<u>Chiantini</u> Advanced Geometry

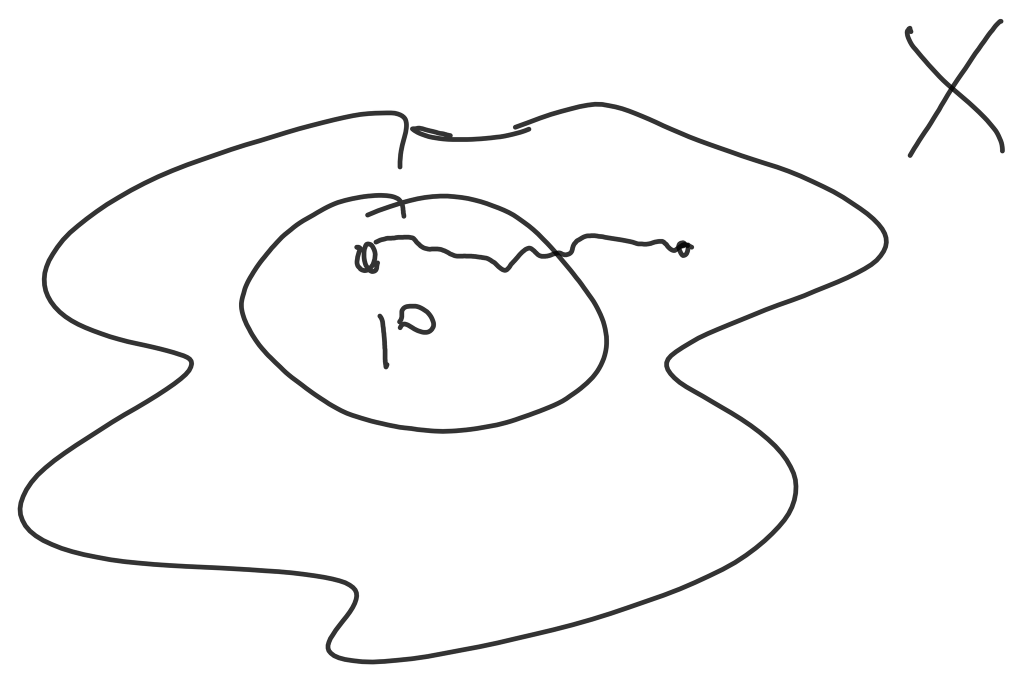
- Topology

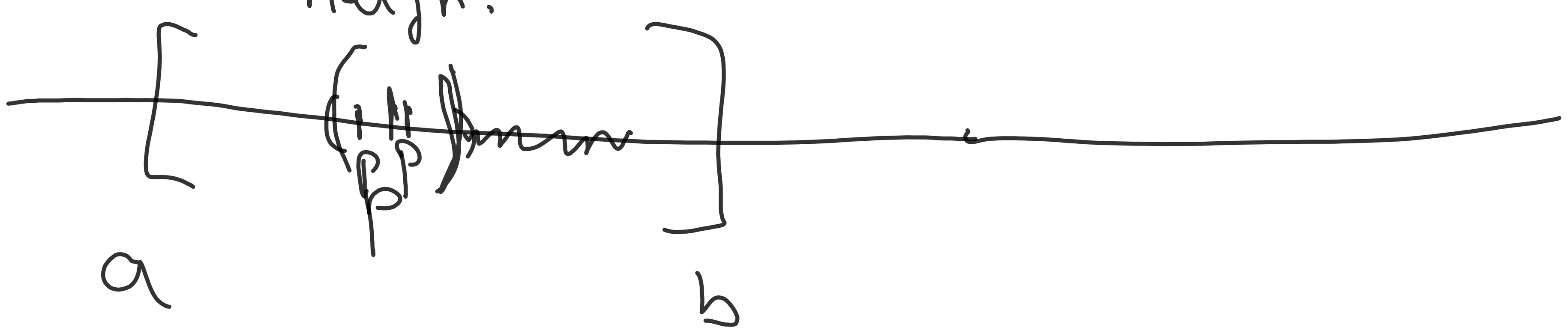
Elementary Geom. and Topology
Singer - Thorpe

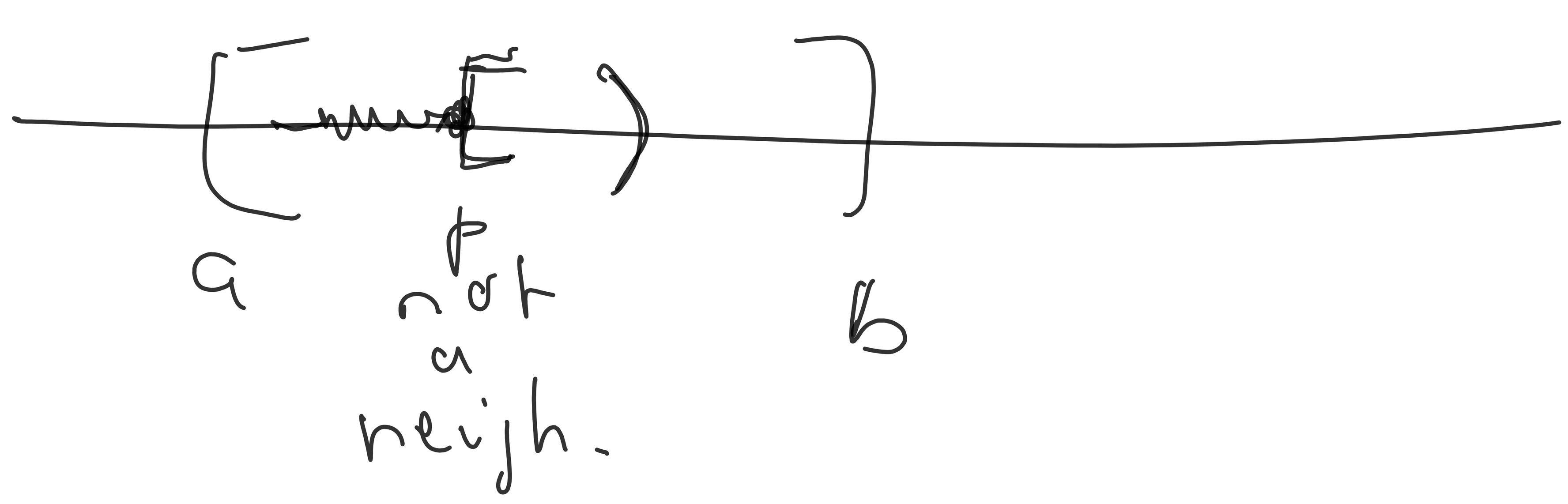
- Algebraic topology
  - pathwise connectedness
  - deformation
  - fundamental group
- Differential geometry
  - differential manifolds
  - tangent bundle
  - vector fields

→ Introduction to ~~alg~~ differential geometry

- Topological space
  neighbourhood







<u>open set A</u> A is a neighbourhood of all points $p \in A$

Continuous functions

$f: (X, T) \longrightarrow (X', T')$ $\quad f: X \to X'$

$\forall A' \in T' \quad f^{-1}(A') \in T$

Ex $(X, T) = (\mathbb{R}, \text{canonical}) = (X', T')$

continuous is as in Analysis

Ex $(\mathbb{R}, \text{canonical}) \longrightarrow (\mathbb{R}, \text{cofinite})$

$f(x) = x^2$

$A' \in$ cofinite $\qquad A' = \mathbb{R} \setminus \{p_1, \dots, p_n\}$

$f^{-1}(A') = \mathbb{R} \setminus \{x : x^2 = p_1 \text{ or } \cdots \text{ or } x^2 = p_n\}$

open in the canonical

$\Downarrow$

f continuous

$f: (\mathbb{R}, \text{cofinite}) \longrightarrow (\mathbb{R}, \text{canonical})$

$f^{-1}(0,1) = \{x : x^2 \in (0,1)\} = (-1, 1) \notin$ cofinite

$\Downarrow$

f not continuous

$id: (X, T) \longrightarrow (X, T')$

continuous $\Rightarrow \forall A' \in T' \quad id^{-1}(A') \in T$?

$id^{-1}(A') = A'$

true $\Leftrightarrow T' \subseteq T$

$\Rightarrow$ $\mathrm{id}\ (X,T) \to (X,\underline{T})$ is continuous

- **Ex** $f : (X, \text{discrete}) \to (Y, T')$

$\forall A' \in T'$ $\quad f^{-1}(A') \in \text{discrete}$

$\Rightarrow f$ continuous.

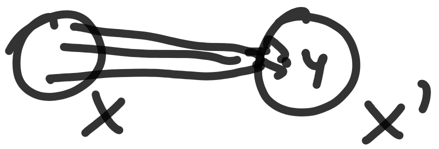
$$f : (X,T) \longrightarrow (Y, \underset{\{\emptyset, Y\}}{\text{trivial}})$$

$f^{-1}(\emptyset) = \emptyset \in T$ $\quad f^{-1}(Y) = X \in T$

$\Rightarrow f$ is continuous.

Ex $f : (X,T) \to (X',T')$ constant

$f(x) = y \quad \forall x \in X$



$$\forall A' \in T' \quad f^{-1}(A') = \begin{cases} y \in A' & f^{-1}(A') = X \in T \\ y \notin A' & f^{-1}(A') = \emptyset \in T \end{cases}$$

$\Rightarrow f$ is continuous.

Prop
$$\begin{tikzcd} (X,T) \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & (X',T') \arrow[r, "g"] & (X'',T'') \end{tikzcd}$$

f, g continuous $\Rightarrow g \circ f$ continuous.

CATEGORY objects maps

1) $\forall$ objects X
$X \xrightarrow{\mathrm{id}} X$ is a good map

2) $X \xrightarrow{f} Y \xrightarrow{g} Z$ $\quad f,g$ good
$\Rightarrow g \circ f$ good

sets | maps

groups | homom.

lin. spaces | lin. maps.

top. spaces | continuous maps

Def ISOMORPHISM

$f : X \to Y$ good
s.t. $\exists \; g : Y \to X$ good
with $g \circ f = id_X$ $\quad f \circ g = id_Y$

$X \sim Y$ iff $\exists \; f : X \to Y$ isom.

In the case of top. spaces
isomorphism $\Rightarrow$ homeomorphism

$id : (\mathbb{R}, \text{canon.}) \to (\mathbb{R}, \text{cofinite})$
is NOT a homeom.

1) $f:(X,T) \longrightarrow X'$

2) $f: X \longrightarrow (X',T')$

1) Find the biggest topology on X' s.t. f contin.

2) " smallest " X " "

② $T = \{ f^{-1}(A') : A' \in T' \}$ is a topology on X

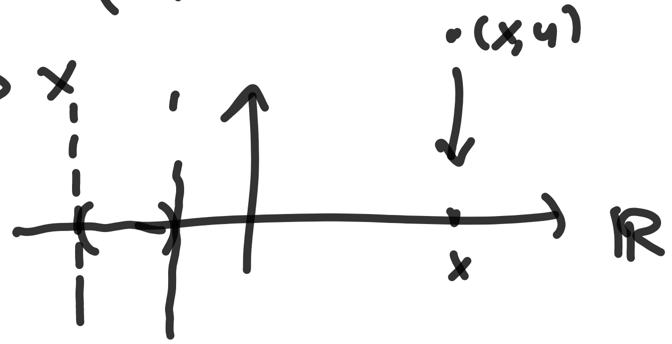
because $f^{-1}(A' \cup B') = f^{-1}(A') \cup f^{-1}(B')$

$f^{-1}(A' \cap B') = f^{-1}(A') \cap f^{-1}(B')$

$T = f^{*}(T')$

<u>Ex</u> $X = \mathbb{R}^2 \xrightarrow{f} X' = (\mathbb{R}, \text{canonical})$

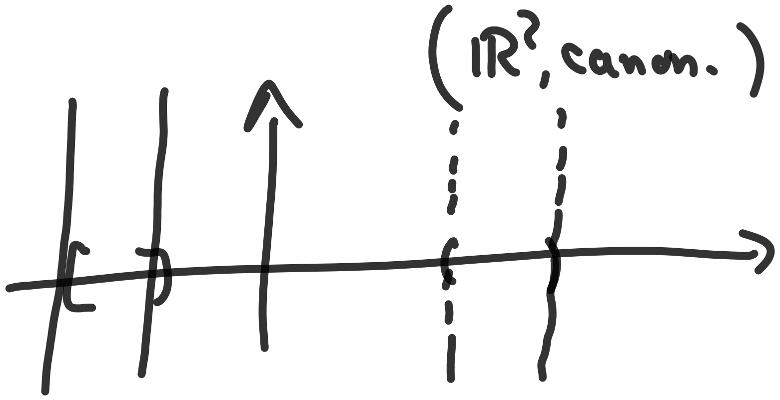
$(x,y) \mapsto x$



$f : (X,T) \longrightarrow X'$

$T' := \{ A' \subseteq X' : f^{-1}(A') \in T \}$ is a topology on X'

$\| \; f_* T$



$f_*(\text{canon.})$
$\| \; \text{canon.}$

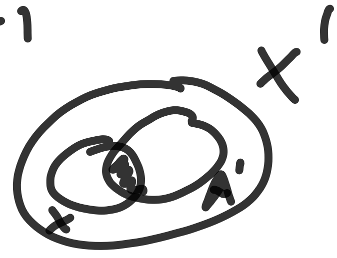
$\mathbb{R}$

(X', T') top. space

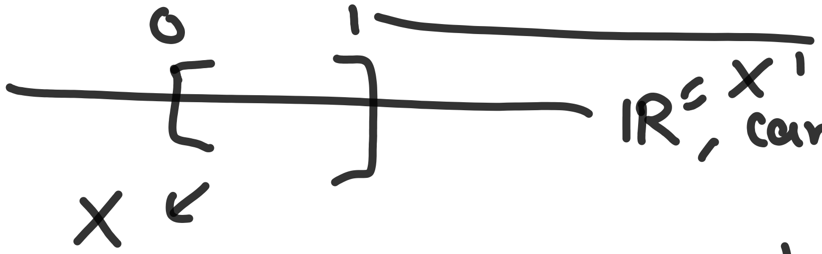
$X \subseteq X'$ $\quad f: X \to X'$ inclusion

$\forall A' \in T'$ $\quad \underset{\overset{\shortparallel}{A' \cap X}}{f^{-1}(A')} \in T = f^{*}T'$

induced topology



Ex



$\mathbb{R} \subseteq X'$, canon. $= T'$

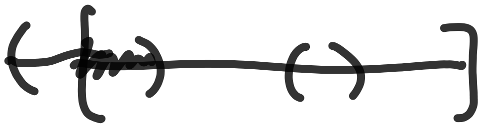
X

induced top. on X $\qquad [0,1] \cap A' \quad A' \in T'$

$(0,1) \cap [0,1] = (0,1)$

$(0,2) \cap [0,1] = (0,1]$

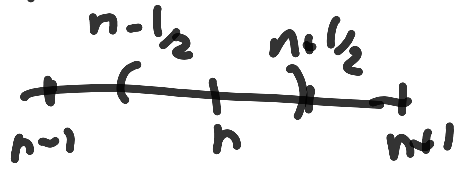
$(\frac{1}{2}, \frac{3}{2}) \cap [0,1] = (\frac{1}{2}, 1]$



Ex $(\underset{\overset{\shortparallel}{X'}}{\mathbb{R}}, \underset{\overset{\shortparallel}{T'}}{\text{canon.}})$ $\qquad X = \mathbb{N}$ $\qquad$ induced ?

$(0,8) \cup (10,11) \cap \mathbb{N} = \{1,2,3,\dots,7\}$

$\forall n \in \mathbb{N}$ $\quad (n-\frac{1}{2}, n+\frac{1}{2}) \cap \mathbb{N} = \{n\}$



every singleton of $\mathbb{N}$ is open

$\Rightarrow$ every subset of $\mathbb{N}$ is open (union of singletons)

$\Rightarrow$ induced top. = discrete top. on $\mathbb{N}$

$X \qquad S \subseteq \mathcal{P}(X)$

∘ $L(S)$ = smallest topol. s.t. all elem. of S are open.

$\Downarrow$ top. generated by S

$$= \left\{ \bigcup_i (A_{i,1} \cap \dots \cap A_{i,n}) \; ; \; A_{i_j} \in S \right\}$$

can. top. on $\mathbb{R}$ = L(open intervals)

S = singletons $\quad L(S)$ = discrete

$$\begin{array}{l} X \xrightarrow{f_1} (X_1', T_1') \\ \; f_2 \searrow \\ \qquad (X_2'', T_2'') \end{array}$$

$L(f_1^{-1}(T_1') \cup f^{-1}(T_2'))$ is the smallest top. s.t. f_1, f_2 are both continuous.

A open set $=$ neighbourhood of all $p \in A$

o

<u>Properties</u>

1) the union of any family of open sets is open

2) the intersection of two open sets is open

3) $\emptyset$, total space are open

<u>Def</u> X set. A TOPOLOGY on X is a collection of subsets $T \subseteq \mathcal{P}(X)$ which satisfies 1) 2) 3)

1) $\forall \{A_i\}$ $A_i \in T$ $\forall i \Rightarrow \bigcup A_i \in T$
2) $A_1, A_2 \in T \Rightarrow A_1 \cap A_2 \in T$
3) $\emptyset, X \in T$

(X, T) is a topological space.

<u>Ex</u> $X = \mathbb{R}$ $\quad T = \{ A : A$ is a union of open intervals $\}$

T is a topology

1) obvious
2) the intersection of two open intervals is an open interval

3) $\emptyset$ is an open interval $\quad \mathbb{R} = (-\infty, +\infty)$

T is called NATURAL (CANONICAL) TOPOLOGY

Ex $\mathbb{R} = X$ $\quad T = \{A : A^c \text{ is finite}\} \cup \{\emptyset\}$

cofinite topology

1) $\{A_i\}$ $\quad A_i \in T$ $\quad (\cup A_i)^c = \cap A_i^c$ is finite

2) $A_1, A_2 \in T$ $\quad A_1^c, A_2^c$ finite

$(A_1 \cap A_2)^c = A_1^c \cup A_2^c$ finite

3) $\emptyset \in T$ $\quad \mathbb{R}^c = \emptyset$ finite $\Rightarrow \mathbb{R}, \emptyset \in T$

Ex $T = \mathcal{P}(X)$ is a topology (DISCRETE TOPOLOGY)

Ex $T = \{X, \emptyset\}$ $\quad$ TRIVIAL TOPOLOGY

Topological spaces are primitive geometric objects

Def X $\quad d$ distance on X

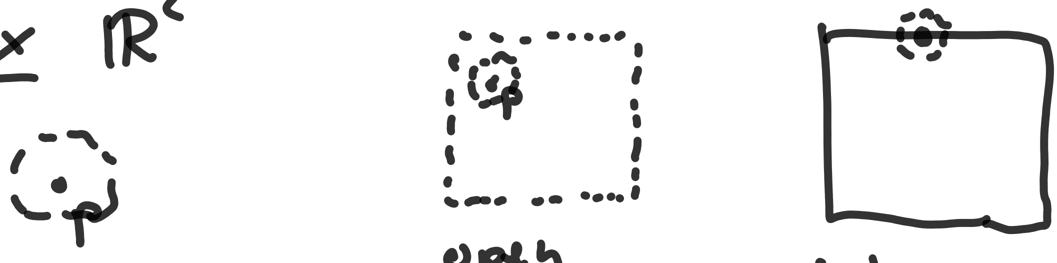
open sphere with center P and radius $r > 0$

is $S(P, r) = \{Q : d(P, Q) < r\}$

$T = \{A ; \forall P \in A \ \exists r > 0 \text{ s.t. } P \in S(P, r) \subseteq A\}$

is a topology

Ex $\mathbb{R}^2$

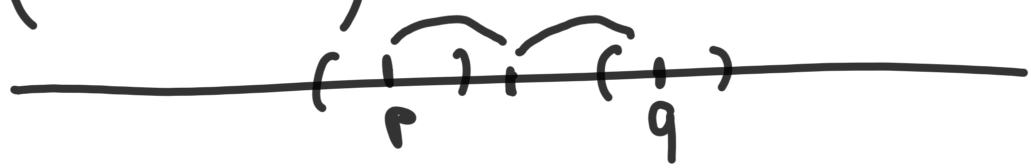


open $\qquad$ not open

Separation

Def (X,τ) is separated (or HAUSDORFF space) if $\forall P,Q$ $P\neq Q$ $\exists A_P, A_Q$ open sets s.t. $P\in A_P$ $Q\in A_Q$ and $A_P\cap A_Q=\emptyset$

Ex ($\mathbb{R}$, natural) is separated



$\varepsilon < \frac{p+q}{2}-p$ $\quad (p-\varepsilon, p+\varepsilon)\cap(q-\varepsilon, q+\varepsilon)$

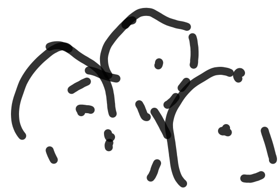
Ex ($\mathbb{R}$, cofinite) not separated

A, A' cofinite $(A\cap A')=\emptyset \Rightarrow (A\cap A')^C=\mathbb{R}$

$(A\cap A')^C = A^C\cup A'^C$ impossible.

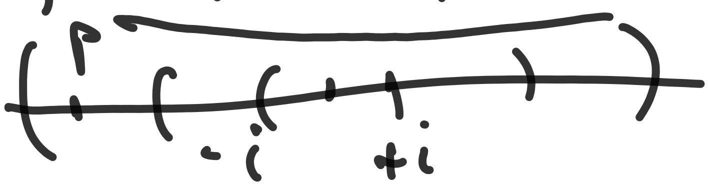
Def Open covering of X = $\{\underset{\in T}{A_i}\}$ s.t. $\bigcup A_i = X$

Ex $\mathbb{R}^2$ $\{S(P,1)\}$ $\forall P\in\mathbb{R}^2$



Def (X,T) is COMPACT if every open covering contains a finite subcovering

Ex ($\mathbb{R}$, natural) not compact



$\{A_i\}$ covering

$A_i=(-i,i)$
$i\in\mathbb{R}_+$

$\{A_{i_1}, \dots, A_{i_n}\}$ $\quad i = \max\{|i_1|, \dots, |i_n|\}$

$i \notin A_{i_1} \cup \dots \cup A_{i_n}$

__Ex__ $(\mathbb{R}, \text{cofinite})$ compact

$\{A_i\}$ covering $\quad A_i$ covers $\mathbb{R}$ except at most for finitely many pts

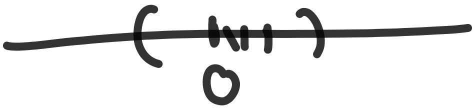
$A_i = \mathbb{R} - \{p_1 \dots p_j\}$ $\quad \forall p_k$ take $A_{i_k} \ni p_k$

$A_i \cup A_{p_1} \cup \dots \cup A_{p_j} = \mathbb{R}$

(X, T) $\qquad \{a_i\}_{i \in \mathbb{N}} = \{a_0, a_1, \dots, a_n, \dots\}$

__Def__ ℓ is a __limit__ for $\{a_i\}$ if $\forall$ open set $A \ni \ell$ $\exists\, n_0$ s.t. $n \geq n_0 \Rightarrow a_n \in A$

$\{\frac{1}{n}\}$ $\quad 0$ is a limit



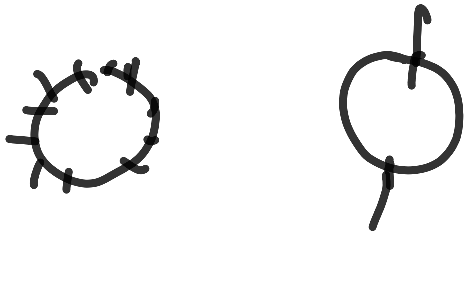
X separated $\Rightarrow$ every sequence has at most one limit

X compact $\Rightarrow$ every sequence has a subsequence with a limit

$\{0, 1, 0, 1, 0, 1 \dots\}$

$0 \quad 0 \quad 0$

$1 \quad 1 \quad 1$



Def $f : (X, T) \longrightarrow (X', T')$ top. spaces

° $f : X \to X'$ f is CONTINUOUS

if $\forall A' \in T'$ $f^{-1}(A') \in T$

(X, T)

Y inherits an induced topol.

$T_Y = \{A \cap Y : A \in T\}$

(X, T) $\sim$ equiv. relation on X

$X' = X/\sim$ $X \xrightarrow{\pi} X'$ standard projection

$\pi_*(T)$ top. on X' quotient topology

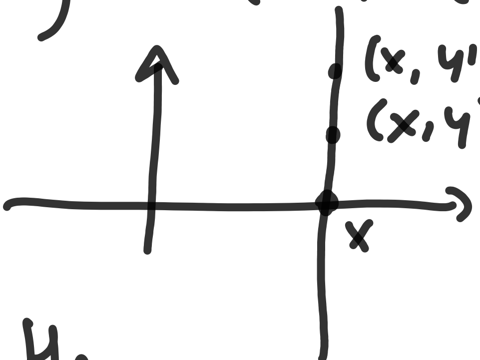
$(\mathbb{R}^2, \text{canon.})$ $(x,y) \sim (x',y') \iff x = x'$

$\mathbb{R}^2/\sim$

$\shortparallel$

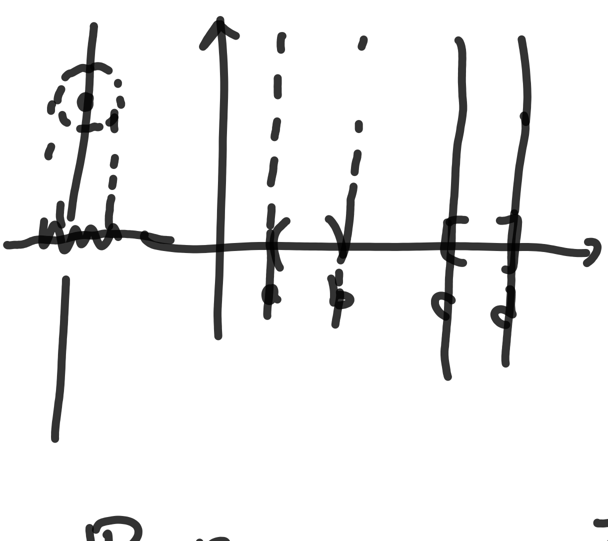
$\mathbb{R}$

$\cup\!\shortmid$



$(x,y') \sim (x,y)$ $\forall y'$

A' is open in the quotient top. $\iff$ $\pi^{-1}(A')$ is open in $\mathbb{R}^2$



$\pi^{-1}(a,b)$ open in $\mathbb{R}^2$

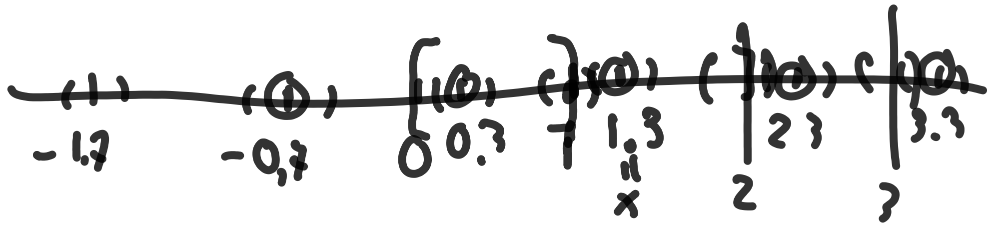
(a,b) " in the quotient

canon. on $\mathbb{R}$ $\subseteq$ quotient top.

indeed canon on $\mathbb{R}$ = quot. top.

$\mathbb{R}$, canon. $\mathbb{Z} \subseteq \mathbb{R}$ $\mathbb{R}/\mathbb{Z}$

$x \sim y \iff x - y \in \mathbb{Z}$

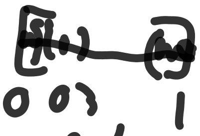


$0 \sim 1$

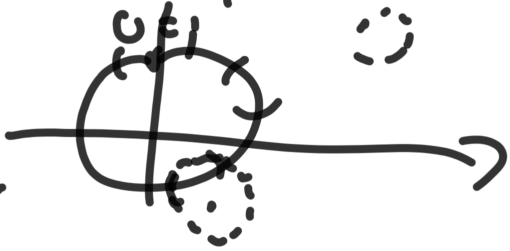
$\mathbb{R}/\mathbb{Z}$

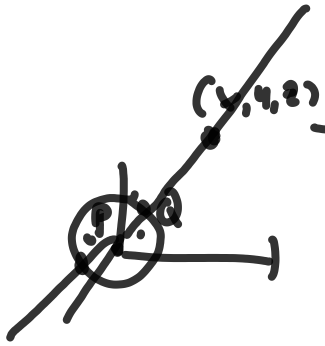
$\shortparallel$

S^1 circle



$0 \sim 1$



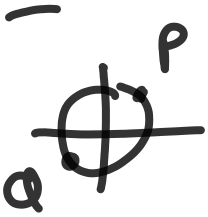


$\{P : d(P,0) = 1\} = S^2$
$\subset \mathbb{R}^3$

$P \sim Q \iff P, Q$ antipodal or $P = Q$

$\{P, Q\}$ equiv. class $\qquad S^2/\sim \; = \mathbb{P}^2_{\mathbb{R}} = \dfrac{\mathbb{R}^3 - \{0\}}{\sim}$

$\mathbb{P}^1 = S^1/\sim$



$(X, T) \qquad (X', T')$

$$\begin{tikzcd} & X \times X' \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X' \end{tikzcd}$$

$p_1(x, y) = x$

$p_2(x, y) = y$

$L\left(\{p_1^{-1}(A)\} \cup \{p_2^{-1}(A')\}\right) \qquad A \in T, \; A' \in T'$

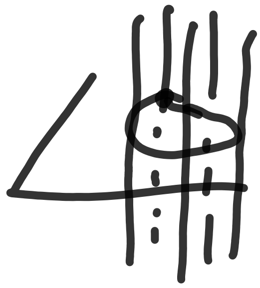
$\shortparallel$

product topology

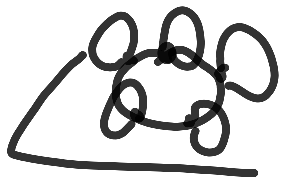
$(X \times X', \text{product top.}) \begin{matrix} \xrightarrow{p_1} X \\ \xrightarrow{p_2} X' \end{matrix}$ both continuous

$S^1 \times \mathbb{R}$

cylinder

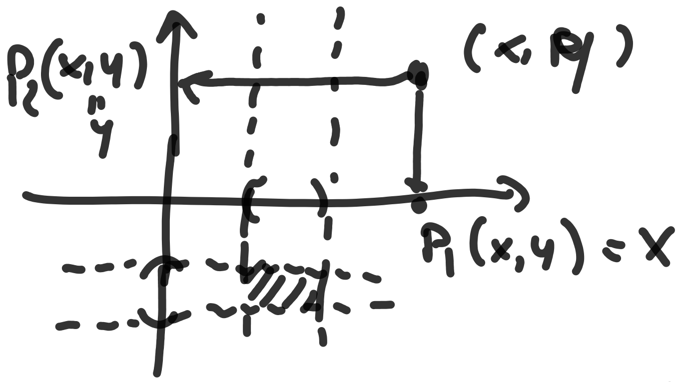


$S^1 \times S^1$

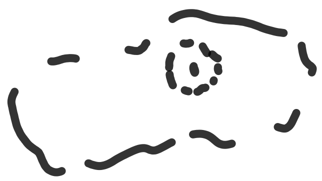


torus

Ex $\mathbb{R}$, canon. $\quad \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$



unions of open rectangles



nat. topol.

Prop (X, T) $\quad Y \subseteq X$

Assume that $\forall p \in Y \ \exists A_p$ open s.t. $p \in A_p \subseteq Y$. Then Y is open in T.

pf $\quad Y = \bigcup_{p \in Y} A_p$ because every $p \in Y$ belongs to A_p so $Y \subseteq \bigcup A_p$

Moreover each $A_p \subseteq Y$ so $\bigcup A_p \subseteq Y$

$\mathbb{R}^2$ $\quad \mathcal{L}\begin{pmatrix}\text{open}\\ \text{rectangles}\end{pmatrix}$ $\quad \mathcal{L}\begin{pmatrix}\text{open}\\ \text{circles}\end{pmatrix}$

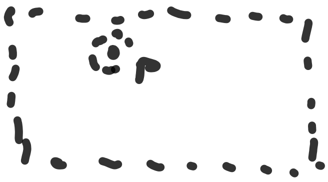
product topol. $\quad$ canon. top.

<u>Rem</u> If $S' \subseteq L(S)$ then $L(S') \subseteq L(S)$

$= Y$ $\quad \forall p \in Y \ \exists$ open rectangle A_p s.t. $p \in A_p \subseteq Y$

$\overset{\text{prop}}{\Longrightarrow}$ Y is open in $L\left(\begin{smallmatrix}\text{open}\\ \text{rect.}\end{smallmatrix}\right)$

$\overset{\text{Rem}}{\Longrightarrow}$ $\underset{\text{can. top.}}{\underset{\parallel}{L\left(\begin{smallmatrix}\text{open}\\ \text{circles}\end{smallmatrix}\right)}} \subseteq \underset{\text{product top.}}{\underset{\parallel}{L\left(\begin{smallmatrix}\text{open}\\ \text{rect.}\end{smallmatrix}\right)}}$

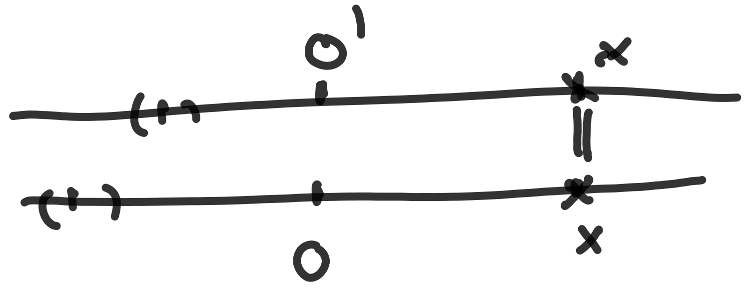


$= Y$ $\quad \forall p \in Y \ \exists$ open circle A_p s.t. $p \in A_p \subseteq Y$

$\overset{\text{prop}}{\Longrightarrow}$ Y is open $L\left(\begin{smallmatrix}\text{open}\\ \text{circle}\end{smallmatrix}\right)$

$\overset{\text{rem.}}{\Longrightarrow}$ $\underset{\text{product top}}{\underset{\parallel}{L\left(\begin{smallmatrix}\text{open}\\ \text{rect.}\end{smallmatrix}\right)}} \subseteq \underset{\text{canon. top.}}{\underset{\parallel}{L\left(\begin{smallmatrix}\text{open}\\ \text{circles}\end{smallmatrix}\right)}}$

Finally $\quad$ product top. $=$ can. top.

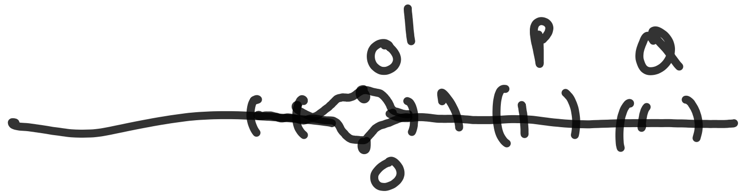


$X =$ two lines

$\underset{\text{line 1}}{x} \sim \underset{\text{line 2}}{x}$ unless $x = 0$.

$\underset{\text{line } i}{x} \sim \underset{\text{line } i}{x}$

$\hookrightarrow$ separated



there are no open sets $A \ni O$ and $A' \ni O'$ s.t. $A \cap A' = \emptyset$

not separated