$f: \underset{x_0 \dots x_n}{\mathbb{P}^n} \to \mathbb{P}^m$ given by Veronese + change of coord.

<u>Th</u> $Y \subseteq \mathbb{P}^n$ projective subvar. $f(Y)$ is a proj subvariety of $\mathbb{P}^m$

$Y = V(f_1 \dots f_n \dots)$ $\qquad f = g \circ V_d \qquad g$ change of coord. = linear map.

$\deg f_i = d_i$ take k_i such that $k_i d > d_i$

replace f_i with $\underbrace{x_j^{k_i d - d_i} f_i}_{\text{degree } k_i d}$ $\forall j = 0, \dots, n$

<u>n.b.</u> $Y = V(\underbrace{x_j^{k_i d - d_i} f_i}\dots)$

$\hookrightarrow$ can be written as a combination of monom. of degree k_i in terms of monom. of degree d in the $x_0 \dots x_n$

e.g. a monomial of degree k_i in the coordinates of the target space of V_d

$f(Y)$ = defined by the previous monomials.

$P \in Y$ $\quad$ P satisfies $f_i(P) = 0 \Rightarrow x_j^{k_i d - d_i} f_i(P) = 0$

$\Downarrow$

$\sum$ mon. in the coord. of $V_d(P)$

$\Rightarrow V_d(P)$ satisfies all the equation

$\cancel{Q \notin f(Y)}$ $Q \in \mathbb{P}^n$ $Q \notin Y \Rightarrow \exists \; x_j^{k_i d - d_i} f_i(Q) \neq 0$

$\Rightarrow V_d(Q)$ does not satisfy

$v_d : \mathbb{P}^n \longrightarrow \mathbb{P}^m$ Veronese map

$v_d(Y)$ = closed in the Zariski topology.

$\Uparrow$

closed in the Zariski topol.

$\Rightarrow$ v_d is <u>closed</u>

$f : \mathbb{P}^n \longrightarrow \mathbb{P}^m$ $\quad f = \pi \circ g \circ v_d$ $\quad \pi$ projection.

$\pi : \mathbb{P}^N \longrightarrow \mathbb{P}^m$

$\pi \underset{\text{surj.}}{\longleftrightarrow} K^{N+1} \xrightarrow{\pi} K^{m+1}$ center $\mathbb{P}(\ker \pi)$

$Y \subseteq \mathbb{P}^N$ proj. subvar. $\quad Y \cap \mathbb{P}(\ker \pi) = \emptyset$ $\quad \pi(Y) \subseteq \mathbb{P}^n$ is well defined

$\pi(Y)$ proj. subvar?

<u>Th</u> (Chow) $\pi(Y)$ is a proj. subvar. of $\mathbb{P}^m$

<u>pf</u> idea for $\mathbb{P}(\ker \pi) = \{1 \text{ point}\}$ $\quad \mathbb{P}^n - \{\text{point}\} \to \mathbb{P}^{n-1}$

$\hookrightarrow (1, 0, \dots, 0)$

$(a_0, a_1, \dots, a_n) \longrightarrow (a_1, \dots, a_n)$

$Y \subseteq \mathbb{P}^n \qquad Y = V(f_i) \qquad f_i \in k[x_0, \dots, x_n]$

$J = \text{rad}\,(L(f_i's)) \qquad Y = V(J)$

ideal in $k[x_0 \dots x_n] \supseteq k[x_1 \dots x_n] \qquad J' = J \cap k[x_1 \dots x_m]$

$\mathbb{P}^{n-1} \supseteq V(J') = \pi(Y)$

$P \in \pi(Y)$ then $\underset{P = \pi(Q)}{}$ Q kills all elem. of J

$Q = (a_0 \dots a_n) \quad \forall f \in J \quad f(a_0 \dots a_n) = 0$

true for $f \in J \cap k[x_1 \dots x_n] = J' \quad \underset{0}{\underset{\shortparallel}{f(a_0 \dots a_n)}} = f(a_1 \dots a_n) = f(\pi(Q)) = f(P)$

$\mathbb{P}^n$ $\qquad \{Y$ proj. subvarieties$\}$

$\emptyset, \mathbb{P}^n$ proj. subvar.

$\{Y_i\}$ proj. subvar.

$Y_i \Leftrightarrow V(f_{i1} \dots f_{is} \dots)$

$\cap Y_i \Leftrightarrow V($all the equation of the f_{ij}'s$)$

Y_1, Y_2 proj var. $\qquad Y_1 = V(f_1 \dots f_r \dots) \quad Y_2 = V(g_1 \dots g_k \dots)$

Prop $\quad Y_1 \cup Y_2 = V(f_i g_j)$

Pf $P \in Y_1 \quad f_i(P) = 0 \ \forall i \quad f_i g_j(P) = 0 \ \forall i,j \quad$ the same if $P \in Y_2$

if $P \notin Y_1$, $P \notin Y_2$ $\ \exists f_i, g_j$ s.t. $f_i(P) \neq 0 \ g_j(P) \neq 0 \Rightarrow f_i g_j(P) \neq 0$

Proj subvar. are the closed subset of a Topology on $\mathbb{P}^n$

ZARISKI TOPOLOGY of $\mathbb{P}^n$

$\mathbb{P}^n \to \mathbb{P}^m$ projective map is CLOSED in the Zariski topology

<u>NB</u> A topological space is <u>irreducible</u> iff $\forall A_1, A_2$ open subsets, $A_1, A_2 \neq \emptyset$ then $A_1 \cap A_2 \neq \emptyset$

<u>Prop</u> The Zariski top. is irreducible.

<u>Cor</u> " is connected.

<u>Th</u> The Zariski topol. is compact.

<u>Th</u> The Hilbert basis theorem.

$R = \mathbb{C}[x_0 \cdots x_n]$ $\qquad J \subseteq R$ ideal

$J = L(f_1 \cdots f_s \cdots)$. Then $J = L(f_{i_0} \cdots f_{i_s})$

i.e
$$\left\{\begin{array}{l} f_1 = 0 \\ \vdots \\ f_s = 0 \\ \vdots \end{array}\right. \longrightarrow \left\{\begin{array}{l} f_{i_0} = 0 \\ \vdots \\ f_{i_s} = 0 \end{array}\right.$$
have the same solutions.

i.e any proj. subvar. can be defined by a finite set of poly equations.

<u>Pf</u> that the Zariski topol. is compact

$\{F_i\}$ $\quad F_i$ closed $\qquad \cap F_i = \emptyset$

$F_i = V(J_i) \qquad J_i = L(f_{i1} \cdots f_{is})$

$\cap F_i = V(\cup J_i) \quad \supset \quad L(f_{11} \cdots\cdots f_{ij} \cdots\cdots) = I$

then $\quad V(I) = \emptyset \quad \Rightarrow \forall x_0 \cdots x_n \ \exists e_0 \cdots e_n$ s.t.

$x_i^{e_i} \in I = L(f_1 \cdots f_s) \qquad f_1 \in J_1 \cdots f_s \in J_s$

$\Downarrow$ Heb. basis thm. $\qquad \Downarrow$

$V(I) = \emptyset \qquad V(J_1) \cap \cdots \cap V(J_s) = \emptyset$

$V(J_1 \cup \cdots \cup J_s) \subseteq V(f_1 \cdots f_s)$

$\Leftarrow$

$F_1 \cap \cdots \cap F_s = \emptyset$

$$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \neq \underline{\mathbb{P}^{n_1+n_2}}$$

$$\mathbb{P}^1 \times \mathbb{P}^1 \neq \mathbb{P}^2$$

$$\mathbb{P}^1 \qquad \mathbb{C} + \{\infty\}$$

$$\mathbb{P}^1 \times \mathbb{P}^1 \rightsquigarrow \big((a_1, a_2)(b_1, b_2)\big) \longrightarrow (a_1, a_2, b_1, b_2) \in \mathbb{P}^3$$

$\sim \quad \exists\, k_1, k_2$

$(a_1', a_2') = k_1 (a_1, a_2)$
$(b_1', b_2') = k_2 (b_1, b_2)$

$\to$ not defined if $k_1 \neq k_2$

$\mathbb{P}^1 \times \mathbb{P}^1 \Rightarrow ((1,2),(3,4)) = ((3,6),(6,8))$
$k_1 = 3 \quad k_2 = 2$

but in $\mathbb{P}^3$

$(1,2,3,4) \neq (3,6,6,8)$ in $\mathbb{P}^3$

C. Segre (1870)

$$((a_1, a_2)(b_1, b_2)) \rightsquigarrow (a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)$$

$((1,2),(3,4)) \longrightarrow (3,4,6,8)$
$((3,6)(6,8)) \longrightarrow (18,24,36,48)$ $\quad \cdot 6$

$$\mathbb{P}^1 \times \mathbb{P}^1 \underset{\text{injective}}{\overset{s}{\hookrightarrow}} \mathbb{P}^3$$

$$\mathbb{P}^n \times \mathbb{P}^m \qquad ((x_0 \cdots x_n)(y_0 \cdots y_m)) \longrightarrow (x_0y_0, x_0y_1 \cdots, x_1y_0, \cdots x_ny_m)$$

$$\hookrightarrow \mathbb{P}^{(n+1)(m+1)-1} = \mathbb{P}^{nm+n+m}$$

$$\mathbb{P}^3 \times \mathbb{P}^3 \rightsquigarrow \mathbb{P}^{11} \qquad \text{the Segre map.}$$

The Segre map is injective

$$s((a_1, a_2)(b_1, b_2)) = s((a_1', a_2')(b_1', b_2')) \qquad a_1 \neq 0 \quad b_1 \neq 0$$

$$\parallel$$

$$(a_1b_1\ a_1b_2\ a_2b_1\ a_2b_2) = (a_1'b_1'\ a_1'b_2'\ a_2'b_1'\ a_2'b_2')$$

i.e. $\exists k$

$$a_1b_1 = ka_1'b_1'$$
$$a_1b_2 = ka_1'b_2'$$
$$a_2b_1 = ka_2'b_1'$$
$$a_2b_2 = ka_2'b_2'$$

$$b_1 = \frac{ka_1'}{a_1} b_1' \qquad k_1 = \frac{ka_1'}{a_1} \qquad (b_1, b_2) = k_1(b_1', b_2')$$

$$b_2 = \frac{ka_1'}{a_1} b_2'$$

$$a_1 = \frac{kb_1'}{b_1} a_1' \qquad k_2 = \frac{kb_1'}{b_1} \qquad (a_1, a_2) = k_2(a_1', a_2')$$

$$a_2 = \frac{kb_1'}{b_1} a_2'$$

$$((a_1, a_2)(b_1, b_2)) = ((a_1', a_2'), (b_1', b_2')) \longleftarrow$$

The Segre map is <u>not</u> surjective

$((a_1, a_2)(b_1, b_2)) \rightsquigarrow (a_1b_1,\ a_1b_2\ a_2b_1,\ a_2b_2)$ $\quad (a_1b_1)(a_2b_2) = (a_1b_2)(a_2b_1)$

i.e. find $(m_1, m_2, m_3, m_4) \in \mathbb{P}^3$ with $\boxed{m_1m_4 = m_2m_3}$

(m_1, m_2, m_3, m_4) with $m_1m_4 = m_2m_3$ $\quad$ alg. eq. homo degree 2

$m_1 \neq 0$

$$m_4 = \frac{m_2m_3}{m_1} = \left(\frac{m_2}{m_1}\right)m_3$$

$$m_2 = \left(\frac{m_2}{m_1}\right)m_1$$

$$s\left((m_1, m_3)\left(1, \frac{m_2}{m_1}\right)\right) = \left(m_1,\ m_1\left(\frac{m_2}{m_1}\right),\ m_3,\ m_3\left(\frac{m_2}{m_1}\right)\right) = (m_1, m_2, m_3, m_4)$$

$\mathbb{P}^2 \times \mathbb{P}^3 \longrightarrow \mathbb{P}^{11}$

$$(x_0, x_1, x_2)\,(y_0, y_1, y_2, y_3) \longrightarrow (\underset{m_{00}}{\overset{m_0}{x_0y_0}}\ \underset{m_{01}}{\overset{m_1}{x_0y_1}}\ \underset{m_{02}}{\overset{m_2}{x_0y_2}}\ \underset{m_{03}}{\overset{m_3}{x_0y_3}}\ \overset{m_4}{x_1y_0}\ \overset{m_5}{x_1y_1}\ \overset{m_6}{x_1y_2}\ \overset{m_7}{x_1y_3}\ \overset{m_8}{x_2y_0}\ \overset{m_9}{x_2y_1}\ \overset{m_{10}}{x_2y_2}\ \overset{m_{11}}{x_2y_3})$$

e.g. $m_2 m_5 = (x_0y_2)(x_1y_1) = (x_0y_1)(x_1y_2) = m_1 m_6$

and get in this way many equations.

$(x_0, x_1, x_2)\,(y_0, y_1, y_2, y_3) \mapsto \begin{pmatrix} x_0y_0 & x_0y_1 & x_0y_2 & x_0y_3 \\ x_1y_0 & & \text{etc.} & \end{pmatrix} \in \mathbb{P}^{11}$ $\quad m_{ij}$ $\quad 0 \le i \le 2$, $0 \le j \le 3$

$m_2 m_5 = m_1 m_6 \qquad m_{02}\, m_{11} = m_{01} m_{12}$

$$\begin{pmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \end{pmatrix}$$

i.e. $s(\mathbb{P}^2 \times \mathbb{P}^3) \subseteq \left(\begin{matrix}\text{set of matrices} \\ \text{of rank } 1\end{matrix}\right)$

Conversely, $M = (m_{ij})$ of rank 1 $\quad R_0 R_1 R_2$ rows $\quad R_0 \neq 0$

$\exists\ \alpha_1 \alpha_2$ s.t. $R_1 = \alpha_1 R_0 \quad R_2 = \alpha_2 R_0 \qquad \underline{m_{ij} = \alpha_i (R_0)_j} \qquad \alpha_0 = 1$

$s((1, \alpha_1, \alpha_2), (R_0)) = M$

$$s\colon(\mathbb{P}^n \times \mathbb{P}^m) \longrightarrow \mathbb{P}^{nm+n+m} \xrightarrow{\varphi} \mathbb{P}^{nm+n+m}$$ φ change of coord.

$$(x_0 \dots x_n)\ (y_0 \dots y_m) \rightsquigarrow (x_i y_j)$$

$$\varphi \circ s\left[(x_0 \dots x_n)(y_0 \dots y_m)\right] \longrightarrow \left[\textstyle\sum a_{ij} x_i y_j, \dots\dots, \right]$$

$$(x_0, x_1)(y_0\, y_1) \rightsquigarrow \left(\underbrace{x_0 y_1 + x_1 y_0}, \underbrace{x_1 y_1 + 2 x_0 y_1}, \underbrace{x_1 y_1 + 3 x_1 y_0}, \underbrace{x_0 y_1}\right)$$

bihomogeneous polinomials of bidegree (1, 1)

$$\mathbb{P}^n \times \mathbb{P}^m \xrightarrow{V_d \times 1} \mathbb{P}^N \times \mathbb{P}^m \xrightarrow{s} \mathbb{P}^M \xrightarrow{\varphi} \mathbb{P}^M$$ φ cambio de coord.

$(x_0 \dots x_n)\ (y_0 \dots y_m) \longrightarrow$ (monomi di grado d nelle x) $(y_0 \dots y_m) \longrightarrow$ prodotti di monomi di grado d nelle x e le y_j $\longrightarrow$ Combin. lin. di

polinomi di bigrado $(d, 1)$

$$\mathbb{P}^n \times \mathbb{P}^m \xrightarrow{V_d \times V_{d'}} \mathbb{P}^N \times \mathbb{P}^M \xrightarrow{\text{Segre}} \mathbb{P}^{NM+N+M} \xrightarrow{\varphi}$$ polinomi di bigrado d, d'

$$\mathbb{P}^n \times \mathbb{P}^m \xrightarrow{v_d \times v_{d'}} \mathbb{P}^N \times \mathbb{P}^M \to \mathbb{P}^{NM+N+M} \xrightarrow{\varphi} \mathbb{P}^{NM+N+M} \xrightarrow{\pi} \mathbb{P}^q$$

$$(z_0 \dots z_q)$$

In the image
$$\begin{cases} z_0 = F_0(x_0 \dots x_n, y_0 \dots y_m) \\ \\ z_q = F_q(x_0 \dots x_n\, y_0 \dots y_m) \end{cases}$$
bi homo poly of bidegree (d, d')

with $F_0 \dots F_q \neq 0$ unless $(x_0 \dots x_n) = 0$ or $(y_0 \dots y_m) = 0$

<u>Teor</u> (Chow) S è chiusa (nella top. di Zariski)

Hence everything described in $\mathbb{P}^q$ by bihomo poly is closed in the Zariski topology, hence it is a proj. subvariety.

Ex $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{S} \mathbb{P}^3$

Im. $\left\{ (m_0, m_1, m_2, m_3) : \begin{pmatrix} m_0 & m_1 \\ m_2 & m_3 \end{pmatrix} \text{ has rank } 1 \right\}$

$$\begin{cases} m_0 = a_0 b_0 \\ m_1 = a_0 b_1 \\ m_2 = a_1 b_0 \\ m_2 = a_1 b_1 \end{cases}$$

eq. of bidegree $(1,1)$ $\Longleftrightarrow$ $m_0 m_3 - m_1 m_2 = 0$

Everything can be generalized to $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$

$(x_{10} \cdots x_{1n_1}) \cdots\cdots (x_{s0} \cdots x_{sn_s})$

$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s} \longrightarrow \mathbb{P}^N$ $\quad N = \left[\prod (n_i+1)\right] - 1$

$(x_{10} \cdots x_{1n_1}) \cdots (x_{s0} \cdots x_{sn_s}) \rightsquigarrow (\cdots, x_{1j_1} x_{2j_2} \cdots x_{sj_s}, \cdots)$

By composing with Veronese + ch. of coord. + projection get all subsets of $\mathbb{P}^N$ defined by multihomo eq. of multideg $(d_1 \cdots d_s)$