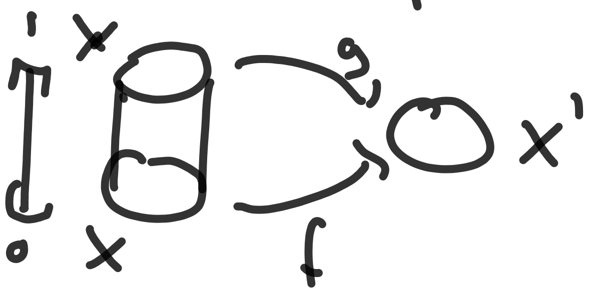
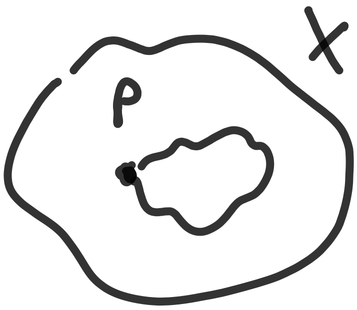
homotopic maps $f,g : X \to X'$

$H : X \times [0,1] \to X'$

$H(x,0) = f(x) \qquad H(x,1) = g(x)$




$f \sim g$




loops around $p$

$\alpha : [0,1] \to X$

$\alpha(0) = \alpha(1) = p$

$\{\text{loops around } p\}$

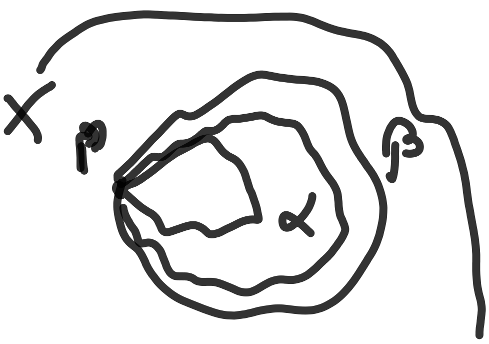
$$\alpha\beta(t) = \begin{cases} \alpha(2t) & t \le \frac{1}{2} \\ \beta(2t-1) & t \ge \frac{1}{2} \end{cases}$$

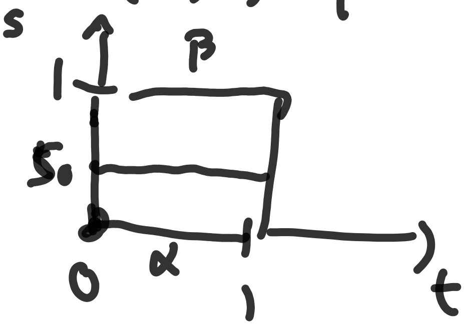
$\{\text{loops around } p\}/\sim$ homotopy

$\alpha, \beta : [0,1] \to X$

$\alpha \sim \beta \qquad H : [0,1] \times [0,1] \to X$

$H(t,0) = \alpha(t), \quad H(t,1) = \beta(t)$




$\alpha \sim \beta$

$H(0,s) = p$

$H(1,s) = p$

$\{\text{loops around } p\}/\sim$

∘ $\overline{\alpha}\,\overline{\beta} = \overline{\alpha\beta}$ $\qquad \overline{\alpha} = \overline{\alpha'} \quad \overline{\beta} = \overline{\beta'}$

$$\Downarrow$$

$$\overline{\alpha'\beta'} = \overline{\alpha\beta}$$

i.e. $\alpha \sim \alpha'$ and $\beta \sim \beta'$ then $\alpha\beta \sim \alpha'\beta'$

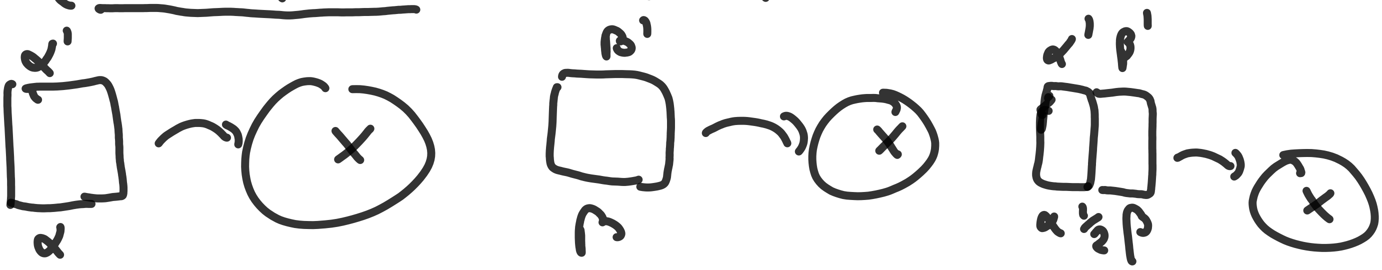
$\Downarrow$

$H_1 : [0,1] \times [0,1] \to X$

$\underline{H_1(t,0) = \alpha(t)}$ $\quad H_1(t,1) = \alpha'(t) \quad H_1(0,s) = H_1(1,s) = p$

$H_2 : [0,1] \times [0,1] \to X$

$H_2\underline{(t,0) = \beta(t)}$ $\quad H_2(t,1) = \beta'(t) \quad H_2(0,s) = H_2(1,s) = p$




$$H(t,s) = \begin{cases} t \le 1/2 & H_1(2t, s) \\ t \ge 1/2 & H_2(2t-1, s) \end{cases}$$

↳ glueing lemma $\Downarrow$ $H$ cont.

$t = 1/2 \qquad H(1/2, s) = H_1(1,s) = p$

$\qquad\qquad \quad\; = H_2(0,s) =$

$$H(t,0) = \begin{cases} t \le 1/2 & H_1(2t,0) = \alpha(2t) \\ t \ge 1/2 & H_2(2t-1,0) = \beta(2t-1) \end{cases} = \alpha\beta(t)$$

$H(t,1) = \alpha'\beta'(t)$

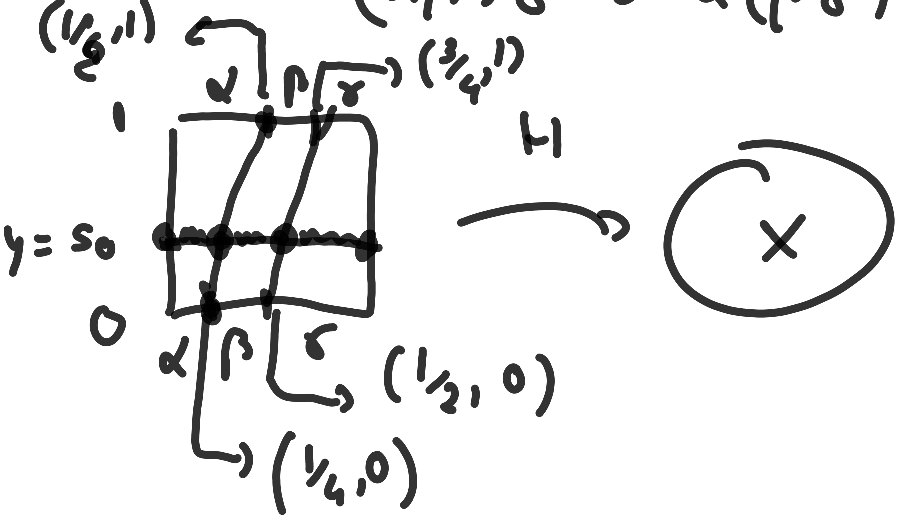
$H(0,s) = H_1(0,s) = p \qquad H(1,s) = H_2(1,s) = p$

$\{\text{loops around } p\}/\sim$ have a operation

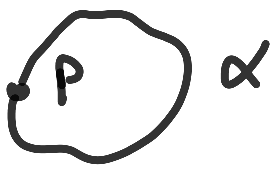
$\circ \quad \bar{\alpha}\,\bar{\beta}\,\bar{\gamma} \Rightarrow (\bar{\alpha}\,\bar{\beta})\,\bar{\gamma} \overset{?}{=} \bar{\alpha}\,(\bar{\beta}\,\bar{\gamma})$

$$\underset{\overline{(\alpha\beta)\gamma}}{\shortparallel} \qquad \underset{\overline{\alpha(\beta\gamma)}}{\shortparallel}$$

need $(\alpha\beta)\gamma \sim \alpha(\beta\gamma)$



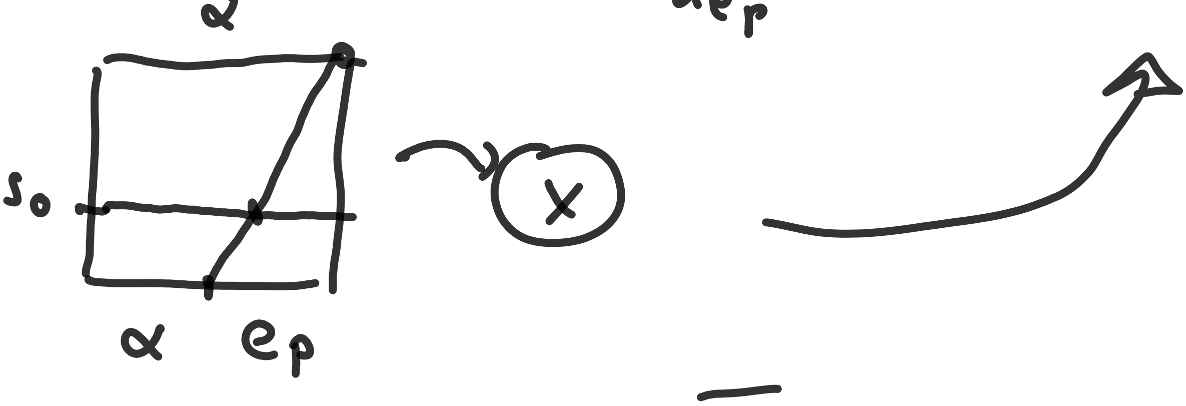

$e_p$ = constant loop



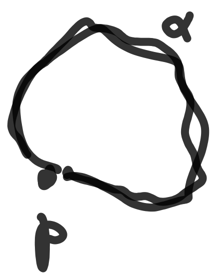

$\bar{\alpha}\,\bar{e}_p = \bar{\alpha} = \overline{e_p\alpha}$

$\shortparallel$

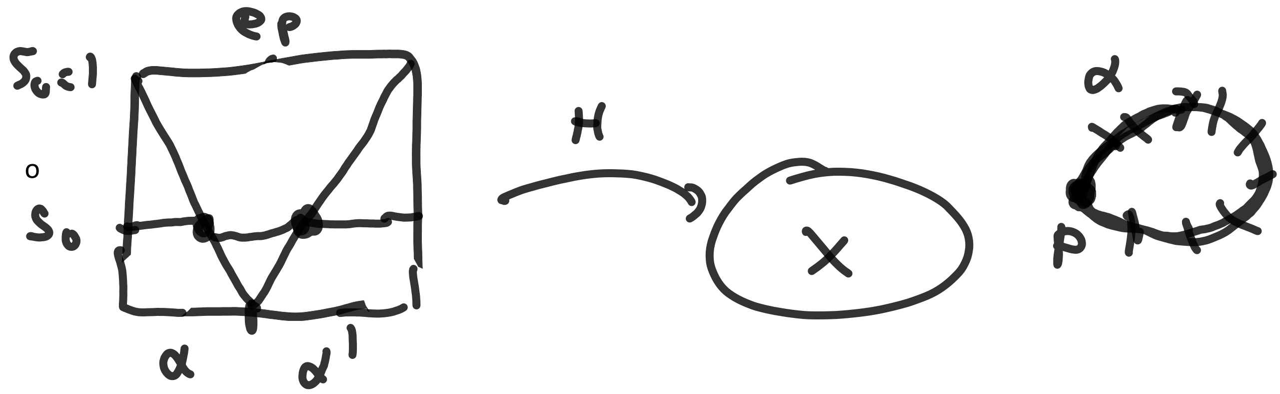
$\overline{\alpha e_p}$ need $\alpha e_p \sim \alpha$




$\bar{\alpha} \qquad \alpha^{-1} \qquad \alpha'(t) = \alpha(1-t)$




$\bar{\alpha}\,\overline{\alpha'} = \bar{e}_p$ i.e. $\alpha\alpha' \sim e_p$

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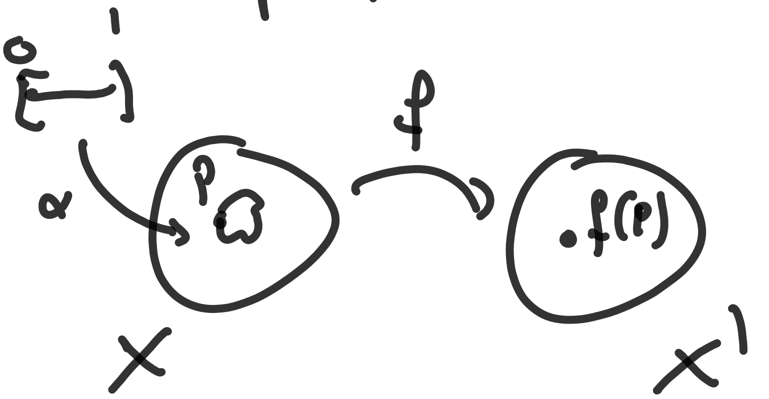
**Def** $(X, \mathcal{T})$ $\quad p \in X$

The quotient $\{\text{loops around } p\}/\sim$ is called the FUNDAMENTAL GROUP of $X$ with base point $p$ $\quad \pi(X, p)$

---


good maps between top. spaces $\xrightarrow{?}$ good maps between groups




$f$ continuous

$$\pi(X,p) \xrightarrow{\pi(f)} \pi(X', f(p))$$

$$\pi(X,p) = \{\text{loops around } p\}/\sim$$

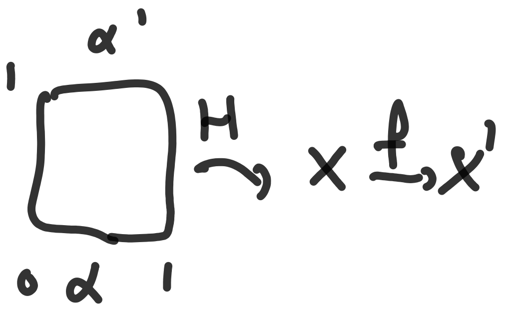
$$\pi(X', f(p)) = \{\text{loops around } f(p)\}/\sim$$

$\alpha : [0,1] \to X \qquad \alpha(0) = \alpha(1) = p$

$f\circ\alpha : [0,1] \to X' \qquad f\circ\alpha(0) = f(\alpha(0)) = p$

$\qquad\qquad\qquad\qquad\qquad f\circ\alpha(1) = f(\alpha(1)) =$

$\alpha \sim \alpha' \Rightarrow f\circ\alpha \sim f\circ\alpha'$ ?





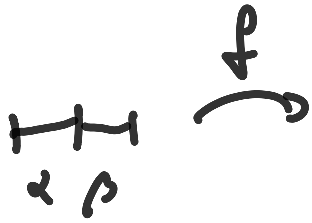
Composition with $f$ $\quad \pi(f) : \pi(X,p) \longrightarrow \pi(X', f(p))$

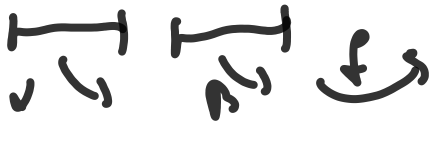
$\pi(f)(\overline{\alpha\beta}) \overset{?}{=} \pi(f)(\bar\alpha)\,\pi(f)(\bar\beta)$

$\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\pi(f)(\overline{\alpha\beta}) \qquad\qquad\qquad \overline{f\circ\alpha}\ \ \overline{f\circ\beta}$

$\qquad = \overline{f\circ(\alpha\beta)} \quad =$




$\pi(f)(\bar\alpha^{-1}) = (\pi(f)(\bar\alpha))^{-1}$ for the same reason

$f(\alpha(1-t)) = (f\circ\alpha)(1-t)$

**Prop**

$\pi(f)$ is a homomorphism of groups

$$\text{top. spaces} \xrightarrow{\pi} \text{groups}$$

$\circ$ continuous map $\longrightarrow$ homomorphism

## FUNCTOR

1) $\begin{array}{ccc} X & \xrightarrow{id} & X \\ \Downarrow & & \Downarrow \\ p & & p \end{array}$ continuous

$\pi(id, p) : \pi(X, p) \longrightarrow \pi(x, p)$

$$\pi(id, p) \overset{?}{=} id_{\pi(x,p)}$$

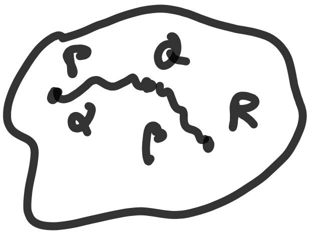
2) $\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \Downarrow & & \Downarrow & & \Downarrow \\ p & & f(p) & & g(f(p)) \end{array}$

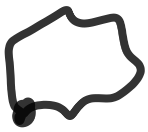
$$\pi(X, p) \xrightarrow{\pi(f)} \pi(Y, f(p)) \xrightarrow{\pi(g)} \pi(Z, g(f(p)))$$

$$\pi(g) \circ \pi(f) \overset{?}{=} \pi(g \circ f)$$

$$\alpha, \beta : [0,1] \to X \qquad \alpha(1) = \beta(0)$$

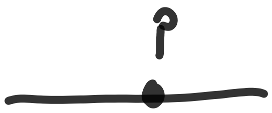
$$\alpha\beta(t) = \begin{cases} \alpha(2t) & t \le 1/2 \\ \beta(2t-1) & t \ge 1/2 \end{cases} \quad : [0,1] \to X$$



$P \in X$

$$\boxed{\begin{array}{c} \{\text{loops around } P\} \\ \downarrow \\ \alpha . \beta \qquad \alpha\beta \end{array}}$$




$\mathbb{R}$

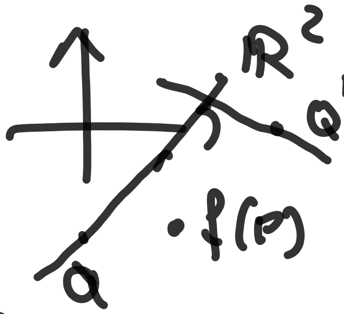
homeom.

$f : \mathbb{R} \to \mathbb{R}^2$

$\Downarrow$

$f|_{\mathbb{R}-\{p\}} : \mathbb{R}-\{p\} \to \mathbb{R}^2 - \{f(p)\}$ homeom.

$P \in \mathbb{R}$




conn. top.

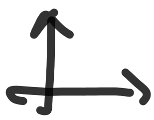
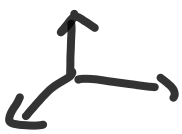
$f^{-1}\ \mathbb{R}^2 \to \mathbb{R}$

$f^{-1}|_{\mathbb{R}^2 - \{f(p)\}} : \mathbb{R}^2 - \{f(p)\} \to \mathbb{R}-\{p\}$

contradiction $\mathbb{R}-\{p\}$ not connected

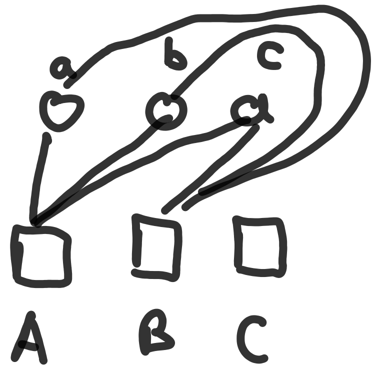
$\mathbb{R}^2 - \{f(p)\}$ conn.

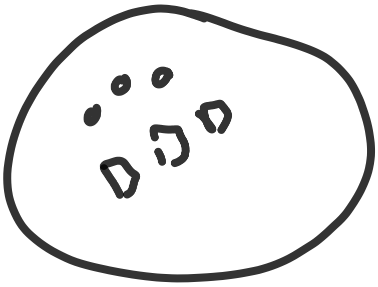
Thus $\mathbb{R}$ and $\mathbb{R}^2$ are <u>NOT</u> homeom.

$\mathbb{R}^2$  $\mathbb{R}^3$ 

need

INVARIANTS



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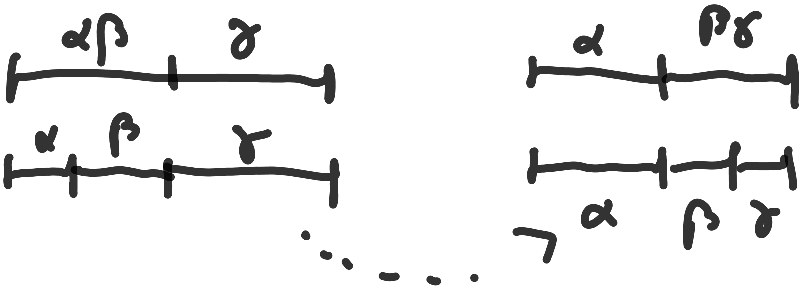
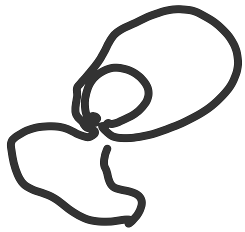
$$(\alpha\beta)\gamma \neq \alpha(\beta\gamma)$$



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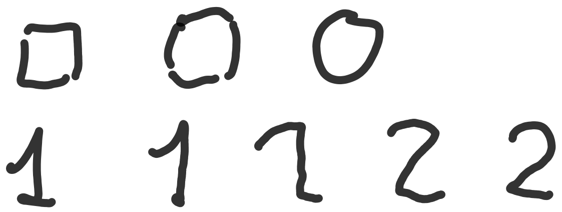
Deformation

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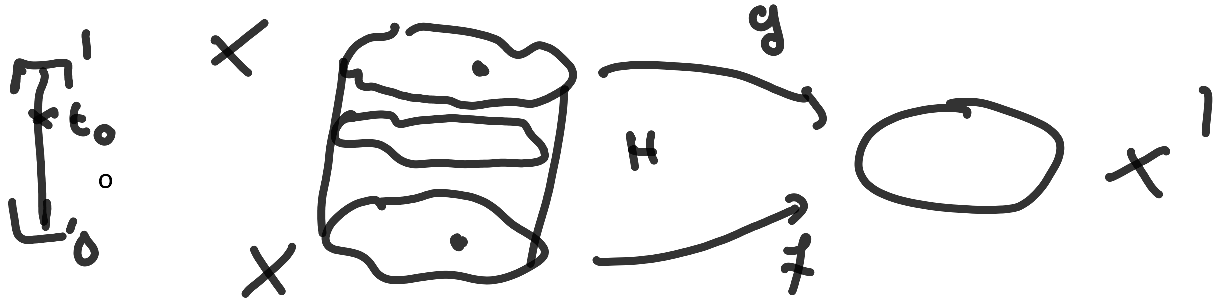
$$(X,T) \underset{g}{\overset{f}{\rightrightarrows}} (X',T') \quad f,g \text{ continuous}$$

<u>Def</u> A HOMOTOPY between $f,g$ is a continuous map

$$H : X \times [0,1] \longrightarrow X' \quad \text{s.t.}$$

$$\forall p \in X \quad H(p,0) = f(p) \quad H(p,1) = g(p)$$




<u>Ex</u> $X = \mathbb{R} = X'$ $\quad f = \sin \quad g = \cos$

$\mathbb{R} \times [0,1] \xrightarrow{H} \mathbb{R} \qquad H(x,t) = (1-t)\sin(x) + t\cos x$

$t = 0 \quad \forall x \in \mathbb{R} \qquad H(x,0) = \sin(x)$

$t = 1 \quad \forall x \in \mathbb{R} \qquad H(x,1) = \cos(x)$

If $\exists$ a homotopy between $f$ and $g$ we say that $f$ and $g$ are <u>homotopic</u>

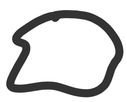
<u>Ex</u> $\mathbb{R}^2 \to \mathbb{R}^2 \quad f = id \qquad g = \text{constant}(0,0)$

$H(P,t) = (1-t)P \qquad t = 0 \quad H(P,t) = P \quad \forall P$

$\qquad\qquad\qquad\qquad\quad t = 1 \quad H(P,t) = 0$

$id$ $\quad$ constant $(0,0)$
are homotopic.

X
conn.



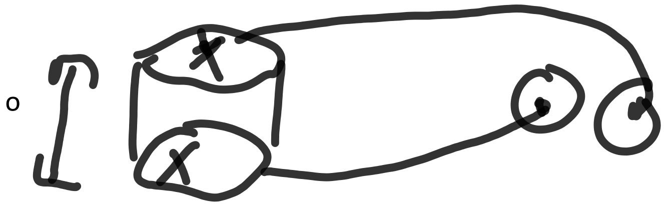
X'


$f(x) = P \quad \forall x \in X \qquad g(x) = Q \quad \forall x \in X$

Then there are no homotopies between $f$ and $g$

Indeed

Assume a homotopy $H$ exists.




$H(x,0) = p$
$H(x,1) = q$ $\forall x \in X$

$H^{-1}(A_1)$ open in $X \times [0,1]$

$H^{-1}(A_2)$ " " "

- since $A_1 \cup A_2 = X'$ $\Rightarrow$ $H^{-1}(A_1) \cup H^{-1}(A_2) =$
  $H^{-1}(A_1 \cup A_2) = X \times [0,1]$

- since $A_1 \cap A_2 = \emptyset$ $\Rightarrow$ $H^{-1}(A_1) \cap H^{-1}(A_2) =$
  $H^{-1}(A_1 \cap A_2) = \emptyset$

- since $p \in A_1 \Rightarrow H^{-1}(A_1) \ni (x,0)$ so $H^{-1}(A_1) \neq \emptyset$
  " $q \in A_2 \Rightarrow H^{-1}(A_2) \ni (x,1)$ so $H^{-1}(A_2) \neq \emptyset$

$\Rightarrow$ $H^{-1}(A_1), H^{-1}(A_2)$ is a disconnection of
$X \times [0,1]$

A contradiction because $X$ conn, $[0,1]$ conn $\Rightarrow$
$X \times [0,1]$ conn.

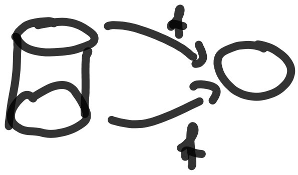
---

$(X, T)$ $(X', T')$ $\{$ cont. maps $X \to X'$ $\}$

$f \sim g$ iff. $f$ is homotopic to $g$

<u>Prop</u> $\sim$ is an equivalence relation

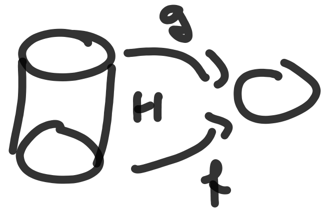
<u>Pf</u> <u>refl.</u> $f \sim f$




$A' \in T'$

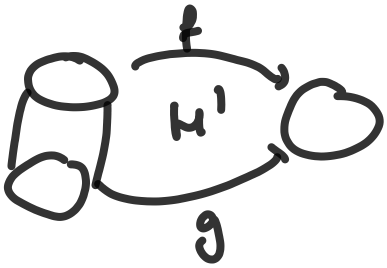
$H(x,t) = f(x)$ $H^{-1}(A') = f^{-1}(A') \times [0,1]$
open in $X \times [0,1]$

**Symm**




$f \sim g$

$g \sim f$




$H'(x,t) = H(x, 1-t)$

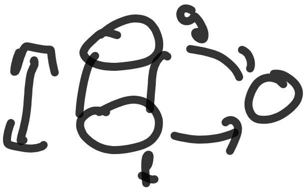
$H'(x,0) = H(x,1) = g(x)$

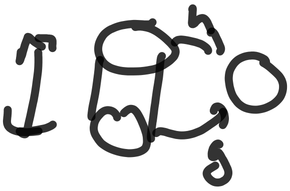
$H'(x,1) = H(x,0) = f(x)$

**trans** $f \sim g$ and $g \sim h$ $\quad f,g,h : X \to X'$

$H : X \times [0,1] \to X'$ $\quad H(x,0) = f(x)$ $\quad H(x,1) = g(x)$

$H' : X \times [0,1] \to X'$ $\quad H'(x,0) = g(x)$ $\quad H'(x,1) = h(x)$




$$H'' : X \times [0,1] \to X' \qquad H''(x,t) \begin{cases} t \leq 1/2 & H(x,2t) \\ t \geq 1/2 & H'(x,2t-1) \end{cases}$$

$g(x) = H(x,1) \leftarrow H''(x, 1/2) \rightarrow H'(x,0) = g(x)$

$H''(x,0) = H(x,0) = f(x)$

$H''(x,1) = H'(x,1) = h(x)$

$$X \times [0,1] = X \times [0, 1/2] \cup X \times [1/2, 1]$$

$\underset{\text{cont.}}{H'' = H(x,2t)}$ $\qquad$ $\underset{\text{cont.}}{H'' = H'(x,2t-1)}$

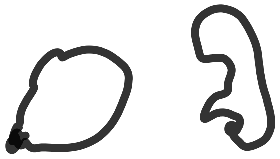
By the gluing lemma $H''$ is cont.

$\Rightarrow f \sim h$

One can create the quotient

$$\{ f : X \to X' \text{ cont.} \} / \sim$$

loop $[0,1] \to X$ $\qquad \{\text{loops}\}/\sim$



$$(\alpha\beta)\gamma \neq \alpha(\beta\gamma)$$

$$(\alpha\beta)\gamma \sim \alpha(\beta\gamma) \ ?$$

$(X, T)$ $F \subseteq X$ pathwise connected

o

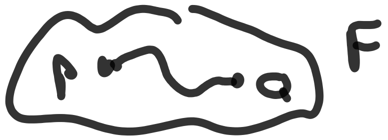
if $\forall P, Q \in F$ $\exists$ path with extremal pts $P, Q$

Rem $F$ pathwise conn. $P$ is an accum. pt. of $F$ $\not\Rightarrow$ $F \cup \{P\}$ is path. conn.


Def $F \subseteq X$ $\forall P, Q$ put $P \sim Q$
if $\exists$ path $\alpha : \underbrace{[0,1] \to F}_{\text{continuous}}$ $\alpha(0) = P$ $\alpha(1) = Q$

reflex. $\alpha : [0,1] \to F$ $\alpha(0) = \alpha(1) = P$
$\alpha(t) = P$ $\forall t \in [0,1]$ $\Rightarrow$ $P \sim P$

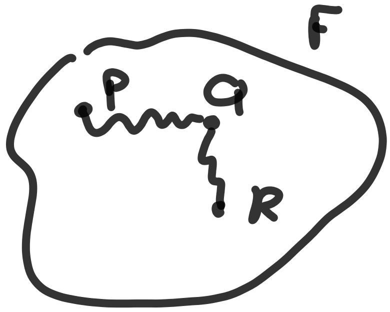
symm $P \sim Q$ i.e. $\exists \alpha : [0,1] \to F$ $\alpha(0) = P$
$\alpha(1) = Q$




$\bar{\alpha} : [0,1] \to F$ $\bar{\alpha}(t) = \alpha(1-t)$

$\bar{\alpha}(0) = \alpha(1) = Q$ $\bar{\alpha}(1) = \alpha(0) = P$
$\bar{\alpha}$ is continuous since compos. of cont. functions
$\Rightarrow Q \sim P$

trans. $P \sim Q$ and $Q \sim R \overset{?}{\Rightarrow} P \sim R$




$\exists$ $\alpha_1 : [0,1] \to F$ $\alpha_1(0) = P$
$\alpha_1(1) = Q$

$\alpha_2 : [0,1] \to F$ $\alpha_2(0) = Q$
$\alpha_2(1) = R$

both continuous

need $\alpha : [0,1] \to F$ contin. $\alpha(0) = P$ $\alpha(1) = R$

$$\alpha_1 \alpha_2 = \alpha(t) = \begin{cases} t \le 1/2 & \alpha(t) = \alpha_1(2t) \\ t \ge 1/2 & \alpha(t) = \alpha_2(2t-1) \end{cases}$$

$\alpha(0) = \alpha_1(0) = P$ $\alpha(1) = \alpha_2(1) = R$

$$\alpha(1/2) \begin{cases} \to \alpha_1(1) = Q \\ \to \alpha_2(0) = Q \end{cases}$$

By the glueing lemma $\alpha$ is continuous

$[0,1] = [0,1/2] \cup [1/2,1]$ $[0,1/2] \cap [1/2,1] = \{1/2\}$

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GLUEING LEMMA

Rem - $(X,T)$ $Y \subseteq X$ $Y$ open

induced topol. $T_Y = \{A \cap Y \cdot A \in T\} = \{A \in T : A \subseteq Y\}$

- $(X,T)$ $Y \subseteq X$ $Y$ closed

$T_Y$ closed subsets in the induced topol.

$Y - (A \cap Y) = Y \cap A^c =$

intersections of $Y$ with closed subsets of $X$

$= \{Z \subseteq Y : Z \text{ closed in } X\}$

- $Y \subseteq X$ is open iff $\forall P \in Y \ \exists A_P \in T$ s.t. $P \in A_P \subseteq Y$ (then $Y = \bigcup A_P$)

- $Y \subseteq X$ is closed if $\exists F_1, \dots, F_n$ each $F_i$ closed and $\forall P \in Y \ \exists j$ with $P \in F_j \subseteq Y$ (then $Y = F_1 \cup \dots \cup F_n$)

## Glueing Lemma, open case

- $(X,T)$ $(X',T')$ top. spaces

$\{A_i\}$ open covering of $X$

$\forall i$ $f_i : A_i \to X'$ continuous

$\forall i,j$ $\forall p \in A_i \cap A_j$ $f_i(p) = f_j(p)$

Then $\exists f : X \to X'$ continuous s.t. $f|_{A_i} = f_i$ $\forall i$

pf Existence is obvious

$\forall p \in X$ find $A_i$ s.t. $p \in A_i$

define $f(p) = f_i(p)$

Continuity: $\forall A' \in T'$

$$f^{-1}(A') = \bigcup f^{-1}(A') \cap A_i = \bigcup f_i^{-1}(A')$$

$\Rightarrow f^{-1}(A')$ is open

## Glueing Lemma. Closed case

$(X,T)$ $(X',T')$ top. spaces

$F_1, \dots, F_n \in X$ closed s.t. $X = F_1 \cup \dots \cup F_n$

$\forall i$ $f_i : F_i \to X'$ continuous, $\forall i j$ $\forall p \in F_i \cap F_j$ $f_i(p) = f_j(p)$

$\exists f : X \to X'$ continuous s.t. $\forall i$ $f|_{F_i} = f_i$

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$(X,T)$ Every equivalence class of $\sim$ is a "pathwise connected component" of $X$

Ex


$X = (\text{lines } x = 1/n) \cup (x \text{ axis}) \cup (0,1)$

pathwise conn. comp.

$\{(0,1)\}$ $\quad\quad$ $(\text{lines } x = 1/n) \cup (x \text{ axis})$

not closed

- Pathwise conn. compon. are pathwise conn. They are maximal pathwise conn. subsets of $X$.

$X = \bigcup X_i$ $\quad$ $X_i$ pathwise conn comp.

**Prop.** $X \xrightarrow{f} X'$ continuous $\quad F \subseteq X$ pathwise conn.

$\Rightarrow f(F)$ is pathwise conn.

**pf** take $P, Q \in f(F)$ $\quad \exists P_0 \in F\ Q_0 \in F$ s.t.

$f(P_0) = P \quad f(Q_0) = Q$.

There $\exists$ $\alpha : [0,1] \to X$ $\quad \alpha(0) = P_0 \quad \alpha(1) = Q_0$

contin.

$f \circ \alpha : [0,1] \longrightarrow X'$ continuous

$f \circ \alpha(0) = f(P_0) = P \qquad f \circ \alpha(1) = f(Q_0) = Q$

**Coroll** $X, X'$ homeom. If $X$ path. conn. $\Rightarrow X'$ path. conn.

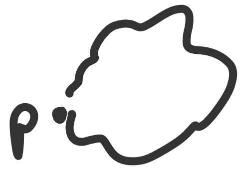
If $X'$ path. conn. $\Rightarrow X$ path. conn.

In any topol. space $X$

- $\alpha_1$ path from $P$ to $Q$
- $\alpha_2$ " " $Q$ to $R$

$$\alpha_1\alpha_2(t) \begin{cases} t \le 1/2 & \alpha_1(2t) \\ t \ge 1/2 & \alpha_2(2t-1) \end{cases}$$ from $P$ to $R$

**Def** $P \in X$. A LOOP around $P$ is any path $\alpha : [0,1] \to X$ s.t. $\alpha(0) = \alpha(1) = P$

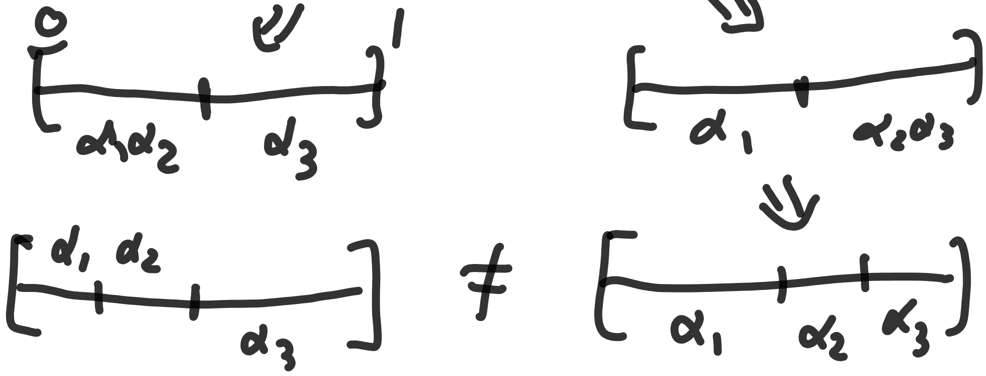



$L_P = \{ \text{loops around } P \}$

$\alpha_1 \quad \alpha_2 \qquad \alpha_1(1) = P = \alpha_2(0)$

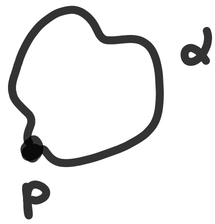
$\alpha_1\alpha_2 \in L_P$

**Associative** $\quad (\alpha_1\alpha_2)\alpha_3 \overset{?}{=} \alpha_1(\alpha_2\alpha_3)$

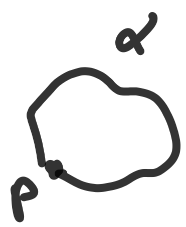



**Neutral elem** $\quad e_P$ = constant path in $P$

$e_P\alpha \overset{?}{=} \alpha$


**Inverse** $\quad \alpha \qquad \bar{\alpha}(t) = \alpha(1-t)$




$\alpha\bar{\alpha} \neq e_P$