Gram-Schmidt su $\mathbb{C}$

$b(v,v) = -1 \qquad v \rightsquigarrow iv \qquad b(iv,iv) = i^2 b(v,v) = 1$

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

$\hookrightarrow$ dim Rad(b)

In particolare se b non-deg. $(\mathrm{Rad}(b) = (0)) \Rightarrow \exists$ base ortonorm.

b forma bilineare $\Rightarrow$ forma quadratica $Q(v) = b(v,v)$

$\Rightarrow$ luogo geometrico dei vettori t.c. $Q(v) = b(v,v) = 0$.

—

Eq. 2° grado

$$ax^2 + bx + c = 0$$

∞

$a, b, c \in \mathbb{R}$

$\Rightarrow$ 2 soluz. complesse coniug.
2 soluz. reali
1 soluz. (doppia) reale

$$x = \frac{x_1}{x_0}$$

$$a\frac{x_1^2}{x_0^2} + b\frac{x_1}{x_0} + c = 0$$

$$\boxed{a x_1^2 + b x_1 x_0 + c x_0^2 = 0}$$

$\mathbb{P}^1_{\mathbb{C}}$ $\quad$ $\mathbb{C}^2$ $\quad$ forma bilin. associata all'equazione

$$\begin{array}{c|cc} & x_1 & x_0 \\ \hline x_1 & a & b/2 \\ x_0 & b/2 & c \end{array}$$

$$\underset{b((x_1,x_0),(x_1,x_0))}{=} (x_1,x_0)\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\begin{pmatrix} x_1 \\ x_0 \end{pmatrix} = ax_1^2 + b\,x_1x_0 + c\,x_0^2$$

—

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

$\mathbb{P}^1_{\mathbb{C}}$

rango 0 $\quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\dim Rad = 1$ rango 1 $\quad$ 2 soluz. coincid.

$\dim Rad = 0$ rango 2 $\quad$ 2 soluz. distinte

rango 1 $\Leftrightarrow \det = 0 \Leftrightarrow \underset{0}{\underset{\shortparallel}{ac - b^2/4}} = -\dfrac{\Delta}{4} = 0 =$ 2 soluz. coincid.

—

<u>Ex</u> $\quad x_1^2 + 2x_1x_0 + x_0^2 \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \det = 0 \qquad Rad = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right)$

$x_1 + x_0 = 0 \qquad x_1 = -x_0$

$Rad = \{(-x_0, x_0) : x_0 \in \mathbb{C}\} = L(-1, 1) \qquad \dfrac{-1}{1}$

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$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

Ex $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$x_1^2 + 2x_1x_0 = 0$$
$$x_1(x_1 + 2x_0) = 0$$

$x_1 = 0$

$x_1 = -2x_0$

$L(-2, 1)$

-2 $\quad 0$ $\quad \mathbb{C}$

$$x_1^2 + 2x_1x_0 = 0 \qquad \frac{x_1^2}{x_0^2} + \frac{2x_1x_0}{x_0^2} = 0$$

$x = \frac{x_1}{x_0}$

$$\boxed{x^2 + 2x = 0}$$

Ex $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$$2x_1x_0 + x_0^2 = 0 \qquad \frac{2x_1x_0}{x_0^2} + \frac{x_0^2}{x_0^2} = 2\frac{x_1}{x_0} + 1 = 2x + 1$$

$$x = -1/2$$

$$x_0(2x_1 + x_0)$$

$x_0 = 0$

$2x_1 + x_0 = 0$

$L(-1/2, 1)$

$-1/2$ $\quad \infty$

Ex $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$x_0^2 = 0$

punto all'∞

contato 2 volte

$$\frac{x_0^2}{x_0^2} = 0 \qquad 1 = 0$$

no soluz.

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- $\mathbb{P}^1_{\mathbb{C}}$
  - rango 2 — 2 soluz. distinte (2 punti distinti)
  - rango 1 — 1 soluz. doppia

- $\mathbb{A}^1_{\mathbb{C}}$
  - rango 2
    - 2 soluz. distinte affini
    - 1 punto affine e 1 punto all'∞ (grado 1)
  - rango 1
    - 1 soluz. doppia affine
    - punto doppio all'∞ (costante $\neq 0$)

- $\mathbb{P}^1_{\mathbb{R}}$
  - rango 2
    - 2 soluz. reali
    - 2 soluz. complesse coniugate ($x^2+1=0$ $x_1^2+x_0^2=0$)
  - rango 1 — 1 soluz. doppia (reale)

- $\mathbb{A}^1_{\mathbb{R}}$
  - rango 2
    - 2 soluz. reali
      - entrambe affini
      - 1 soluz. affine (+ pto all'∞) (grado 1)
    - 2 soluz. complesse coniugate (affini)
  - rango 1
    - 1 soluz. doppia (reale) affine
    - 0 soluz. affini (punto doppio reale all'∞) (costante $\neq 0$)

Dal punto di vista della segnatura (su $\mathbb{R}$)

rango 1 (0 è autoval.)

l'altro autoval. può essere
- pos. b semidef. positiva
- neg. b " negativa; $-b$ " positiva

vettori isotropi = Rad(b)

rango 2 (0 <u>non</u> è autoval.)
- autoval. concordi: def positiva / def negativa — NO vettori isotropi
- autoval. discordi: 2 autosp. dim 1 — 2 punti reali

$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ $\det > 0$, $tr > 0$ — 2 autoval. concordi

$x_1^2 + 4x_1x_0 + 5x_0^2$ $\quad x^2 + 4x + 5$ no soluz. reali

$\Delta = 16 - 20$ (2 soluz. complesse coniugate)

$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ $\det < 0$, $tr > 0$ — autoval. discordi — 2 punti reali

$x^2 + 4x + 3$ $\quad \Delta = 16 - 12$

$x = -2 \pm \sqrt{4-3} = \begin{cases} -1 \\ -3 \end{cases}$

Eq. di 2° grado in 2 variabili x, y

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad x = \frac{x_1}{x_0} \quad y = \frac{x_2}{x_0}$$

$$A\frac{x_1^2}{x_0^2} + B\frac{x_1}{x_0}\frac{x_2}{x_0} + C\frac{x_2^2}{x_0^2} + D\frac{x_1}{x_0} + E\frac{x_2}{x_0} + F = 0$$

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1x_0 + Ex_2x_0 + Fx_0^2 = 0$$

$$\begin{array}{c|ccc} & x_1 & x_2 & x_0 \\ \hline x_1 & & & \\ x_2 & & \left(\begin{matrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{matrix}\right) & \\ x_0 & & & \end{array} = M \qquad (x_1\ x_2\ x_0)\begin{pmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_0 \end{pmatrix}$$

$$Dx_1x_0 + Ex_2x_0 + Fx_0^2 = 0 \qquad x_0(Dx_1 + Ex_2 + Fx_0) = 0$$

$$x = \frac{x_1}{x_0} \quad y = \frac{x_2}{x_0} \qquad Dx + Ey + F = 0$$

$\mathbb{P}^2_{\mathbb{C}}$ $\quad$ b forma bilin. simm. su $\mathbb{C}^3$

- $\dim Rad(b) = 3$ $\quad$ rango $M = 0$ $\quad$ $M = 0$ $\quad$ $b = 0$
- $\dim Rad(b) = 2$ $\quad$ rango $M = 1$ $\quad$ b semidef.

Prop Non vi sono vettori isotropi fuori del radicale

dm per assurdo sia $v \notin Rad(b)$ $\quad$ v isotropo

Base del radicale u_1, u_2

u_1, u_2, v sono lin. indip. perché $v \notin L(u_1, u_2) = Rad(b)$

$\Downarrow$

base di $\mathbb{C}^3$

$\forall w \in \mathbb{C}^3$ $\quad$ $w = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 v$

$$b(v,w) = b(v, \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 v) = \alpha_1 \underset{\underset{0 \to u_1, u_2 \in Rad}{\parallel}}{b(v,u_1)} + \alpha_2 \underset{\underset{0}{\parallel}}{b(v,u_2)} + \alpha_3 \underset{\underset{0 \; v \text{ isotropo}}{\parallel}}{b(v,v)} = 0$$

$\Rightarrow v \in Rad(b)$. Assurdo

rango $M = 1 \Rightarrow A x_1^2 + B x_1 x_2 + C x_2^2 + D x_1 x_0 + E x_2 x_0 + F x_0^2 = 0$

rappresenta una retta proiettiva

Ex

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$x_1^2 + 2x_1x_2 + x_2^2 + 2x_1x_0 + 2x_2x_0 + x_0^2 = 0$$

$$= (x_1 + x_2 + x_0)^2$$

retta doppia

$x_1 + x_2 + x_0 = 0$

$x + y + 1 = 0$

Ex

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad x_1^2 = 0 \qquad (\text{asse } y)^2$$

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad x_0^2 = 0 \qquad (\text{retta all'}\infty)^2$$

$$1 = 0$$

(b è DOPPIAMENTE DEGENERE)

$$\begin{cases} Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1x_0 + Ex_2x_0 + Fx_0^2 = 0 \\ x_0 = 0 \end{cases}$$

$$\begin{pmatrix} A & B/2 & \\ B/2 & C & \\ & & \end{pmatrix}$$

- dim Rad$(b) = 1$ rango $M = 2$

  ↳ rappresenta un punto di $\mathbb{P}^2_{\mathbb{C}}$

Con un cambio di coord. Rad(b) = origine $= L(0,0,1)$

$$0 = (x_1\ x_2\ x_0)\begin{pmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \forall (x_1\ x_2\ x_0)$$

$$= (x_1\ x_2\ x_0)\begin{pmatrix} D/2 \\ E/2 \\ F \end{pmatrix}$$

$$0 = (1\,0\,0)\begin{pmatrix} D/2 \\ E/2 \\ F \end{pmatrix} = D/2 \Rightarrow D = 0$$

$$0 = (0\,1\,0)\begin{pmatrix} D/2 \\ E/2 \\ F \end{pmatrix} = E/2 \Rightarrow E = 0$$

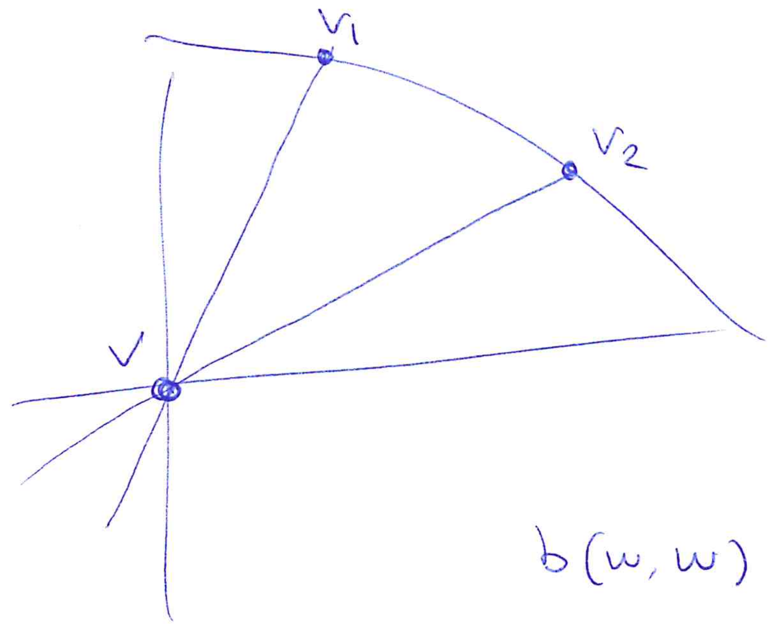
$$0 = (0\,0\,1)\begin{pmatrix} D/2 \\ E/2 \\ F \end{pmatrix} = F \Rightarrow F = 0$$

$\Leftarrow$

rango 2

$$\begin{pmatrix} A & B/2 & 0 \\ B/2 & C & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 = 0 \longrightarrow 2 \text{ punti distinti all'}\infty$$



v_1, v_2 isotropi all'∞

$v = (0,0,1)$

$w \in \overset{\dim 2}{L(v, v_1)}$ $\quad w = \alpha v + \beta v_1$

$$b(w,w) = b(\alpha v + \beta v_1, \alpha v + \beta v_1) =$$

$$0 = \alpha^2 \underset{\substack{\| \\ 0}}{b(v,v)} + 2\alpha\beta \underset{\substack{\| \\ 0}}{b(v,v_1)} + \beta^2 \underset{\substack{\| \\ 0}}{b(v_1,v_1)}$$

$v \in \text{Rad}$ $\qquad$ v_1 isotropo

tutti i vettori di $\underset{\substack{\| \\ \text{retta}}}{L(v,v_1)}$ sono isotropi

anche $L(v,v_2)$ retta è formato da vettori isotropi

$\cup$ 2 rette distinte

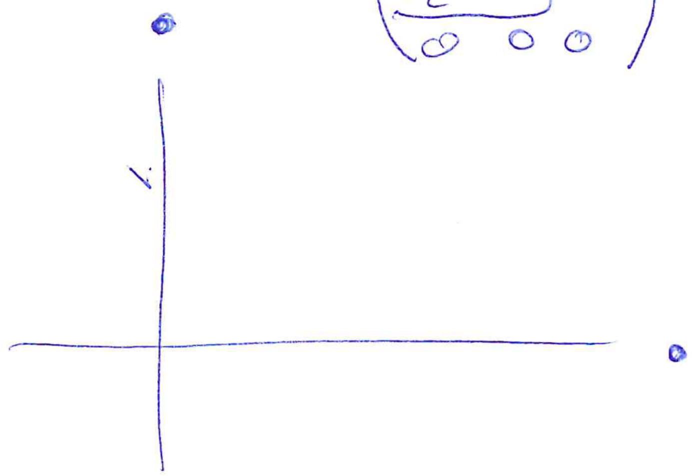
Ex $xy = 0$ $\frac{x_1}{x_0} \frac{x_2}{x_0} = 0$ $\boxed{x_1 x_2 = 0}$ $\left(\begin{array}{cc|c} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$

all'∞

(asse ~~x~~y)(asse x) = 0

$(1, 0, 0)$

$(0, 1, 0)$



$Rad(b) = (0)$ rango $M = 3$

Con un cambio di base (di coord.) $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

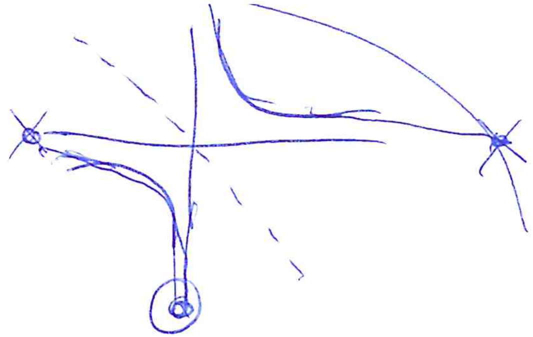
v_1, v_2, v_3

$x_1^2 + x_2^2 + x_0^2 = 0$ $x^2 + y^2 + 1 = 0$

v_1 v_2 iv_3 $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $x_1^2 + x_2^2 - x_0^2 = 0$ $x^2 + y^2 - 1 = 0$

Ex

$xy = 1$



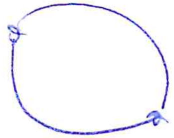
$x_1 x_2 - x_0^2 = 0$

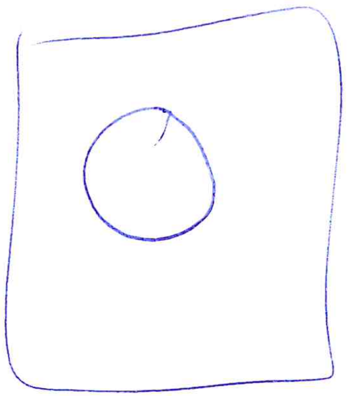
$$\begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

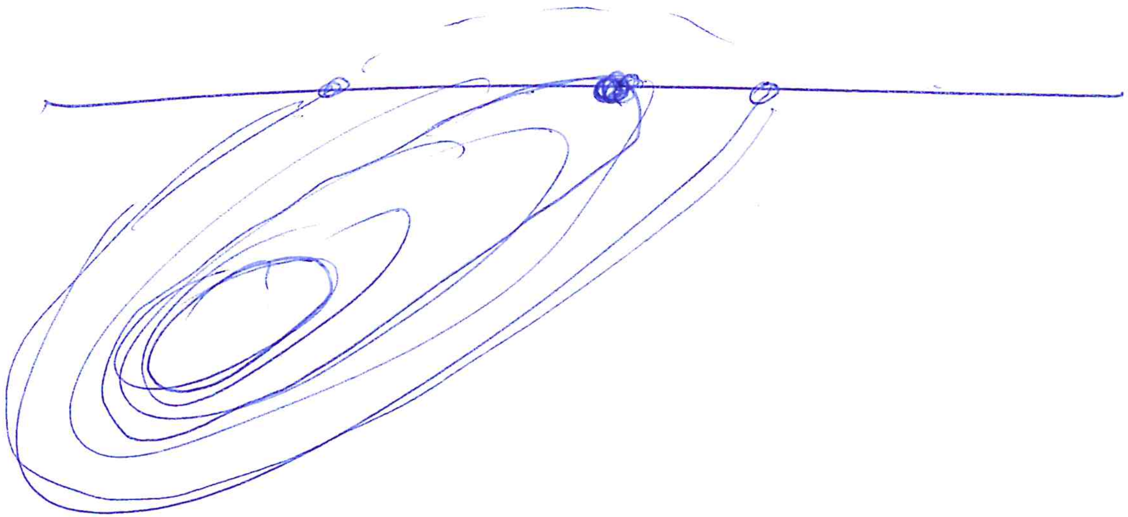
ранг 3

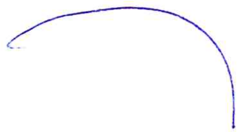
$$\begin{cases} x_1 x_2 - x_0^2 = 0 \\ x_0 = 0 \end{cases}$$

$x_1 x_2 = 0$









$y = x^2$

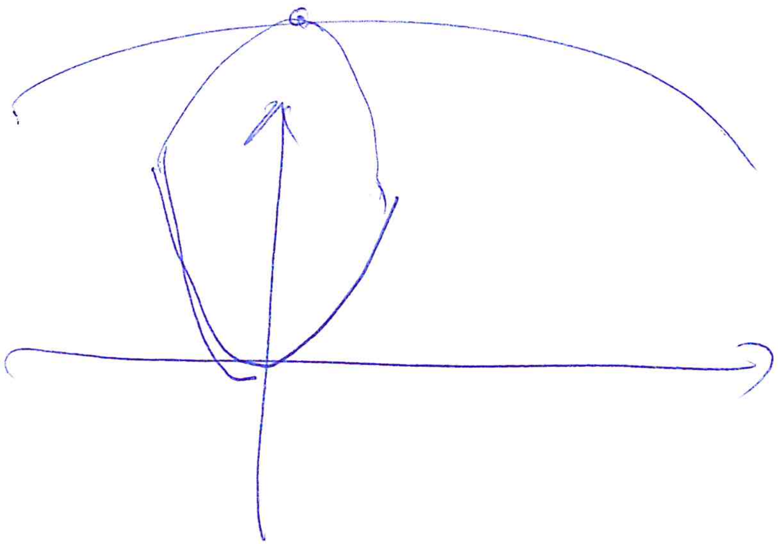
$x_1^2 - x_2 x_0 = 0$

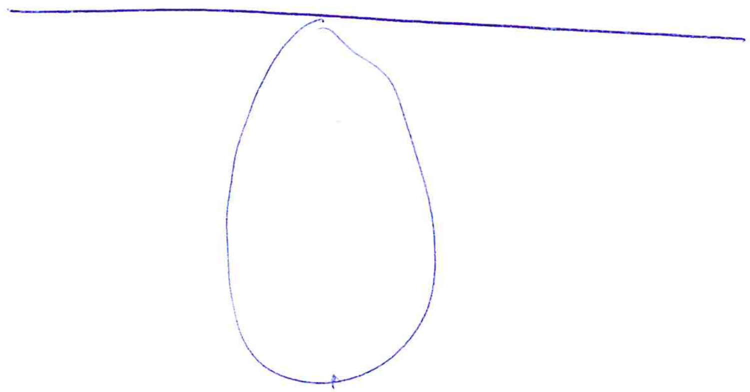
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix}$$

$$\begin{cases} x_1^2 - x_2 x_0 = 0 \\ x_0 = 0 \end{cases}$$

$$\begin{cases} x_1^2 = 0 \\ x_0 = 0 \end{cases}$$

$(0, 1, 0)$





Su $\mathbb{P}^2_{\mathbb{C}}$ luogo di $A x_1^2 + B x_1x_2 + C x_2^2 + D x_1x_0 + G x_2x_0 + F x_0^2 = 0$ è

- rango $r = 1$ retta doppia
- rango $r = 2$ 2 rette distinte
- rango $r = 3$ $x^2 + y^2 - 1 = 0$

Su $\mathbb{P}^2_{\mathbb{R}}$

- rango $r = 1$ retta doppia reale
- rango $r = 2$ 2 rette distinte < 2 reali / complesse coniugate

$x_1^2 + x_2^2 = 0$ $x^2 + y^2 = 0$ $(x+iy)(x-iy) = 0$

- rango $r = 3$ segnatura
  - $(3,0,0)$ b def pos. $\emptyset$
  - $(2,0,1)$ $\neq \emptyset$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $x^2 + y^2 - 1 = 0$
  - $(1,0,2)$ $\neq \emptyset$
  - $(0,0,3)$ b def neg ($-b$ def. pos.) $\emptyset$

$\emptyset$ autoval. concordi

$\neq \emptyset$ autoval. discordi

circonf. $\quad \underline{x^2+y^2+ax+by+c=0}$

$$\begin{pmatrix} 1 & 0 & a/2 \\ \cancel{a/2} & 1 & b/2 \\ a/2 & b/2 & c \end{pmatrix}$$

$$\begin{cases} x_1^2+x_2^2+a x_1 x_0 + b x_2 x_0 + c x_0^2 = 0 \\ x_0 = 0 \end{cases}$$

$$\begin{cases} x_1^2+x_2^2 = 0 \\ x_0 = 0 \end{cases} \qquad \begin{cases} x_1 = \pm i x_2 \\ x_0 = 0 \end{cases}$$

$$\underbrace{\begin{matrix} (i, 1, 0) \\ (-i, 1, 0) \end{matrix}}_{\text{punti ciclici}}$$

raggio 0