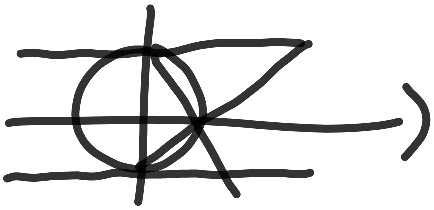
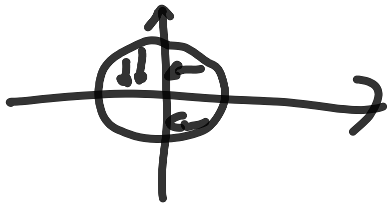
S^1  

X $\underset{\{(U_i,\varphi_i)\}}{\overset{}{\mathcal{A}}}$ diff. structure on X

$\mathcal{A} = \{(U_i,\varphi_i)\}$

<u>Def</u> A chart (U,φ) is COMPATIBLE with $\mathcal{A}$ if $\dim(U,\varphi) = \dim \mathcal{A}$ and $\forall i$

$\varphi_i \circ \varphi^{-1}$ and $\varphi \circ \varphi_i^{-1}$ are both C^∞

In other words (U,φ) compatible $\Leftrightarrow$

$\mathcal{A} \cup \{(U,\varphi)\}$ is again a diff. structure

<u>Ex</u> $\mathcal{A} = (\mathbb{R}, \{(\mathbb{R}, id)\})$ is a diff. structure on $\mathbb{R}$

take $((0,1), id)$ is compatible with $\mathcal{A}$.

i.e. $id \circ id^{-1}$ is C^∞

<u>Ex</u> $(X,\mathcal{A})$ diff. manifold $\mathcal{A} = \{(U_i,\varphi_i)\}$

(U_0,φ_0) $\underset{\text{open}}{U \subseteq U_0}$ $\varphi = \varphi_0|_U$

Then (U,φ) is compatible with $\mathcal{A}$.

<u>Ex</u> $(\mathbb{R}, \{(\mathbb{R}, id)\})$ $(\mathbb{R},\varphi)$ $\varphi(x) = x^3$

$\varphi: \mathbb{R} \to \mathbb{R}$ homeom.

<u>not</u> compatible

$$\varphi \circ id^{-1}(x) = \varphi(x) = x^3 \quad C^\infty$$

$$id \circ \varphi^{-1}(x) = id(\sqrt[3]{x}) = \sqrt[3]{x} \quad \underset{\uparrow}{\overset{\text{not}}{C^\infty}} \quad \frac{d}{dx} = \frac{1}{3}\frac{1}{\sqrt[3]{x^2}}$$

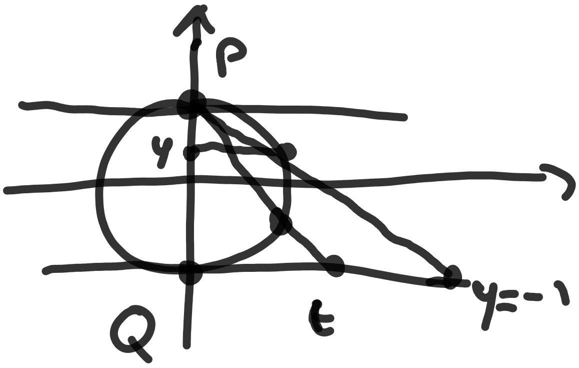
$\varphi(\mathbb{R} \cap \mathbb{R}) = \mathbb{R} \longrightarrow \mathbb{R}$ not defined in 0

$\{(\mathbb{R}, id), (\mathbb{R}, \varphi)\}$ is NOT a diff. structure on $\mathbb{R}$.

<u>Ex</u> $S^1 = X$ $\quad A = \{(U_1, \varphi_1), (U_2, \varphi_2)\}$ stereo proj.

(U, φ) $\quad U = S^1 \cap \{x > 0\}$ $\quad \varphi$ proj along x

$\varphi(x, y) = y$



$\varphi \circ \varphi_1^{-1}(t) =$

$$\varphi\left(\frac{4t}{t^2+4}, \frac{t^2-4}{t^2+4}\right) = \frac{t^2-4}{t^2+4} \quad \underline{C^\infty}$$

$\varphi_1 \circ \varphi^{-1}(y) : \varphi(U \cap U_1) \to \mathbb{R}$

$\varphi(U) = (-1, 1)$

$\varphi_1 \circ \varphi^{-1}(y) = (\sqrt{1-y^2}, y)$ $\qquad y = mx + 1$ line $P, \varphi^{-1}(y)$

$$y = m\sqrt{1-y^2} + 1$$

$$m = \frac{y-1}{\sqrt{1-y^2}}$$

$$y = \frac{y-1}{\sqrt{1-y^2}} x + 1$$

$$-1 = \frac{y-1}{\sqrt{1-y^2}} x + 1 \qquad x = \frac{-2\sqrt{1-y^2}}{y-1} = \varphi_1 \circ \varphi^{-1}(y) \quad y \in (-1, 1)$$

$$\left.\begin{array}{l} y < 1 \Rightarrow y - 1 \neq 0 \quad \frac{1}{y-1} \; C^\infty \\ -1 < y < 1 \Rightarrow -2\sqrt{1-y^2} \; C^\infty \end{array}\right\} \Rightarrow \varphi_1 \circ \varphi^{-1} \; \underline{C^\infty}$$

$\varphi \circ \varphi_2^{-1}$ $\quad \varphi_2 \circ \varphi^{-1}$ $\quad C^\infty$

$$\Downarrow$$

(U, φ) is compatible with A.

Def X $\quad A_1, A_2$ diff. structures on X

∘ A_1 is <u>compatible</u> with A_2 if all charts of A_1 are.

This is equiv. to say that $A_1 \cup A_2$ is a diff. structure

Ex on S^1 $\quad A_1$ stereo proj

A_2 implicit functions thm. $\quad A_1 \cup A_2$ is diff. struct.

X $\quad A_1, A_2$ diff. struct. on X

(X, A_1) $\quad (X, A_2)$ diff. manifolds

Prop
$id : (X, A_1) \to (X, A_2)$
$id : (X, A_2) \to (X, A_1)$
diff. maps $\Leftrightarrow A_1$ is compatible with A_2

Indeed id is a diff. map iff. $\forall$ charts $(U, \varphi) \in A_1$ and $(U', \varphi') \in A_2$

$$\left.\begin{array}{ll} \varphi' \circ id \circ \varphi^{-1} = \varphi' \circ \varphi^{-1} & \text{is } C^\infty \\ \varphi \circ id \, \varphi'^{-1} = \varphi \circ \varphi'^{-1} & \text{is } C^\infty \end{array}\right| \Rightarrow \text{every chart of } A_1 \text{ is compatible with } A_2$$

X $\quad \{\text{diff. structures on } X\} \ni A_1, A_2$

$A_1 \sim A_2 \Leftrightarrow A_1$ is compatible with A_2

Thm $A_1 \sim A_2$ is an equivalence relation pf <u>reflex.</u> $A_1 \sim A_1$ because $A_1 \cup A_1 = A_1$ diff. struct.

<u>symm.</u> $A_1 \sim A_2 \Rightarrow A_1 \cup A_2$ diff. struct.

$\Rightarrow A_2 \cup A_1$ diff. struct. $\Rightarrow A_2 \sim A_1$

<u>trans.</u> $A_1 \sim A_2$ and $A_2 \sim A_3 \overset{?}{\Rightarrow} A_1 \sim A_3$

$(U, \varphi) \in A_1 \qquad (U'', \varphi'') \in A_3$

Is $\varphi'' \circ \varphi^{-1}$ C^∞? $\quad \varphi'' \circ \varphi^{-1} : \varphi(U \cap U'') \to \dots$

$p \in U \cap U''$ $\varphi(p) \in \varphi(U \cap U'') \qquad p \in (U', \varphi') \in A_2$

$p \in U \cap U'' \cap U'$ open neigh. of p

$\varphi(p) \in \varphi(U \cap U'' \cap U')$ " " $\varphi(p)$

In $\varphi(U \cap U'' \cap U')$ $\quad \underset{\substack{\downarrow \\ C^\infty}}{\varphi'' \circ \varphi^{-1}} = \underbrace{(\varphi'' \circ \varphi'^{-1})}_{\substack{C^\infty \\ \text{because} \\ A_2 \sim A_3}} \circ \underbrace{(\varphi' \circ \varphi^{-1})}_{\substack{C^\infty \\ \text{because } A_1 \sim A_2}}$

The same holds for

$\varphi \circ \varphi''^{-1} = (\varphi \circ \varphi'^{-1}) \circ (\varphi' \circ \varphi''^{-1})$

ZORN'S LEMMA

<u>Thm</u> Every diff. structure A on X is contained in a <u>UNIQUE</u> <u>MAXIMAL</u> structure A'.

<u>Rem</u> A_1, A_2 diff. structures on X, if $A_1 \subseteq A_2$ then $A_1 \sim A_2$ because $A_1 \cup A_2 = A_2$ diff. struct.

<u>pf</u> $Z = \{$diff. structures that contain $A\}$ $\subseteq$

Assume A_1, A_2 are max elem. in Z

$A_1 \supseteq A$ so $A_1 \sim A$

$A_2 \supseteq A$ " $A_2 \sim A$ $\longmapsto A_1 \sim A_2$

$\Downarrow$

$A_1 \cup A_2$ diff. structure

$A_1 = A_1 \cup A_2$ $\quad$ $A_2 = A_1 \cup A_2$

$\to A_1 = A_2 \leftarrow$

For the $\exists$ of a max elem. Use the Zorn's Lemma

$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots \subseteq A_n \subseteq \dots$ chain of elem. of Z

$\bar{A} = \bigcup A_i$ Just need $\bar{A} \in Z$ i.e.

- $\bar{A}$ is a diff. struch.

$\underline{(u,\varphi)\ (u',\varphi')} \in \bar{A}$ $\quad$ $i \leq j \Rightarrow A_i \subseteq A_j$

$\underset{A_i}{\in}$ $\quad$ $\underset{A_j}{\in}$ $\quad$ $(u,\varphi),(u',\varphi') \in A_j$ diff. struch

$\varphi' \circ \varphi^{-1}$ and $\varphi \circ \varphi'^{-1}$ are both C^∞

X $\quad A_1, A_2$ diff. structures

$\mathring{A}_1, A_2$ compatible $\Leftrightarrow A_1 \cup A_2$ diff. structure

$(X, A_1) \sim (X, A_2)$ (equivalent)

$f: (X, A) \longrightarrow (X', A')$ differ. map

Def $(X, A), (X', A')$ are DIFFEOMORPHIC

if $\exists$ $(X, A) \xrightarrow{f} (X', A')$ diff. maps
$(X', A') \xrightarrow{g} (X, A)$

s.t. $g \circ f = id_X \quad f \circ g = id_{X'}$

Ex If A_1, A_2 compatible then (X, A_1) (X, A_2) are diffeomorphic

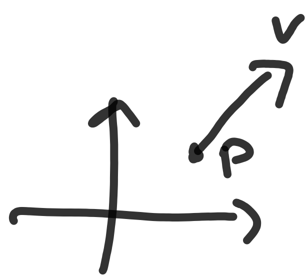
Indeed $id: (X, A_1) \longrightarrow (X, A_2)$ are
$id: (X, A_2) \to (X, A_1)$ diff. maps.

Ex $S^1 = X$ $\quad A_1 \Leftrightarrow$ stereo. proj
$A_2 \Leftrightarrow$ implicit functions thm

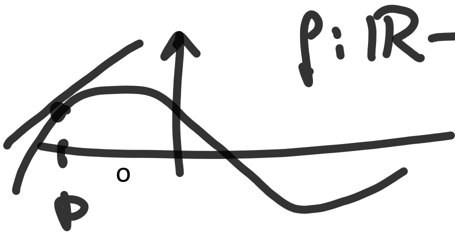
(X, A_1) is diffeom to (X, A_2)

$\underline{(X, A)} \xrightarrow{f} \mathbb{R}$ $\qquad \underline{\mathbb{R}}^n \xrightarrow{f} \mathbb{R}$

$\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ directional derivative

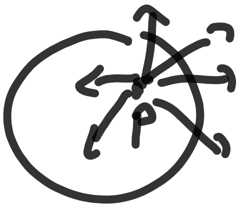


$| \dfrac{\partial f}{\partial v} = (\text{grad} f)_p \cdot v$



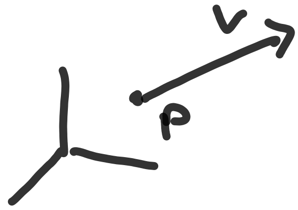
$f: \mathbb{R} \to \mathbb{R}$ $\qquad$ $f: \mathbb{R}^2 \to \mathbb{R}$

S^2

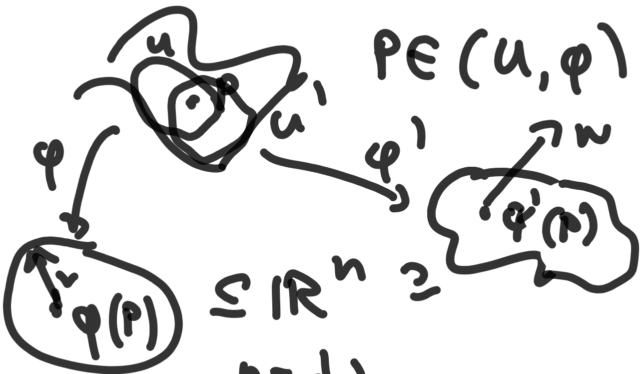


TANGENT VECTORS

(X, A) $\quad$ $p \in X$ $\qquad$ $\mathbb{R}^n = X$



$p \in (U, \varphi)$



$\subseteq \mathbb{R}^n \supseteq$

$n = \dim.$

$\varphi' \circ \varphi^{-1} : \varphi(U \cap U') \to \varphi'(U \cap U')$

C^∞ $\qquad \subseteq \mathbb{R}^n \qquad\qquad \subseteq \mathbb{R}^n$

$F : \mathbb{R}^n \to \mathbb{R}^m$ $\quad C^\infty$ $\qquad p \quad V_p =$ vectors applied in p

$F(p) \quad W_{F(p)} =$ vectors applied in $F(p)$

dF = differential of F

$dF_p : V_p \to W_{F(p)}$ linear $\qquad F = (F_1, \dots, F_m)$

Jacobian matrix $J = \left(\partial F_i / \partial x_j \right)$ $\quad m \times n$

$J_p : \underset{\dim n}{V_p} \to \underset{\dim m}{W_{F(p)}}$

Ex $F : \mathbb{R}^2 \to \mathbb{R}^3$ $\quad F(x,y) = (\underbrace{xy}_{F_1}, \underbrace{x^2+y^2}_{F_2}, \underbrace{\sin(x+y)}_{F_3})$

- $\partial F_1/\partial x = y \quad \partial F_1/\partial y = x$

$\partial F_2/\partial x = 2x \quad \partial F_2/\partial y = 2y$

$\partial F_3/\partial x = \cos(x+y) = \partial F_3/\partial y$

$$J = \begin{pmatrix} y & x \\ 2x & 2y \\ \cos(x+y) & \cos(x+y) \end{pmatrix}$$

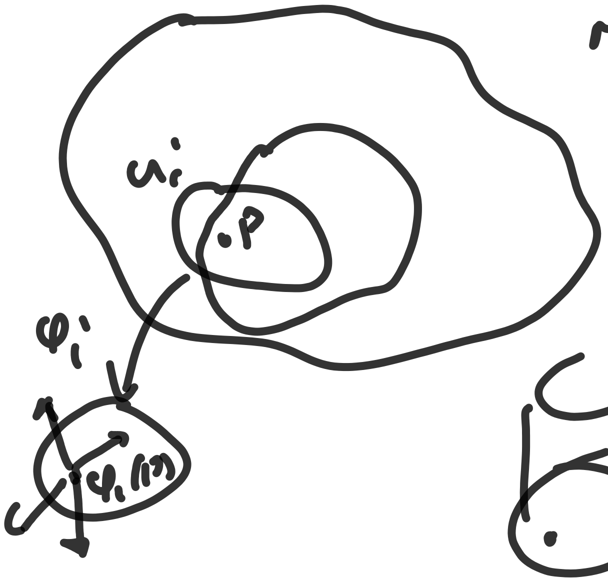
$p = (0,0) \qquad J_p = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$

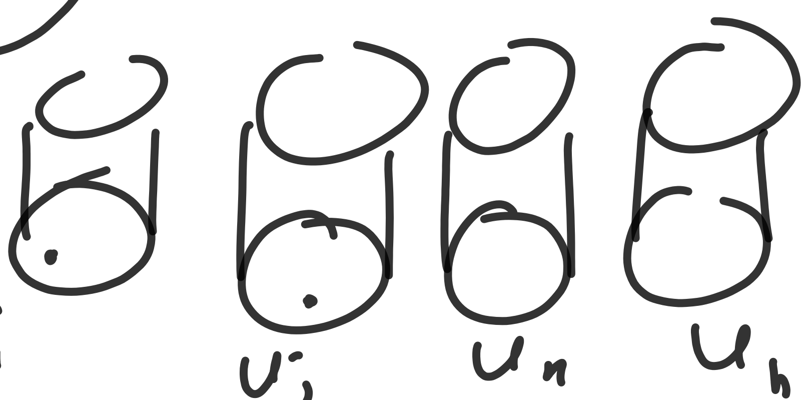
$dF(v) \quad v = (2,3) \quad dF(v)$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$$

$M = (X, A) \qquad A = \{(U_i, \varphi_i)\}$

$n = \dim M$





$\bigsqcup U_i \times \mathbb{R}^n = T$

$\downarrow$

$(p,v)_{U_i \times \mathbb{R}^n}$

$(p,v)_{U_i \times \mathbb{R}^n} \sim (q,w)_{U_j \times \mathbb{R}^n}$

iff $p = q \quad w = d(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(p)} v$

- $d(id) = id$
- $d(g \circ f)_p = (dg)_{f(p)} \circ (df)_p$
- $(df)_p^{-1} = (df^{-1})_{f(p)}$

<u>refl.</u> $(P,v)_{U_i \times \mathbb{R}^n} \sim (P,v)_{U_i \times \mathbb{R}^n}$

indeed $P = P$ $\quad d(\varphi_i \circ \varphi_i^{-1})_{\varphi_i(P)} = d(id) = id$

$$v = d(\varphi_i \circ \varphi_i^{-1})_{\varphi_i(P)} v$$

<u>simm</u> $(P,v)_{U_i \times \mathbb{R}^n} \sim (Q,w)_{U_j \times \mathbb{R}^n} \Rightarrow P = Q$

$$w = d(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(P)} v$$

$\overset{?}{\Rightarrow} (Q,w)_{U_j \times \mathbb{R}^n} \sim (P,v)_{U_i \times \mathbb{R}^n}$

$Q = P$ $\quad \varphi_i \circ \varphi_j^{-1} = (\varphi_j \circ \varphi_i^{-1})^{-1}$ $\quad \underline{v} = \left(d(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(P)}\right)^{-1} w =$

$(\varphi_j \circ \varphi_i^{-1})(\varphi_i(P)) = \varphi_j(P)$ $\quad = d\left((\varphi_j \circ \varphi_i^{-1})^{-1}\right)_{\varphi_j(P)} w =$

$$d(\varphi_i \circ \varphi_j^{-1})_{\varphi_j(P)} \underline{w}$$

<u>trans.</u> $(P,v)_{U_i \times \mathbb{R}^n} \sim (Q,w)_{U_j \times \mathbb{R}^n} \sim (R,u)_{U_k \times \mathbb{R}^n}$

$(P,v)_{U_i \times \mathbb{R}^n} \sim (R,u)_{U_k \times \mathbb{R}^n}$?

$P = Q = R \Rightarrow P = R$

$u = d(\varphi_k \circ \varphi_j^{-1})_{\varphi_j(P)} w \qquad w = d(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(P)} v$

$u = d(\varphi_k \circ \varphi_j^{-1})_{\varphi_j(P)} \, d(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(P)} v =$

$= d(\varphi_k \circ \cancel{\varphi_j^{-1}} \circ \cancel{\varphi_j} \circ \varphi_i^{-1})_{\varphi_i(P)} v$

Def The TANGENT BUNDLE of $M=(X,A)$ is the quotient $\mathbb{T}/\sim \; = T_M$

Elements of T_M are tg vectors

$\tau \in T_M \qquad \tau = \{ (P,v)_{U_i \times \mathbb{R}^n}, (Q,w)_{U_j \times \mathbb{R}^n}, \dots \}$

$P = Q = \dots$ is the application point of τ

v = vector part of τ in $(U_i; \varphi_i)$ $\quad P \in U_i$

w = " " in (U_j, φ_j) $\quad P \in U_j$

$\dots \qquad\qquad \dots$

$T_{M,P}$ = tg vectors applied in $P \sim \mathbb{R}^n$

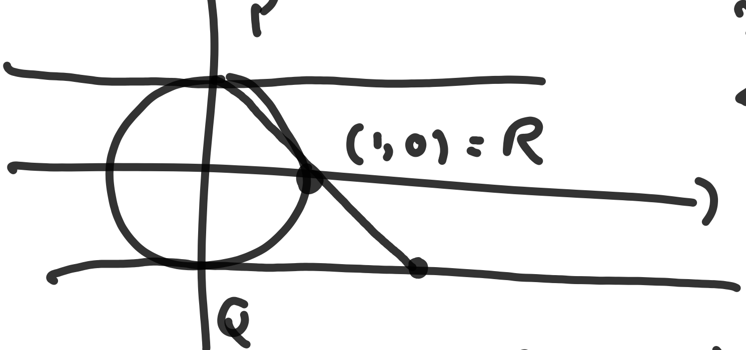
vector space of dim. n.

Ex $T_{\mathbb{R}^n} = \mathbb{R}^n \times \mathbb{R}^n$

Ex T_{S^1}

$U_1 = S^1 - \{P\}$

$U_2 = S^1 - \{Q\}$



τ applied in R

τ with vector part v in (U_1, φ_1)

vector part w in (U_2, φ_2)

$v, w \in \mathbb{R} \qquad w = d(\varphi_2 \circ \varphi_1^{-1})_{\varphi_1(R)} v$

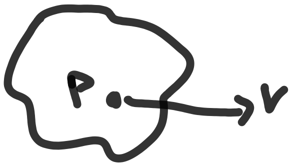
$$\varphi_2 \circ \varphi_1^{-1}(t) = \frac{4}{t} \qquad d\left(\varphi_2 \circ \varphi_1^{-1}\right) = -\frac{4}{t^2}$$

$$\varphi_1^{o}(R) = 2 \qquad w = d\left(\varphi_2 \circ \varphi_1^{-1}\right)_{\varphi_1(R)} v = -\frac{4}{2^2} v = -v$$

$M = (X, \mathcal{A})$ $\qquad T_M$ tg bundle $\qquad n = \dim M$

$\circ \quad \tau \in T_M \qquad P =$ applic. pt. $\in M$

$v =$ vector part $\mathbb{R}^n$ in (U_i, φ_i)

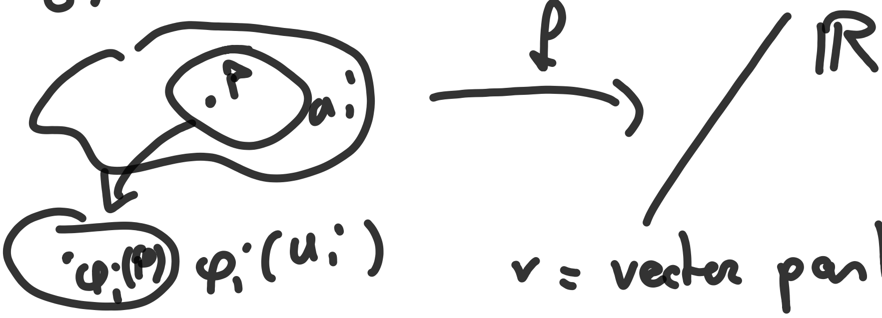


$w =$ " $\qquad (U_j, \varphi_j)$

$$w = d(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(P)} v$$

$M \xrightarrow{f} \mathbb{R} \qquad \tau \in T_M \qquad \tau \in T_{M,P} \qquad P$ appl. pt.

<u>Def</u> $\dfrac{\partial f}{\partial \tau}$. Pick a chart $(U_i, \varphi_i) \ni P$



$v =$ vector part of τ in (U_i, φ_i)

$$\partial f / \partial \tau = \frac{\partial (f \circ \varphi_i^{-1})_{\varphi_i(P)}}{\partial v} \qquad f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{R}$$

$$\overset{\shortparallel}{\text{grad}} (f \circ \varphi_i^{-1})_{\varphi_i(P)} \cdot v \qquad \varphi_i(U_i) \subseteq \mathbb{R}^n$$

Pick $(U_j, \varphi_j) \ni P \qquad w =$ vector part of τ in (U_j, φ_j)

$$\frac{\partial (f \circ \varphi_j^{-1})_{\varphi_j(P)}}{\partial w} = \text{grad} (f \circ \varphi_j^{-1})_{\varphi_j(P)} w$$

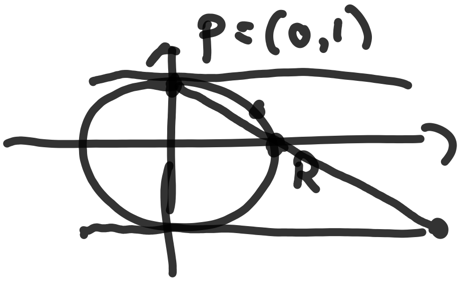
but $w = d(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(P)} v$

$$\text{grad} (f \circ \varphi_j^{-1})_{\varphi_j(P)} \, d(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(P)} v$$

$= d(f\circ\varphi_j^{-1})_{\varphi_j(p)}\, d(\varphi_j\circ\varphi_i^{-1})_{\varphi_i(p)}\, v$

$= d(f\circ\cancel{\varphi_j^{-1}}\circ\cancel{\varphi_j}\circ\varphi_i^{-1})_{\varphi_i(p)}\, v = \mathrm{grad}(f\circ\varphi_i^{-1})_{\varphi_i(p)}\, v$

Ex



$f: S^1 \to \mathbb{R}$

$f(x,y) = y$

$\varphi_1(R) = 2$

τ = tg vector applied in $R=(1,0)$, whose vector part in $S^1-\{P\}$ is 3

$(f\circ\varphi_1^{-1})(t) = \left(f\left(\frac{4t}{t^2+4}, \frac{t^2-4}{t^2+4}\right)\right) = \frac{t^2-4}{t^2+4}$

$\mathrm{grad}(f\circ\varphi_1^{-1}) = \frac{d}{dt}\left(\frac{t^2-4}{t^2+4}\right) = \frac{2t(t^2+4)-2t(t^2-4)}{(t^2+4)^2}$

$= \frac{16t}{(t^2+4)^2}$

$\mathrm{grad}(f\circ\varphi_1^{-1})_{\varphi_1(R)} = \frac{32}{64} = \frac{1}{2}$

$\frac{\partial f}{\partial \tau} = \frac{1}{2}\cdot 3 = 3/2$

NB $\mathrm{grad}(f\circ\varphi_2^{-1})_{\varphi_2(R)} = -\frac{1}{2}$

vector part of τ in $\underline{(S^1-\{Q\}, \varphi_2)}$ is -3

Rem $T_M \xrightarrow{\pi} M$ $\quad \pi(\tau)$ = applic. point

π = natural projection.

π surjective

Rem $\pi^{-1}(P) = T_{M,P}$ = tg vectors applied in P

- $T_{M,P}$ = tg <u>space</u> of M at P

$T_{M,P}$ is a vector space of dim. $n = \dim M$

$\tau_1, \tau_2 \in T_{M,P}$ Fix $(U_i, \varphi_i) \ni P$

τ_1 has vector part v_1 in (U_i, φ_i)
τ_2 " " v_2 " (U_i, φ_i)

$\tau_1 + \tau_2$ = tg vector applied in P whose vector part in (U_i, φ_i) is $v_1 + v_2$

if $P \in (U_j, \varphi_j)$
τ_1 has vector part w_1 in (U_j, φ_j) $w_1 = d(\varphi_j \circ \varphi_i^{-1}) v_1$
τ_2 " " w_2 " $w_2 = d(\varphi_j \circ \varphi_i^{-1}) v_2$

$$w_1 + w_2 = d(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(P)} v_1 + d(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(P)} v_2 +$$

$$= d(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(P)} (v_1 + v_2)$$

Similarly $\forall \alpha \in \mathbb{R}$ $\alpha \tau_1$ = tg vector applied in P whose vector part in (U_i, φ_i) is αv_1

$\bar{0} \in T_{M,P}$ is the tg vector applied in P whose vector part is 0.

A basis (canonical bases w.r.t. (U_i, φ_i)) for $T_{M,P}$ is the set $(\tau_1, \dots, \tau_n)$ of tg vectors s.t. the vector part of τ_k in (U_i, φ_i) is e_k
$(0 - \underset{k\text{-th place}}{1} \quad 0)$

$d(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(P)}$ $\Rightarrow$ dim of $T_{M,P}$ is n

---

Rem T_M has a natural structure of diff. manifold of dim $= 2 \dim M$

$M = (X, \underset{\{(U_i, \varphi_i)\}}{\underset{\shortparallel}{A}})$

$\forall (U_i, \varphi_i)$ consider $\pi^{-1}(U_i) = \{$tg vectors whose appl. pt. is $\in U_i\}$ is open in T_M

$T_M \xrightarrow{\pi} M$ on T_M take the pull back of the top. on X

$\forall \tau \in \pi^{-1}(U_i)$ define

$$\Phi_i(\tau) = (\underset{\mathbb{R}^n}{\underset{\cap}{\varphi_i(\pi(\tau))}}, \underset{\mathbb{R}^n}{\underset{\cap}{v_i}}) \in \mathbb{R}^{2n}$$

v_i = vector part of τ in (U_i, φ_i)

On T_M there is the atlas $\{(\pi^{-1}(U_i), \Phi_i)\}$

$$\Phi_j \circ \Phi_i^{-1}(x_1 \dots x_n, v_1 \dots v_n) = \Phi_j(\tau)$$

where $\varphi_i(\pi(\tau)) = (x_1 \dots x_n)$ and $(v_1 \dots v_n)$ is the vector part of τ in (U_i, φ_i)

$\pi(\tau) = \varphi_i^{-1}(x_1 \dots x_n)$

$$\Phi_j(\tau) = (\underline{y_1 \dots y_n}, \underline{w_1, \dots, w_n}) = \Phi_j \circ \Phi_i^{-1}(\underline{x_1 \dots x_n}, v_1 \dots v_n)$$

$$(y_1 \dots y_n) = \varphi_j(\pi(\tau)) = \underbrace{\varphi_j \circ \varphi_i^{-1}}(x_1 \dots x_n) \quad C^\infty$$

$(w_1, \dots, w_n)$ = vector part of τ in (U_j, φ_j)

$$= d(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(\pi(\tau))}(v_1 \dots v_n) =$$

$$= d(\varphi_j \circ \varphi_i^{-1})_{(x_1 \dots x_n)}(v_1 \dots v_n) \qquad C^\infty$$

As a consequence $\{(\pi^{-1}(U_i), \phi_i)\}$ is a diff. structure on TM.

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