal{E}_m) = m$ , for  $m \in \{11, 12, 13\}$ .

*Proof.* In view of the previous discussion, it suffices to check that a general hypersurface  $Y_4$  contains an aG subscheme  $Z_m$  with the required Hilbert function.

By Theorem 5.1, over  $Y_4$  one has rank 2 ACM bundles  $\overline{\mathcal{E}}_{\ell}$  with  $c_1(\overline{\mathcal{E}}_{\ell}) = 1$ and  $c_1(\overline{\mathcal{E}}_{\ell}) = \ell$ , for  $\ell \in \{3, 4, 5\}$ . By Lemma 2.2, we have:

$$c_1(\overline{\mathcal{E}}_\ell(1)) = 3,$$
  $c_2(\overline{\mathcal{E}}_\ell(1)) = \ell + 8 \in \{11, 12, 13\}.$ 

Set  $\mathcal{F}_{\ell+8} := \overline{\mathcal{E}}_{\ell}(1)$ . Of course, the bundle  $\mathcal{E}_m$  is also ACM, so the zero locus of a general section of  $\mathcal{E}_m$  vanishes along the required aG subscheme.  $\Box$ 

**Case 2.** On a general surface X of degree 5 there exists an ACM stable initialized rank 2 bundle  $\mathcal{E}$  with  $c_1(\mathcal{E}) = 2$  and  $c_2(\mathcal{E}) = 14$ .

*Proof.* We proceed similarly, considering a general hypersurface  $Y_4$ . This time we have to provide 14 aG points of index 4 in  $Y_4$ , with difference Hilbert function 1, 3, 6, 3, 1. Equivalently, we have to provide an initialized rank 2 ACM bundle with  $c_1 = 3$ ,  $c_2 = 14$ . As we have already pointed out, this question has been addressed by Beauville, see also Theorem 5.1. The statement follows.

The proof of Proposition 6.3 is thus established.

6.2. The submaximal case:  $c_1 = 3$ , 20 points. Unfortunately, we cannot use the degeneration technique of the previous section, because on a quartic surface, an aG set of 20 points with index 4 yields a bundle with  $c_1 = 4$ . Instead, we construct a set of 20 points with the *wrong* Hilbert function on a general quintic surface and deform it to the required set of points.

**Remark 6.4.** We would like to point out that the existence of such  $\mathcal{E}$  over the general quintic surface can be addressed by Macaulay2, see Schreyer's appendix in [Bea00]. Indeed, it suffices to take a random skew-symmetric matrix  $f: \mathcal{O}_{\mathbb{P}^3}^5(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}^5$  and check that the differential of the rational map Pf is surjective at [f]. This happens if the space of quintic forms in the four variables  $x_0, \ldots, x_3$  is generated by the polynomials  $P_{ijk} = x_k \operatorname{Pf}(f_{i,j})$ , where  $f_{i,j}$  is obtained by f removing the *i*-th column and the *j*-th row.

This computation is carried out easily by Macaulay 2, see [GS], and in fact it proves that on the general surface of degree  $d \leq 13$  it is defined an initialized rank 2 ACM bundle  $\mathcal{E}$  with  $c_1(\mathcal{E}) = d - 2$  and  $c_2(\mathcal{E}) = d(d - 1)(d - 2)/3$ .

In view of the previous remark, the question about existence is thus settled. However, we give here an abstract proof, in the hope of clarifying why the phenomenon occurs. We will use the notation  $C_g^d$  for a reduced connected curve of arithmetic genus g and degree d contained in  $\mathbb{P}^3$ .

**Lemma 6.5.** A general quintic surfaces in  $\mathbb{P}^3$  contain a smooth set of 10 points  $A_0$  in uniform position, with difference Hilbert function of the form 1,3,4,2. Moreover we may assume that  $A_0$  lie in a smooth irreducible complete intersection curve  $\mathcal{C} = C_6^4$ .

Proof. Take the intersection of a general quartic or quintic surface with a general quartic elliptic curve  $\mathcal{D} = C_4^1$  and consider a length 10 subscheme  $A_0$  of the intersection. Since  $\mathcal{D}$  is complete intersection of two general quadrics and  $A_0$  lies in no hyperplanes, one computes  $\Delta h(R_{A_0}, 2) = 4$ . The claim on the Hilbert function follows. Notice that  $A_0$  is contained in a smooth quadric surface  $Q \supset \mathcal{D}$ . Consider the linear system  $|\mathcal{L}|$  cut on Q by cubic surfaces through  $A_0$ . Since cubics separate the points of  $A_0$ ,  $|\mathcal{L}|$  has no fixed components. Furthermore,  $|\mathcal{L}|$  contains properly all the unions of  $C_4^1$  with hyperplane sections of Q, so  $|\mathcal{L}|$  is not composed with a pencil. Thus a general element  $\mathcal{C}$  in the linear system  $|\mathcal{L}|$  is irreducible, by Bertini's theorem. The curve  $\mathcal{C}$  is also smooth, since  $A_0$  is smooth and separated (sheaf-theoretically) by cubics.

**Corollary 6.6.** A general quintic surface in  $\mathbb{P}^3$  contains a set of 20 distinct points A, in uniform position, with difference Hilbert function of the form 1, 3, 5, 6, 4, 1.

*Proof.* Fix a general quintic X and consider the set  $A_0 \subset X$  and the complete intersection curve  $\mathcal{C}$ , of degree 6, containing  $A_0$ , given by the previous lemma.

Then the residual intersection  $A = (C \cap X) \setminus A_0$  is formed by 20 points. It is easy to compute the Hilbert function of A and check that it has the desired form.

The linear series cut on C by quintics through  $A_0$  has no base points, for  $A_0$  is cut sheaf-theoretically by quintics. Therefore its general element is smooth. Then, replacing X with another quintic surface X' in a neighbourhood of X, we may assume A consists of distinct points in uniform position.

Finally, since the curve C is contained in a smooth quartic surface, the same happens to A.

Recall the notation introduced at the beginning of section 3:  $\mathcal{G}(m, i)$  denotes the subset  $\operatorname{Hilb}_m(\mathbb{P}^3)$  parametrizing aG sets of points with index *i*. Denote by  $\mathcal{U}$  the locally closed subvariety of  $\operatorname{Hilb}_{20}(\mathbb{P}^3)$  parametrizing smooth sets of 20 points given by the previous lemma.

We want to prove that  $\mathcal{U}$  sits in the closure of  $\mathcal{G}(20, 4)$ . To perform the task, we take a general  $A \in \mathcal{U}$  and we examine carefully the behavior of sections of a bundle on  $Y_4$  associated to A. We need a series of lemmas.

**Lemma 6.7.** Take a general  $A \in \mathcal{U}$ , contained in a smooth general quartic  $Y_4$ . Then:

- i) the subscheme A induces a unique semistable bundle  $\mathcal{F}_A$  of rank 2 on  $Y_4$  with  $c_1(\mathcal{F}_A) = 4$ ;
- ii) the bundle  $\mathcal{F}_A(-2)$  has a global section vanishing on a smooth set B of 4 non-aligned points lying in some plane  $\pi$ ;
- iii) the bundle  $\mathcal{F}_A(-1)$  has a 5-dimensional space of sections, the general one vanishing on a smooth set W of 8 points in  $\pi$ , with difference Hilbert function 1, 2, 3, 2 (i.e. with general Hilbert function on  $\pi$ );

iv) we may assume that the curve  $\Gamma = \pi \cap Y_4$  is smooth.

*Proof.* From the difference Hilbert function, one sees immediately that A is not separated by quartics, and  $\mathrm{H}^1(Y_4, \mathcal{I}_A(4)) = 1$ . Then by Serre duality,  $\mathrm{Ext}^1(\mathcal{I}_A(4), \mathcal{O}_{Y_4})$  is 1-dimensional and provides an extension, as in sequence (2.6):

(6.3) 
$$0 \to \mathcal{O}_{Y_4} \to \mathcal{F}_A \to \mathcal{I}_A(4) \to 0,$$

where, by Theorem 2.3,  $\mathcal{F}_A$  is a rank 2 bundle with Chern classes  $c_1(\mathcal{F}_A) = 4$ ,  $c_2(\mathcal{F}_A) = 20$ .

Looking at the Hilbert function of A, we get  $\mathrm{H}^{0}(Y_{4}, \mathcal{F}_{A}(-3)) = 0$ ,  $\mathrm{h}^{0}(Y_{4}, \mathcal{F}_{A}(-2)) = 1$ ,  $\mathrm{h}^{0}(Y_{4}, \mathcal{F}_{A}(-1)) = 5$ . It turns out then that  $\mathcal{F}_{A}$  is semistable, and  $\mathcal{F}_{A}(-2)$  has a global section s vanishing on a finite set B. We have  $c_{1}(\mathcal{F}_{A}(-2)) = 0$ ,  $c_{2}(\mathcal{F}_{A}(-2)) = 4$ , so B has length 4. We have an exact sequence:

(6.4) 
$$0 \to \mathcal{O}_{Y_4} \xrightarrow{s} \mathcal{F}_A(-2) \to \mathcal{I}_B \to 0.$$

We obtain that  $h^0(Y_4, \mathcal{I}_B(1)) = h^0(Y_4, \mathcal{F}_A(-1)) - h^0(Y_4, \mathcal{O}_{Y_4}(1)) = 5 - 4 = 1$ , so *B* is not aligned. Since  $h^0(Y_4, \mathcal{F}_A(-1)) > h^0(Y_4, \mathcal{O}_{Y_4}(1))$ , then  $\mathcal{F}_A(-1)$  has global sections which are not multiple of *s*. Consequently, a

general global section  $t \in \mathrm{H}^{0}(Y_{4}, \mathcal{F}_{A}(-1))$  vanishes on a set W of finite length  $8 = c_{2}(\mathcal{F}_{A}(-1))$ , and we have an exact sequence:

(6.5) 
$$0 \to \mathcal{O}_{Y_4} \xrightarrow{t} \mathcal{F}_A(-1) \to \mathcal{I}_W(2) \to 0.$$

One easily computes the value of  $h^0(Y_4, \mathcal{F}_A(t))$  for all t, hence the Hilbert functions of B and W are known. Let us show that W belongs to  $\pi$ . By construction the plane  $\pi$  is induced in  $H^0(Y_4, \mathcal{I}_B(1))$  by a general section  $t \in H^0(Y_4, \mathcal{F}_A(-1))$ , thus it corresponds to the vanishing set of the wedge product  $s \wedge t$ . More precisely, W corresponds to the intersection of the vanishing loci of s f and t, for any choice of  $f \in H^0(Y_4, \mathcal{O}_{Y_4}(1)))$ . Since the plane containing W is induced by any global section  $s \in H^0(Y_4, \mathcal{F}_A(-2))$ , the two planes coincides.

Finally, we need to show that we may assume B, W and  $\Gamma = \pi \cap Y_4$ to be smooth. Indeed on  $\pi$  the Hilbert scheme of subschemes of finite length is smooth. Then we may move generically B in  $\pi$  to a smooth set of 4 points. Since the Hilbert function is constant in the family, the space  $\mathrm{H}^0(Y_4, \mathcal{I}_{B,\mathbb{P}^2}(4))$  defines a bundle over the deformation. Thus  $Y_4$  moves in a family of quartic surfaces containing the 4 points. Similarly,  $\mathrm{Ext}^1(\mathcal{I}_B, \mathcal{O}_{Y_4})$ , which is dual to  $\mathrm{H}^1(Y_4, \mathcal{I}_B)$ , has constant dimension through the deformation. Thus also  $\mathcal{F}_A$  moves in a family of bundles. Possibly replacing Awith some neighbouring general element of  $\mathcal{U}$ , we can now assume that the schemes B and  $\Gamma$  are smooth. A similar procedure proves that we may presume W to be smooth.  $\Box$ 

**Lemma 6.8.** Fix the previous notation. The sections  $\mathrm{H}^{0}(Y_{4}, \mathcal{F}_{A}(-1))$  determine a non-complete linear series of degree 8 and dimension at most 4 on the curve  $\Gamma = \pi \cap Y_{4}$ . The dimension of the series is 4, unless  $\mathcal{F}_{A}$  has infinitely many sections (modulo the  $\mathbf{k}^{*}$  action) vanishing on W.

*Proof.* The datum of W and B corresponds to a linear section of  $\Gamma$ . Indeed we have the natural exact diagram:



The claim follows, since  $h^0(Y_4, \mathcal{F}_A(-1)) = 5$ .

Next, we need a technical result, for deformations of 8 points. We are grateful to Ciro Ciliberto, who pointed us an elegant a quick proof for it.

**Proposition 6.9.** Let W be a general set of 8 points in a plane  $\pi \subset \mathbb{P}^3$ . Then in Hilb<sub>8</sub>( $\mathbb{P}^3$ ), W sits in the closure of the subvariety defined by complete intersections of three quadrics.

*Proof.* In fact, we are going to prove that W is the projection to  $\pi$  of a complete intersection W' of three quadrics  $Q_1, Q_2, Q_3 \subset \mathbb{P}^3$ . Then the procedure of projection determines a deformation of the quadrics which, in turn, defines a deformation of W' to W.

To show the claim, fix a cubic plane curve  $\mathcal{B} \subset \pi$  which contains W. As W is general, we may assume  $\mathcal{B}$  smooth. Call  $\mathcal{N}$  the linear series cut on  $\mathcal{B}$  by the lines of  $\pi$  and consider the complete linear series  $\mathcal{M} = \mathcal{O}_{\mathcal{B}}(W - 2\mathcal{N})$  on  $\mathcal{B}$ . This series is a pencil for it has degree 2. Fix a Weierstrass point y for  $\mathcal{M}$  and consider the complete linear series  $\mathcal{N}' = \mathcal{O}_{\mathcal{B}}(\mathcal{N} + y)$  on  $\mathcal{B}$ . Since  $\mathcal{N}'$  has degree 4, it defines a map  $\phi : \mathcal{B} \to \mathbb{P}^3$ , whose image is a quartic curve  $\mathcal{B}' \subset \mathbb{P}^3$ . The curve  $\mathcal{B}$  corresponds to the projection of  $\mathcal{B}'$  from  $z = \phi(y)$ . The subscheme W sits in the series  $2\mathcal{N}'$  by construction. Hence  $W' = \phi(W)$  is the intersection of  $\mathcal{B}'$  and a quadric. Since  $\mathcal{B}'$  is itself a complete intersection of two quadrics and the projection from z maps W' to W, the claim follows.

**Remark 6.10.** In general one may ask for which values of a, b, c, a general set of a b c points in  $\pi$  is the limit of a complete intersection of type (a, b, c) in  $\mathbb{P}^3$ . A simple parameter count proves that the answer is negative as soon as (a, b, c) > (2, 3, 3). The previous lemma provides a positive answer for (a, b, c) = (2, 2, 2). A similar argument works for (a, b, c) = (2, 2, n).

Now we are ready to prove:

**Proposition 6.11.** The subscheme  $\mathcal{U} \subset \operatorname{Hilb}_{20}(\mathbb{P}^3)$  sits in the closure of  $\mathcal{G}(20, 4)$ .

*Proof.* Call A a general element in  $\mathcal{U}$  and fix a quartic surface Y containing A. Let  $\mathcal{F}$  be a rank 2 bundle on Y associated to A, with Chern classes  $c_1 = 4$  and  $c_2 = 20$ . Consider also a smooth set W of 8 points in a plane  $\pi$ , given by a general section of  $\mathcal{F}(-1)$  (see Lemma 6.7).

By Proposition 6.9, we have a flat family  $\eta : \mathcal{W} \to \Delta$  of smooth length-8 subschemes of  $\mathbb{P}^3$ , parametrized by a neighbourhood  $\Delta$  of 0, whose fibre  $\eta^{-1}(0)$  is W and whose general fibre  $\eta^{-1}(g)$  is a set  $W_g$ , complete intersection of three quadrics. Since  $h^0(\mathbb{P}^3, \mathcal{I}_{W,\mathbb{P}^3}(4)) = h^0(\mathbb{P}^3, \mathcal{I}_{W_g,\mathbb{P}^3}(4))$ , we may lift Yto a family of quartic surfaces whose general element  $Y_g$  contains  $W_g$ . Notice that the family  $W_g$  is produced via the choice of a cubic plane curve through W (which moves in a pencil) plus a Weierstrass point on the curve (they are interchanged in the pencil). Thus it depends continuously on one parameter.

The rank 2 bundle  $\mathcal{F}$  corresponds, up the  $\mathbf{k}^*$  action, to the choice of an element in the 2-dimensional space  $\operatorname{Ext}^1(\mathcal{I}_W(2), \mathcal{O}_Y) \cong \operatorname{H}^1(Y, \mathcal{I}_W(2))^*$ . Our statement amounts to the following:

**Claim 1.** The deformation  $\eta$  of the 8 points  $W \subset \Gamma = Y \cap \pi$  can be chosen in such a way that  $\mathcal{F}$  extends to a rank 2 bundle  $\mathcal{F}_g$  on  $Y_g$ , where  $\mathcal{F}_g(-1)$  is associated with a complete intersection set of 8 points.

 $\Box$ 

Notice that  $\operatorname{Ext}^1(\mathcal{I}_{W_g}(2), \mathcal{O}_{Y_g}) \cong \operatorname{H}^1(Y_g, \mathcal{I}_{W_g}(2))^*$ , thus  $\operatorname{Ext}^1(\mathcal{I}_{W_g}(2), \mathcal{O}_{Y_g})$  defines a 1-dimensional subset of the 2-dimensional space  $\operatorname{Ext}^1(\mathcal{I}_W(2), \mathcal{O}_Y)$ , as  $W_g$  goes to W. So the claim is non trivial.

Proof Claim 1. The statement is obvious when there are infinitely many sections of  $\mathcal{F}(-1)$  vanishing on  $W \pmod{\mathbf{k}^*}$ , for in this case all the elements of  $\operatorname{Ext}^1(\mathcal{I}_W(2), \mathcal{O}_Y)$  correspond to the same rank 2 bundle  $\mathcal{F}$  (only the section varies). Assume this is not the case. Thus  $\mathcal{F}$  defines a 4-dimensional linear series in |W|.

Changing the element in  $\operatorname{Ext}^1(\mathcal{I}_W(2), \mathcal{O}_Y)$  has the effect of choosing a bundle  $\mathcal{F}'$  on Y, associated with a 4-dimensional linear series in |W|. This linear series cannot be fixed as we vary the extension, for |W| has dimension 5, so it is not the union of disjoint 4-dimensional subspaces, while a general set of 8 points on  $\Gamma$  determines a rank 2 bundle on Y, hence a 4-dimensional linear subseries. Since a general set of 8 points can be lifted to a *ci* subscheme in  $\mathbb{P}^3$ , we have at least a 1-dimensional family of bundles on Y, associated with W, which lifts to a deformation of a set of 8 points on  $\Gamma$ . Possibly replacing now W with a neighbouring element in |W|, we may assume that W admits a deformation to a complete intersection. This completes the proof.  $\Box$ 

We are now in position to prove:

**Theorem 6.12.** The map  $p = p_{20,4,5} : \mathcal{G}(20,4,5) \to |\mathcal{O}_{\mathbb{P}^3}(5)|$  of diagram (3.1) is dominant. In other words, on a general quintic surface  $X \subset \mathbb{P}^3$  it is defined an initialized indecomposable ACM bundle  $\mathcal{E}$  of rank 2, with Chern classes  $c_1 = 3$  and  $c_2 = 20$ .

*Proof.* Consider the set  $\mathcal{U}'$  of pairs (A, Y) where  $A \in \mathcal{U}$  and Y is a quintic surface containing A. Since  $\mathcal{U}$  sits in the closure of  $\mathcal{G}(20, 4)$  and elements of both  $\mathcal{U}$  and  $\mathcal{G}(20, 4)$  are separated by quintics, then  $\mathcal{U}'$  sits in the closure of  $\mathcal{G}(20, 4, 5)$ . So Corollary 6.6 proves that the map  $p_{|\mathcal{U}'} : \mathcal{U}' \to |\mathcal{O}_{\mathbb{P}^3}(5)|$  is dominant.  $\Box$ 

6.3. The maximal case:  $c_1 = 4$ , 30 points. Here we prove the existence of an initialized ran 2 ACM bundle on the general surface X, achieving the maximal value of  $c_1$  and  $c_2$ . This result is already known, and due to Beauville and Schreyer, see the appendix of [Bea00]. However their proof relies on a Macaulay2 computation, while we outline a geometric approach which makes use of Beilinson's theorem and a deformation argument.

**Theorem 6.13.** On a general quintic surface X there exists an initialized indecomposable rank 2 ACM bundle  $\mathcal{E}$  with Chern classes  $c_1 = 4$  and  $c_2 = 30$ .

The rest of this section contains the proof of this theorem. Consider the sheaves of differentials  $\Omega_{\mathbb{P}^3}^p(p) = \wedge^p \Omega_{\mathbb{P}^3}(p)$ , for  $0 \leq p \leq 3$ . Define the bundle  $\mathsf{P} = \mathcal{O}_{\mathbb{P}^3}^{10} \oplus \Omega_{\mathbb{P}^3}^1(1) \oplus \Omega_{\mathbb{P}^3}^2(2)$  and the vector space  $V = \mathrm{H}^0(\mathbb{P}^3, \wedge^2(\mathsf{P})(1))$ . We can view an element in V as a skew-symmetric matrix  $\mathsf{f} : \mathsf{P}^*(-1) \to \mathsf{P}$ . So we can define a rational map  $\mathrm{Pf} : \mathbb{P}(V) \dashrightarrow \mathbb{P}(\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5)))$ , which associates to an element of V the square root of its determinant, defined up to a nonzero scalar.

**Lemma 6.14.** Assume that on the general quintic surface X there exists an initialized rank 2 bundle  $\mathcal{F}$ , with  $c_1(\mathcal{F}) = 2$ ,  $c_2(\mathcal{F}) = 15$ , and with  $h^1(X, \mathcal{F}(t)) = 0$ , for each  $t \in \mathbb{Z}$ , except for  $h^1(X, \mathcal{F}) = h^1(X, \mathcal{F}(-1)) = 1$ . Then the rational map Pf defined above is dominant.

*Proof.* Let  $\iota : X \to \mathbb{P}^3$  be the natural inclusion. Applying Beilinson's theorem to the sheaf  $\iota_*(\mathcal{F}(1))$ , one can write down the following resolution:

(6.7) 
$$\begin{array}{cccc} \mathcal{O}_{\mathbb{P}^3}(-1)^{10} & \mathcal{O}_{\mathbb{P}^3}^{10} \\ \oplus & \oplus \\ \Omega^2_{\mathbb{P}^3}(2) & \xrightarrow{\mathsf{f}(\mathcal{F}(1))} & \Omega^1_{\mathbb{P}^3}(1) & \to \iota_*(\mathcal{F}(1)) \to 0. \\ \oplus & \oplus \\ \Omega^1_{\mathbb{P}^3}(1) & \Omega^2_{\mathbb{P}^3}(2) \end{array}$$

We still denote by  $\mathsf{P}(\mathcal{F}(1))$  the target of the map  $\mathsf{f}(\mathcal{F}(1))$ , while we write  $\mathsf{Q}(\mathcal{F}(1))$  for the domain of  $\mathsf{f}(\mathcal{F}(1))$ . Notice that  $\mathsf{P}(\mathcal{F}(1)) \simeq \mathsf{P}$ , and  $\mathsf{Q}(\mathcal{F}(1)) \simeq \mathsf{P}^*(-1)$ , with  $\mathsf{P}$  defined above. Observe that there is no obstruction to the lifting of any map  $\mathsf{P}(\mathcal{F}(1)) \to \mathcal{F}(1)$  to an endomorphism of  $\mathsf{P}(\mathcal{F}(1))$ , thanks to the vanishing:

$$\operatorname{Ext}^{1}(\mathsf{P}(\mathcal{F}(1)), \mathsf{Q}(\mathcal{F}(1))) = 0.$$

Using this fact one can easily prove, following step by step the proof of Theorem 2.9 contained in [Bea00], that there is an isomorphism  $\phi$  :  $\mathsf{P}(\mathcal{F}(1)) \to \mathsf{Q}(\mathcal{F}(1))^*(-1)$  such that the following diagram commutes:

where  $\kappa$  is a skew-symmetric duality of  $\mathcal{F}$ . Equivalently, the matrix  $f(\mathcal{F}(1))$  is skew-symmetric, i.e.  $f(\mathcal{F}(1))$  sits in the vector space V defined above. Now  $Pf(f(\mathcal{F}(1)))$  is the equation of the support of  $\mathcal{F}$ , which is defined over the general quintic surface. So our hypothesis implies that the rational map Pf is dominant.

Now set  $\mathsf{P}' = \mathcal{O}_{\mathbb{P}^3}^{10}$ ,  $V' = \mathrm{H}^0(\mathbb{P}^3, \wedge^2(\mathsf{P}')(1))$  and consider the restriction  $\mathrm{Pf}'$ of the rational map  $\mathrm{Pf}$  to  $\mathbb{P}(V')$ , where we think of an element  $[\mathsf{f}']$  of  $\mathbb{P}(V')$ in  $\mathbb{P}(V)$  as a block matrix made up of  $\mathsf{f}'$ , the identity map on  $\Omega_{\mathbb{P}^3}^1(1)$  and  $\Omega_{\mathbb{P}^3}^2(2)$ , and zero everywhere else. Roughly speaking, the next lemma allows to deform the matrix  $f(\mathcal{F}(1))$  in such a way that we can *erase* the undesired summands from the resolution - namely, the copies of  $\Omega^1_{\mathbb{P}^3}(1)$  and  $\Omega^2_{\mathbb{P}^3}(2)$ . This will provide us with an initialized rank 2 ACM bundle as a generalization of  $\mathcal{F}(1)$ .

**Lemma 6.15.** Set hypothesis as in the previous lemma. Then the rational map  $Pf' : \mathbb{P}(V') \dashrightarrow |\mathcal{O}_{\mathbb{P}^3}(5)|$  is dominant. In particular, Theorem 6.13 holds.

*Proof.* Recall by Lemma 6.14 that Pf is dominant, so its differential at a general element [f] of  $\mathbb{P}(V)$  is surjective. Representing [f] as a skew-symmetric matrix  $f : \mathsf{P}^*(-1) \to \mathsf{P}$ , we can write f in the form:

$$\left(\begin{array}{ccc} \mathbf{f}_0 & a & b \\ -a^\top & c & d \\ -b^\top & -d^\top & 0 \end{array}\right),$$

with:

$$\begin{split} a : \Omega_{\mathbb{P}^{3}}^{2}(2) \to \mathsf{P}', & a^{\top} : (\mathsf{P}')^{*}(-1) \to \Omega_{\mathbb{P}^{3}}^{1}(1), \\ b : \Omega_{\mathbb{P}^{3}}^{1}(1) \to \mathsf{P}', & b^{\top} : (\mathsf{P}')^{*}(-1) \to \Omega_{\mathbb{P}^{3}}^{2}(2), \\ c : \Omega_{\mathbb{P}^{3}}^{2}(2) \to \Omega_{\mathbb{P}^{3}}^{1}(1), & c^{\top} : \Omega_{\mathbb{P}^{3}}^{2}(2) \to \Omega_{\mathbb{P}^{3}}^{1}(1), \\ d : \Omega_{\mathbb{P}^{3}}^{1}(1) \to \Omega_{\mathbb{P}^{3}}^{1}(1), & d^{\top} : \Omega_{\mathbb{P}^{3}}^{2}(2) \to \Omega_{\mathbb{P}^{3}}^{2}(2). \end{split}$$

and with  $f_0 = -f_0^{\top}$  and  $c = -c^{\top}$ . Consider a matrix f of the form (6.7) provided by Lemma 6.14. It satisfies d = 0, and the differential of Pf is surjective at the point represented by such f. In general d must be a scalar multiple of the identity, so we may set  $f_{\varepsilon} = f + \varepsilon \operatorname{id}_{\Omega^1_{\mathbb{P}^3}(1)} - \varepsilon \operatorname{id}_{\Omega^2_{\mathbb{P}^3}(2)}$ . The differential of Pf will be surjective at the point  $[f_{\varepsilon}]$  of the space  $\mathbb{P}(V)$ .

Since c is skew-symmetric, we can choose  $\zeta \in \text{Hom}(\Omega^2_{\mathbb{P}^3}(2), \Omega^1_{\mathbb{P}^3}(1))$  with  $\zeta - \zeta^{\top} = c$ . Consider now the automorphism g of P, written in the form of a block matrix like:

$$\left(\begin{array}{ccc} 1 & -b/\varepsilon & 1/\varepsilon^2(\varepsilon \, a - b \, c) \\ 0 & \varepsilon & \zeta \\ 0 & 0 & 1 \end{array}\right)$$

We obtain the matrix  $h_{\varepsilon} = g^{\top} \cdot \mathbf{f}_{\varepsilon} \cdot g$ . The only nonvanishing blocks of the matrix  $h_{\varepsilon}$  are a block of linear forms  $\mathbf{f}'$  and identity maps of  $\Omega^1_{\mathbb{P}^3}(1)$  and  $\Omega^2_{\mathbb{P}^3}(2)$ , which we can now factor out. This means that  $[h_{\varepsilon}]$  sits in  $\mathbb{P}(V')$ , and the differential of Pf is surjective at  $[h_{\varepsilon}]$ , hence we are done.  $\Box$ 

**Lemma 6.16.** On the general quintic surface X it is defined a rank 2 bundle  $\mathcal{F}$  which satisfies the hypothesis of Lemma 6.14.

*Proof.* Our claim is equivalent to the fact that the general quintic surface contains a length 15 subscheme Z having difference Hilbert function of the form 1, 3, 6, 4, 1.

We need a slight refinement of Lemma 3.2, where we allow the subscheme Z to be of the above form, (in particular Z is not aG). Namely we claim that if Z is given as the vanishing locus of a section of a rank 2 bundle  $\mathcal{G}$  with  $c_1 \leq 3$  defined on the general quartic surface  $Y_4$ , then the general quintic surface X contains a subscheme with the same Hilbert function as Z. The proof of this claim is similar to that of Lemma 3.2, so we omit it here.

Counting dimensions, we see that our claim amounts to the fact that on a given smooth quartic surface  $Y_4$  containing Z, with  $\operatorname{Pic}(Y_4) = \mathbb{Z}$ , the family of subschemes of  $Y_4$  with difference Hilbert function 1, 3, 6, 4, 1 has dimension 24. Notice that  $\mathcal{G}$  is stable and has 7 independent sections, hence it suffices to show that on such  $Y_4$  the moduli space  $M_{Y_4}(2; 3, 15)$  has dimension 18. But since  $Y_4$  is a smooth K3 surface and  $\mathcal{G}$  is a stable bundle, this is indeed the case, see for instance [HL97, Part II, Chapter 6].

**Corollary 6.17.** Let  $X \subset \mathbb{P}^3$  be a general quintic surface, and for let  $c_1, c_2$  be integers given in the table (1.1) of Chern classes of an ACM rank 2 bundle  $\mathcal{E}$  on X, with  $c_1 \geq 1$ . Then the component of the moduli space  $\mathsf{M}_X(2; c_1(\mathcal{E}), c_2(\mathcal{E}))$  containing  $[\mathcal{E}]$  is reduced and of the expected dimension.

Proof. For all such integers  $c_1$ ,  $c_2$ , set  $i = c_1 + 1$ ,  $m = c_2$ . By our main result, the map  $p_{m,i,5}$  is dominant, so its differential has maximal rank at a general point  $(\mathbb{Z}, \mathbb{X})$ . The fibre  $p_{m,i,5}^{-1}(\mathbb{X})$  is thus generically smooth of the expected dimension. Now, Theorem 2.3 and Theorem 2.5 provide a rational map  $\zeta : p_{m,i,5}^{-1}(\mathbb{X}) \longrightarrow \mathsf{FM}^{\mathsf{s}}_X(2; c_1(\mathcal{E}), c_2(\mathcal{E}))$ , where the target space is the so-called moduli space of framed sheaves, i.e. pairs  $[\mathcal{F}, s]$  with  $\mathcal{F} \in \mathsf{M}_X(2; c_1(\mathcal{E}), c_2(\mathcal{E}))$  and  $s \in \mathbb{P}(\mathsf{H}^0(\mathbb{X}, \mathcal{F}))$ . The map  $\zeta$  is a locally closed immersion. Since being ACM is an open condition, the differential at  $[\mathcal{E}]$ of the natural projection  $\mathsf{FM}^{\mathsf{s}}_X(2; c_1(\mathcal{E}), c_2(\mathcal{E})) \to \mathsf{M}_X(2; c_1(\mathcal{E}), c_2(\mathcal{E}))$  is surjective on the tangent space of  $\mathsf{M}_X(2; c_1(\mathcal{E}), c_2(\mathcal{E}))$  at  $[\mathcal{E}]$ . It follows that the moduli space  $\mathsf{M}_X(2; c_1(\mathcal{E}), c_2(\mathcal{E}))$  is smooth in a neighbourhood of  $[\mathcal{E}]$ . It easily follows that the component of  $\mathsf{M}_X(2; c_1(\mathcal{E}), c_2(\mathcal{E}))$  containing  $[\mathcal{E}]$  has the expected dimension.

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