GEOMETRIA PROIETTIVA

Perchè non si può dividere per 0.

$\cancel{1/0}$

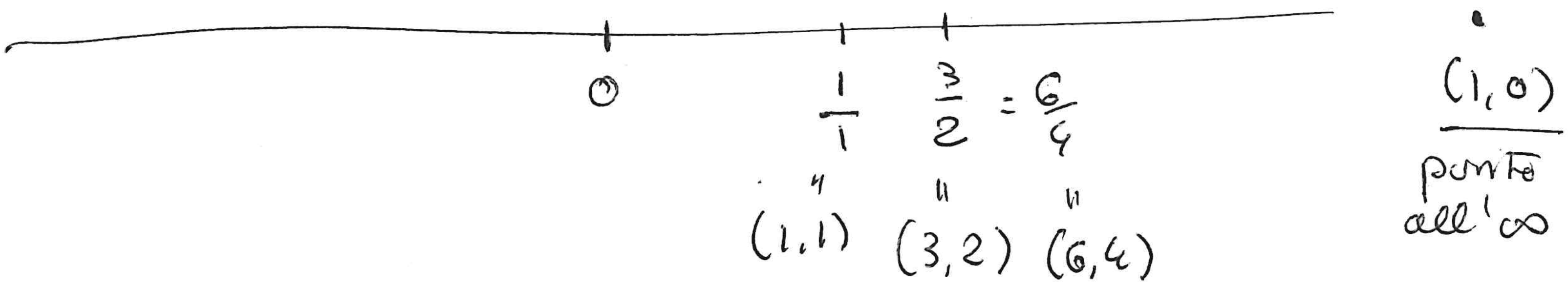
$\mathbb{Q} = \{ \frac{m}{n},\ m, n \in \mathbb{Z}\ n \neq 0 \}$

$\frac{1}{0} = \infty \qquad \infty \cdot 0 = 1$

distributiva

$$1 = \infty \cdot 0 = \infty \cdot (0+0) = \infty \cdot 0 + \infty \cdot 0 = 1 + 1 = 2$$

Möbius



Retta proiettiva $\quad \mathbb{R}^2 - \{(0,0)\}/\sim\ = \mathbb{P}^1$ Retta proiettiva

$(a,b) \sim (c,d) \Leftrightarrow \cancel{\frac{a}{b} = \frac{c}{d}} \quad ad = bc$ cioè $ad - bc = 0$ cioè $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$

$a : b = c : d$

$(1,0) \sim (-1,0) \qquad (1,0) \sim (2,0)$

P2

$(a,b) \sim (c,d) \iff ad - bc = 0$

rif. $(a,b) \sim (a,b)$? $\quad ab - ab = 0$

simm. $(a,b) \sim (c,d) \Rightarrow (c,d) \sim (a,b)$? $\quad ad - bc = 0 \Rightarrow bc - ad = 0$

trans. $(a,b) \sim (c,d) \quad (c,d) \sim (e,f) \Rightarrow (a,b) \sim (e,f)$?

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$$

det $= 0$

det $= 0$

$\det\begin{pmatrix} a & b \\ e & f \end{pmatrix} \neq 0$ è possibile solo se $(c,d) = (0,0)$

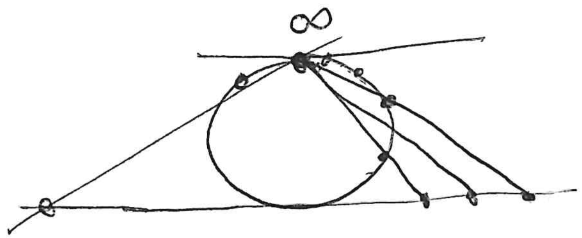
$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 2 & 3 \end{pmatrix} \qquad \det\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \neq 0$$

$(c,d) \neq (0,0) \qquad ad - bc = 0 \qquad cf - de = 0$

$\underline{d \neq 0} \qquad a = \frac{bc}{d} \qquad e = \frac{cf}{d}$

$$\det\begin{pmatrix} \frac{bc}{d} & b \\ \frac{cf}{d} & f \end{pmatrix} = 0$$

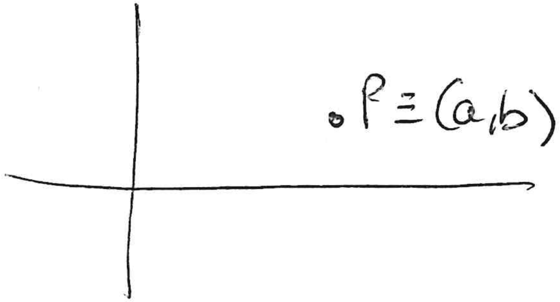
conto simile se $c \neq 0$.



$\frac{0}{0}$ = coso che moltiplicato per 0 dà 0

ma tutti i numeri moltiplicati per 0 danno 0!

E nel piano?



$$(a,b) = \underbrace{\left(\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}\right)}_{\text{stesso denominatore}} \rightsquigarrow (\alpha,\beta,\gamma)$$

$$\mathbb{P}^2 = \mathbb{R}^3 - \{(0,0,0)\} /\sim$$

$$(a,b,c) \sim (a',b',c') \iff \exists\, \lambda \neq 0 \text{ t.c. } (a,b,c) = \lambda (a',b',c')$$

$$\iff \begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix} \text{ ha rango } 1.$$

$$\underset{\text{vecchio}}{(1,2)} = \left(\frac{1}{1}, \frac{2}{1}\right) \longrightarrow (1,2,1)$$

$$\left(\frac{3}{3}, \frac{6}{3}\right) \rightsquigarrow (3,6,3)$$

P4

$(x,y) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \rightsquigarrow (x_0, x_1, x_2) \neq (0,0,0) \to [x_0 : x_1 : x_2]$

vecchio piano (piano affine)

nuovo piano (piano proiettivo $\mathbb{P}^2_{\mathbb{R}}$)

$(3,2,1) \longleftrightarrow \left(\frac{2}{3}, \frac{1}{3}\right)$

$\quad \shortparallel \qquad\qquad\qquad \shortparallel$

$(6,4,2) \longleftrightarrow \left(\frac{4}{6}, \frac{2}{6}\right)$

$(0,1,0) \Rightarrow \left(\frac{1}{0}, \frac{0}{0}\right)$

ind?

$\mathbb{P}^3 \qquad (x,y,z) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) \to (x_0, x_1, x_2, x_3)$

V spazio vettoriale $\qquad V - \{0\} \big/ \sim \; = \mathbb{P}(V)$

$v \sim w \iff v = \lambda w \quad \lambda \in \mathbb{R}$

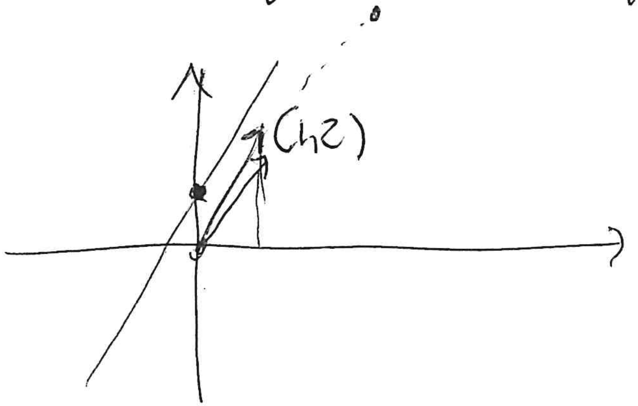
dim proiettiva di $\mathbb{P}(V)$ $=$ $\dim V - 1$

$\mathbb{P}^2$ $\quad ax+by+c=0$ retta

$x = \frac{x_1}{x_0} \qquad y = \frac{x_2}{x_0} \qquad a\frac{x_1}{x_0} + b\frac{x_2}{x_0} + c = 0$

$$\boxed{ax_1 + bx_2 + cx_0 = 0}$$

$2x - y + 1 = 0 \qquad y = 2x+1$



$2x_1 - x_2 + x_0 = 0$

$(1,3) \in$ retta nel piano affine $\qquad (1,3) \to \left(\frac{2}{2},\frac{6}{2}\right) \to \begin{array}{c} x_0\ x_1\ x_2 \\ (2,2,6) \end{array}$

$2(2) - 6 + 2 = 0$

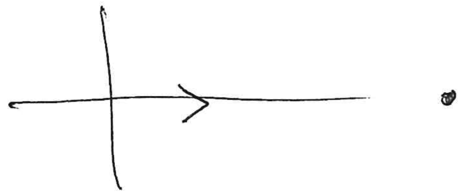
$x_0 = 0 \implies 2x_1 - x_2 = 0 \qquad x_2 = 2x_1 \qquad (0, x_1, 2x_1) = x_1(0,1,2)$

un punto all'∞

$ax_1 + bx_2 + cx_0 = 0 \qquad x_0 = 0 \qquad ax_1 + bx_2 \quad b \neq 0 \qquad x_2 = \overbrace{\boxed{-\frac{a}{b}}}^{m} x_1 \qquad (0, x_1, mx_1)$

$x_1(0,1,m)$

asse x $y=0$ $\frac{x_2}{x_0}=0$ $x_2=0$ $x_0=0$ $(0,x_1,0)$ (P6)



$x_1(\underset{\text{all'}\infty}{0},\underbrace{1,0}_{\text{direz.}})$

$\left(\underset{\text{lungo l'asse x}}{\frac{1}{0}},\underset{\text{ind}}{\frac{0}{0}}\right) \Leftarrow$

$(0,1,0)$ punto all'∞ dell'asse x

$(0,0,1)$ " y

$(1,0,0)$ origine

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r_1 $\quad 2x-y+1=0$ all'∞ ha $(0,1,2)$

r_2 // $2x-y+2=0$ $\quad 2x_1-x_2+2x_0=0$ $\quad x_0=0$ $\quad 2x_1-x_2$ $\quad (0,1,2)$

$r_1 \cap r_2 = \{(0,1,2)\}$

$$\begin{matrix} r_1 \\ r_2 \end{matrix}\begin{cases} ax+by+c=0 \\ a'x+b'y+c'=0 \end{cases} \qquad \begin{cases} ax_1+bx_2+cx_0=0 \\ a'x_1+b'x_2+c'x_0=0 \end{cases} \qquad \begin{array}{ccc} x_0 & x_1 & x_2 \end{array} \begin{pmatrix} c & a & b \\ c' & a' & b' \end{pmatrix}=M$$

rango $M=2 \Rightarrow$ soluz $=$ $\boxed{\text{un}}$ punto proiettivo   rango $M=1$ $\quad r_1=r_2$

P7

rango 2 soluz. $\left(\det\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}, -\det\begin{pmatrix} c & b \\ c' & b' \end{pmatrix}, \det\begin{pmatrix} c & a \\ c' & a' \end{pmatrix} \right)$ e i suoi multipli

viene un pto all'∞ $\Leftrightarrow$ $\det\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} = 0 \Leftrightarrow r_1 // r_2$

Due rette sono // $\Leftrightarrow$ si incontrano all'∞.

$(x_1, y_1) = \left(\frac{x_1}{1}, \frac{y_1}{1}\right)$ $\quad (x_2, y_2) = \left(\frac{x_2}{1}, \frac{y_2}{1}\right)$

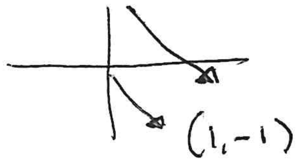
$\det\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{pmatrix} = 0$ rette per i 2 punti

retta $P \equiv \left(1, \frac{1}{2}\right)$ piano affine $\quad Q$ all'∞ $(0, 1, -1)$

$(2, 2, 1) \leftarrow \left(\frac{2}{2}, \frac{1}{2}\right)$

$$\det\begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & -1 \\ x_0 & x_1 & x_2 \end{pmatrix} = 2(x_2 + x_1) + x_0(-3) = 2x_1 + 2x_2 - 3x_0 = 0$$

$2x + 2y - 3 = 0 \ni P$



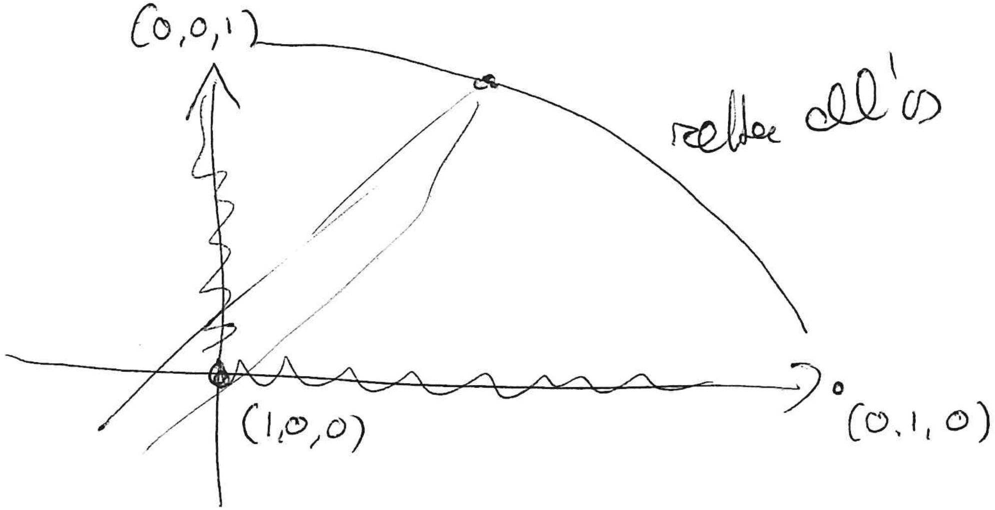
retta per $P \equiv (0,1,1)$ $\quad Q \equiv (0,2,1)$

$$\det \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 1 \\ x_0 & x_1 & x_2 \end{pmatrix} = x_0(-1)$$

$$\boxed{\begin{matrix} -x_0 = 0 \\ x_0 = 0 \end{matrix}}$$ retta

$$0x_1 + 0x_2 - 1x_0 = 0$$

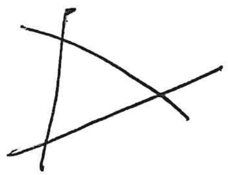
retta formata da tutti e soli i punti all'∞



p9

$$\begin{cases} y_0 = a x_1 + b x_2 + c x_0 \\ y_1 = a' x_1 + b' x_2 + c' x_0 \\ y_2 = a'' x_1 + b'' x_2 + c'' x_0 \end{cases}$$

$$\begin{cases} a x_1 + b x_2 + c x_0 = 0 \\ a' x_1 + b' x_2 + c' x_0 = 0 \\ a'' x_1 + b'' x_2 + c'' x_0 = 0 \end{cases}$$



non ha soluz. proiettive

quindi ha solo la soluz. $(0,0,0)$

$$\det \begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix} \neq 0 .$$

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rette $\quad a x_1 + b x_2 + c x_0 = 0 \quad \forall (a,b,c) \neq (0,0,0)$

P10

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^2$$
(x_0, x_1, x_2) $\quad$ (y_0, y_1, y_2)

$$\mathbb{R}^3 \xrightarrow[\text{lineare}]{f} \mathbb{R}^3$$

$$f(x_0, x_1, x_2) = (a x_1 + b x_2 + c x_0, \; a' x_1 + b' x_2 + c' x_0, \; a'' x_1 + b'' x_2 + c'' x_0)$$

CAMBIAMENTO DI COORDINATE

$$\begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix} \quad \det \neq 0$$

$\mathbb{R}^3 - \{0\}/\sim \quad \mathbb{R}^3 - \{0\}/\sim$

→ funzione $\mathbb{P}^2 \to \mathbb{P}^2$ associata a una applic lineare iniettiva $\mathbb{R}^3 \to \mathbb{R}^3$

$(a,b,c) \sim (a',b',c') \Rightarrow (a',b',c') = \lambda(a,b,c) \Rightarrow f(a',b',c') = \lambda f(a,b,c)$

$f(a',b',c') \sim f(a,b,c)$

In generale dati V, W spazi vettoriali ogni funzione lineare $V \to W$ iniettiva induce una funzione

$$\mathbb{P}(V) \longrightarrow \mathbb{P}(W)$$

(TRASFORMAZIONI PROIETTIVE)

P11

$\mathbb{P}^3 = \mathbb{R}^4 - \{0\}/\sim$ $\qquad (x_0, x_1, x_2, x_3) \qquad x_0 = 0 \Rightarrow$ punti all'∞.

$\mathbb{R}^3 \qquad \mathbb{P}^2 \ni P$

classe di equiv. in $\mathbb{R}^3$ $\longleftrightarrow P$

$=$

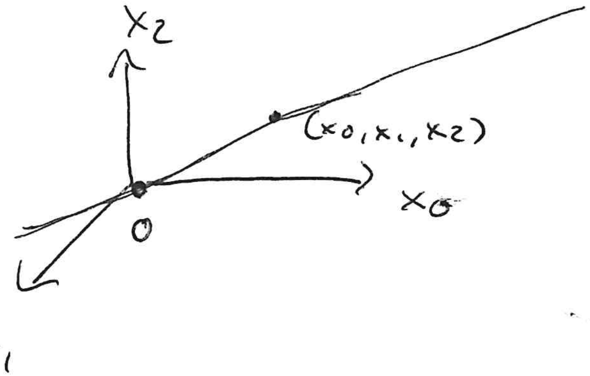
$\{\lambda(x_0, x_1, x_2) : \lambda \in \mathbb{R}\}$

$=$

rette per l'origine

$=$

sottospazio di dim 1



$r =$ retta per $(a_0\ a_1\ a_2)$ e $(b_0\ b_1\ b_2)$

$$\det \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ x_0 & x_1 & x_2 \end{pmatrix} = 0$$

$$\Updownarrow$$

$$(x_0, x_1, x_2) \in \underbrace{L((a_0, a_1, a_2), (b_0, b_1, b_2))}_{\dim 2}$$

$$r \longleftrightarrow L((a_0, a_1, a_2), (b_0, b_1, b_2))$$

Ex Le rette proiettive in $\mathbb{P}^2$ sono definite da $ax_1+bx_2+cx_0=0$ cioè dall'annullamento di un polinomio omogeneo di I° grado

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forma bilin. in $\mathbb{R}^3$ simmetrica

$$\begin{array}{c|c} & \begin{matrix} x_0 & x_1 & x_2 \end{matrix} \\ \begin{matrix} x_0 \\ x_1 \\ x_2 \end{matrix} & \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 0 \\ 3 & 0 & -1 \end{pmatrix} \end{array} = M \longleftrightarrow$$

$$2x_0^2+2x_0x_1+6x_0x_2+x_1^2-x_2^2$$

$$= (x_0\ x_1\ x_2)\, M \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

polinomio omogeneo di 2° grado

Ha senso parlare in $\mathbb{P}^2$ del luogo definito dai vettori di $\mathbb{R}^3$ isotropi resp. alla forma bilin. associata a M.

$\mathbb{P}^2$ $\quad$ $p(x_0,x_1,x_2) \in \mathbb{R}[x_0,x_1,x_2]$ $\quad$ P13

$\Downarrow$

(x_0,x_1,x_2) $\quad$ $\underline{ex.}$ $x_0^3 - 3x_1^2 + x_1x_2 - 4 = p$

$p: \mathbb{R}^3 \to \mathbb{R}$

$p: \mathbb{P}^2 \to \mathbb{R}$? $\quad$ $p(1,1,1) = -5$ $\quad$ $\underline{NO}$

$p(3,3,3) = 5$

$x_0^3 - 3x_1^2 + x_1x_2 - 4 = 0$ non ha senso in $\mathbb{P}^2$

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se p è $\underline{\text{OMOGENEO}}$ $\quad$ $p = x_0^2 + 3x_1^2 + x_1x_2$ $\quad$ polinomio omog. di 2° grado

$$p(x_0,x_1,x_2) = x_0^2 + 3x_1^2 + x_1x_2$$

$$p(\lambda x_0,\lambda x_1,\lambda x_2) = \lambda^2 x_0^2 + 3\lambda^2 x_1^2 + \lambda^2 x_1x_2 = \lambda^2(x_0^2 + 3x_1^2 + x_1x_2)$$

Luogo in cui p si annulla è ben definito in $\mathbb{P}^2$

$$p(x_0,x_1,x_2) = 0 \iff p(\lambda x_0,\lambda x_1,\lambda x_2) = 0$$

In generale se p omogeneo di grado d $\cancel{p(x_0 \dots x_n) = \lambda^d p}$

$$p(\lambda x_0,\dots,\lambda x_n) = \lambda^d p(x_0 \dots x_n) \quad \Leftrightarrow \quad p(x_0 \dots x_n) = 0 \iff p(\lambda x_0,\dots,\lambda x_n) = 0$$