forma di Minkowski

$\mathbb{R}^4$ $\qquad \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}$ $\qquad$ segnatura $(3,0,1)$

---

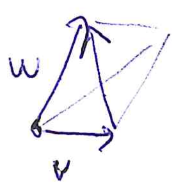
b forma bilin. simm. def. positiva V (su $\mathbb{R}$)

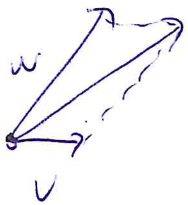
$\forall v \in V \qquad |v|_b = \sqrt{b(v,v)}$

1) $|v|_b \geq 0$ e $|v|_b = 0 \iff v = 0$

2) $|\alpha v|_b = \sqrt{b(\alpha v, \alpha v)} = \sqrt{\alpha^2 b(v,v)} = \sqrt{\alpha^2}\sqrt{b(v,v)} = \sqrt{\alpha^2}\,|v|_b = |\alpha|\,|v|_b$

3) $|v+w|_b \leq |v|_b + |w|_b$ (disuguaglianza triangolare
" di Schwarz)





dim di 3)

$$\sqrt{b(v+w,\, v+w)} \overset{?}{\leq} \sqrt{b(v,v)} + \sqrt{b(w,w)}$$

$$\| $$

$$\sqrt{b(v,v)+2b(v,w)+b(w,w)}$$

$$\cancel{b(v,v)} + \cancel{2}b(v,w) + \cancel{b(w,w)} \overset{?}{\leq} \cancel{b(v,v)} + \cancel{b(w,w)} + \cancel{2}\sqrt{b(v,v)b(w,w)}$$

$$b(v,w) \overset{?}{\leq} \sqrt{b(v,v)\,b(w,w)}$$

se $b(v,w) \leq 0$ OK

se $b(v,w) > 0$

$$b(v,w)^2 \overset{?}{\leq} b(v,v)\,b(w,w)$$

$$0 \leq b(xv+yw,\, xv+yw) = x^2 b(v,v) + y^2 b(w,w) + 2xy\, b(v,w) \qquad \forall x,y$$

$x = b(w,w)$ $\qquad y = -b(v,w)$

$$0 \leq b(w,w)^2 b(v,v) + b(v,w)^2 b(w,w) - 2\, b(w,w)\, b(v,w)^2$$

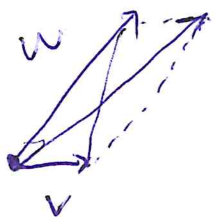
$$0 \leq b(w,w)^2 b(v,v) - b(v,w)^2 b(w,w)$$

se $w = 0$ OK

altrimenti $b(w,w) > 0$

$$0 \leq b(w,w)\,b(v,v) - b(v,w)^2$$

V $\quad b \quad$ def. pozitivna $\qquad |v|_b$



$\mathcal{B} = (v, w) \qquad b \qquad M_b^{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$|v|_b = \sqrt{b(v,v)} = 1 \qquad b(v,w) = 0 \qquad v \perp_b w$

$|w|_b = \sqrt{b(w,w)} = 1$

$$|v+w|_b = \sqrt{b(v+w, v+w)} = \sqrt{b(v,v) + b(w,w) + \underbrace{2b(v,w)}} = \sqrt{2}$$

$$|v-w|_b = \sqrt{b(v-w, v-w)} = \sqrt{b(v,v) + \underbrace{b(-w,-w)}_{\overset{\shortparallel}{b(w,w)}} \underbrace{-2b(v,w)}_{\overset{\shortparallel}{0}}} = \sqrt{1+1} = \sqrt{2}$$

distanza

$$d_b(v,w) = |v-w|_b$$

$d_b(v,w) \geq 0$ e $d_b(v,w) = 0 \iff v = w$

$d_b(v,w) = d_b(w,v)$

$$d_b(w,v) = |w-v|_b =$$

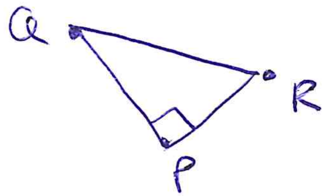
$$= |(-1)(v-w)|_b = |-1|\,|v-w|_b$$

$$|v-w|_b = d_b(v,w)$$

$d_b(v,u) \leq d_b(v,w) + d_b(w,u) \quad \forall w$

$$d_b(v,u) = |v-u|_b = |(v-w)+(w-u)|_b \leq |v-w|_b + |w-u|_b = d_b(v,w) + d_b(w,u)$$

$PQ \perp_b PR \Leftrightarrow P-Q \perp_b P-R$



$$d_b(Q,R)^2 = d_b(Q,P)^2 + d_b(P,R)^2$$

Teorema di Pitagora

O

(V, b) spazio metrico = spazio euclideo

$(V, b) \xrightarrow{f} (W, b')$

Def ISOMETRIE

$f: V \to W$ lineari

$\forall\, v_1, v_2 \in V$

$$b(v_1, v_2) = b'(f(v_1), f(v_2))$$

$$v \perp_b w \iff f(v) \perp_{b'} f(w)$$

$$|v|_b = \sqrt{b(v,v)} = \sqrt{b'(f(v), f(v))} = |f(v)|_{b'}$$

$$d_b(v,w) = |v-w|_b = |f(v-w)|_{b'} = |f(v) - f(w)|_{b'} = d_{b'}(f(v), f(w))$$

$f: \mathbb{R}^2 \to \mathbb{R}^2 \qquad \mathcal{B} = (e_1, e_2) \qquad M_f^{\mathcal{B}} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

b = prodotto scalare

$e_1 \cdot e_1 = 1 \qquad f(e_1) \cdot f(e_1) = (2,0) \cdot (2,0) = 4$ non è isometria

---

$id: (V, b) \to (V, b)$ è un'isometria

$\mathbb{R}^2$ $\quad \mathcal{B}=(e_1,e_2)$ $\quad b=\cdot$ $\quad M_f^{\mathcal{B}}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$

$f(e_1)=e_2 \quad f(e_2)=e_1$

$\forall \quad v=(\alpha,\beta) \quad w=(\gamma,\delta) \quad f(v)=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}=\begin{pmatrix}\beta\\\alpha\end{pmatrix} \quad f(w)=\begin{pmatrix}\delta\\\gamma\end{pmatrix}$

$v\cdot w \quad f(v)\cdot f(w)=(\beta,\alpha)\cdot(\delta,\gamma)=\beta\delta+\alpha\gamma$

$\parallel$

$\alpha\gamma+\beta\cdot\delta$ $=$

isometria

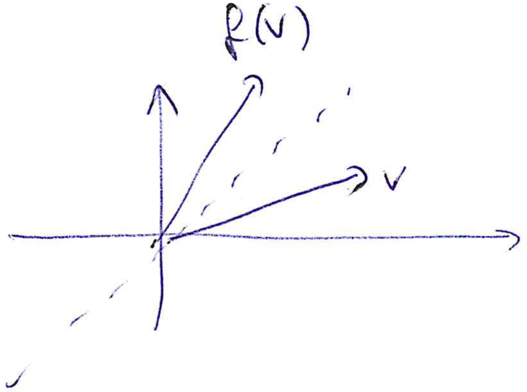
$$M_f^{\mathcal{B}}=\begin{pmatrix}\sqrt{2}/2 & -\sqrt{2}/2\\ \sqrt{2}/2 & \sqrt{2}/2\end{pmatrix}$$

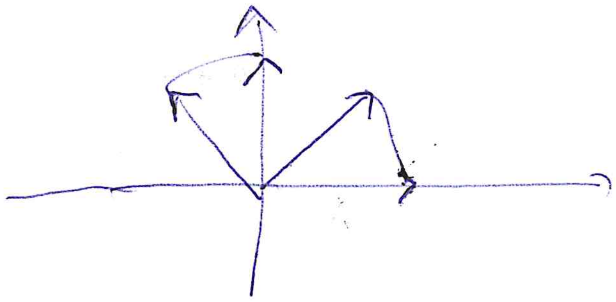
$v=(\alpha\beta) \quad w=(\gamma,\delta)$

$f(v)=\begin{pmatrix}\sqrt{2}/2 & -\sqrt{2}/2\\ \sqrt{2}/2 & \sqrt{2}/2\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}=\left(\frac{\sqrt{2}}{2}\alpha-\frac{\sqrt{2}}{2}\beta,\ \frac{\sqrt{2}}{2}\alpha+\frac{\sqrt{2}}{2}\beta\right) \quad f(w)=\left(\frac{\sqrt{2}}{2}\gamma-\frac{\sqrt{2}}{2}\delta,\ \frac{\sqrt{2}}{2}\gamma+\frac{\sqrt{2}}{2}\delta\right)$

$f(v)\cdot f(w)=\frac{1}{2}\alpha\gamma-\frac{1}{2}\alpha\delta-\frac{1}{2}\beta\gamma+\frac{1}{2}\beta\delta+\frac{1}{2}\alpha\gamma+\frac{1}{2}\alpha\delta+\frac{1}{2}\beta\gamma+\frac{1}{2}\beta\delta=\alpha\gamma+\beta\delta$

Isometria





$f(x,y) = (x,y,0)$ $\quad \mathbb{R}^2 \to \mathbb{R}^3$

$(V,b) \xrightarrow{f} (W,b')$ $\quad$ f isometria

$\mathcal{B} = (v_1 - - - v_n)$ base ortonorm. per V

$b(v_i,v_i) = 1$ $\quad b(v_i,v_j) = 0$ $\quad i \neq j$

$|v_i|_b = 1$ $\quad v_i \perp_b v_j$

$f(v_1), \ldots, f(v_n)$

$f(v_i) \perp_{b'} f(v_j)$ $\quad i \neq j$

$b'(f(v_i), f(v_i)) = 1$

$|f(v_i)|_{b'} = 1$

Isometrie $f: (V, b) \longrightarrow (W, b')$

$\forall v, w \in V \qquad b(v,w) = b'(f(v), f(w))$

quindi $\qquad |v|_b = |f(v)|_{b'} \qquad d_b(v,w) = d_{b'}(f(v), f(w))$

$\mathcal{B} = (v_1, \dots, v_n)$ base di V

<u>Prop</u> se $\forall i,j \qquad b(v_i, v_j) = b'(f(v_i), f(v_j)) \iff f$ è isometria

$\Rightarrow \quad \forall v, w \in V \qquad v = \alpha_1 v_1 + \dots + \alpha_n v_n \qquad w = \beta_1 v_1 + \dots + \beta_n v_n$

$f(v) = \alpha_1 f(v_1) + \dots + \alpha_n f(v_n) \qquad f(w) = \beta_1 f(v_1) + \dots + \beta_n f(v_n)$

$b'(f(v), f(w)) = b'(\alpha_1 f(v_1) + \dots + \alpha_n f(v_n), \beta_1 f(v_1) + \dots + \beta_n f(v_n)) =$

$\sum \alpha_i \beta_j \, b'(f(v_i), f(v_j)) = \sum \alpha_i \beta_j \, b(v_i, v_j) = b(v, w)$

Prop Ogni isometria è iniettiva.

dim Se per assurdo avessimo $v \neq 0$ $v \in \ker f$

$b(v,v) = b'(f(v), f(v)) = b'(0,0) = 0 \Rightarrow$ impossibile perché b def. pos.

Cor Se V, W hanno la stessa dim.

Un isometria è un isomorfismo

$f : (V, b) \to (W, b')$ isometria

$\mathcal{B} = (v_1 - v_n)$ base di V ortonorm. per b

$\mathcal{B}' =$ base di W ortonorm. per b'

$M_f^{\mathcal{B},\mathcal{B}'}$ $f(v_1)$ coord. $f(v_1)_{\mathcal{B}'}$

$\vdots$

$f(v_n)$ $f(v_n)_{\mathcal{B}'}$

in colonna

$$\begin{array}{ccc} f(v_1)_{\mathcal{B}'} & \cdots & f(v_n)_{\mathcal{B}'} \\ \downarrow & & \downarrow \end{array}$$

$$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} = M_f^{\mathcal{B}\mathcal{B}'}$$

$b(v_1, v_2) = 0 \quad \Rightarrow \quad b'(f(v_1), f(v_2)) = 0$

$f(v_1)_{B'} \overset{''}{\cdot} f(v_2)_{B'} \quad \Rightarrow$ le colonne C_1 e C_2 della $M_f^{BB'}$ soddisfano $C_1 \cdot C_2 = 0$

$b(v_1, v_1) = 1 \qquad b'(f(v_1), f(v_1)) = 1 \quad \Rightarrow \quad C_1 \cdot C_1 = 1$

Prop $M_f^{BB'}$ è tale che le sue colonne C_i soddisfano $C_i \cdot C_i = 1 \qquad C_i \cdot C_j = 0 \quad i \neq j$

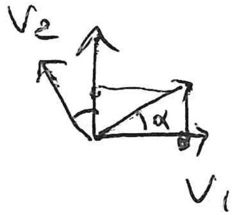
$\Leftarrow$ viceversa

Es $(\mathbb{R}^2, \cdot) \longrightarrow (\mathbb{R}^2, \cdot)$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$ sono isometrie

in entrambe $\quad C_1 \cdot C_1 = 1 \qquad C_2 \cdot C_2 = 1 \qquad C_1 \cdot C_2 = 0$

Ex Rotazione del piano di angolo α



$$f(v_1) = (\cos\alpha)\, v_1 + (\operatorname{sen}\alpha)\, v_2$$
$$f(v_2) = (-\operatorname{sen}\alpha)\, v_1 + (\cos\alpha)\, v_2$$

$$\begin{pmatrix} \cos\alpha & -\operatorname{sen}\alpha \\ \operatorname{sen}\alpha & \cos\alpha \end{pmatrix}$$

$C_1 \cdot C_1$ $\quad (\cos\alpha, \operatorname{sen}\alpha) \cdot (\cos\alpha, \operatorname{sen}\alpha) = \cos^2\alpha + \operatorname{sen}^2\alpha = 1$

$C_2 \cdot C_2$

$C_1 \cdot C_2$ $\quad (\cos\alpha, \operatorname{sen}\alpha) \cdot (-\operatorname{sen}\alpha, \cos\alpha) = -\operatorname{sen}\alpha\cos\alpha + \operatorname{sen}\alpha\cos\alpha = 0$

Def Data una matrice $n \times m$ M diremo che è ORTOGONALE se le sue colonne C_i soddisfano

$C_i \cdot C_i = 1 \qquad C_i \cdot C_j = 0 \quad i \neq j.$

Teor. f è isometria $(V, b) \to (W, b')$ <u>se</u> $M_f^{\mathcal{B}\mathcal{B}'}$ è ortogonale con $\mathcal{B}$ base ortonorm. per b e $\mathcal{B}'$ base ortonorm. per b'.

Matrici ortogonali $n \times n$

- Sono tutte invertibili
- $M^t M = (m_{ij})$

$m_{ij} = \begin{pmatrix} \text{riga } i\text{-esima} \\ \text{di } M^t \end{pmatrix} \cdot \begin{pmatrix} \text{colonna } j\text{-esima} \\ \text{di } M \end{pmatrix}$

$\quad\quad = \begin{pmatrix} \text{colonna } i\text{-esima} \\ \text{di } M \end{pmatrix} \cdot \begin{pmatrix} \text{colonna } j\text{-esima} \\ \text{di } M \end{pmatrix}$

$\Downarrow$

1 se $i = j$

0 se $i \neq j$

se M è ortogonale $\Leftrightarrow M^t M = I$

$$(M^t M) M^{-1} = I M^{-1} = M^{-1}$$

$$\parallel$$

$$M^t (M M^{-1}) = M^t I = M^t \quad \Rightarrow \quad M^{-1} = M^t$$

Teo M $n \times n$ è ortogonale $\Leftrightarrow M^{-1} = M^t$

M ortogonale $\quad M^{-1} = M^t \quad MM^t = M^t M = I$

$\Rightarrow (M^t)^{-1} = M = (M^t)^t$

Coroll $\quad M$ ortog. $\Leftrightarrow M^t$ ortog. $\Leftrightarrow (M^{-1})$ ortog.

M^t ortog.

Prop $\quad M$ ortog $\Leftrightarrow M^t$ ortog $\Leftrightarrow$ le righe R_i di M soddisfano

$$R_i \cdot R_j = \begin{cases} 0 & \text{se } i \neq j \\ 1 & \text{se } i = j \end{cases}$$

Ex $\quad M$ ortogonale $\Leftrightarrow$ M' ottenuta da M scambiando certe colonne

M' ortogonale

Lo stesso vale se M' è ottenuta da M scambiando righe

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \cos\alpha & -\operatorname{sen}\alpha \\ \operatorname{sen}\alpha & \cos\alpha \end{pmatrix} \qquad \begin{pmatrix} \operatorname{sen}\alpha & \cos\alpha \\ \cos\alpha & -\operatorname{sen}\alpha \end{pmatrix}$$

M ortogonale quadrata

$$\det M = \det M^t$$

$$M M^t = I$$

$$\det(M M^t) = \det(I) = 1$$

$$\overset{\shortparallel}{}$$

$$\det M \cdot \det M^t = (\det M)^2$$

<u>Prop</u> M ortogonale quadrata $\Rightarrow \det M = \pm 1$

$$\det \begin{pmatrix} \cos\alpha & -\operatorname{sen}\alpha \\ \operatorname{sen}\alpha & \cos\alpha \end{pmatrix} = \cos^2\alpha + \operatorname{sen}^2\alpha = 1 \qquad \det \begin{pmatrix} \operatorname{sen}\alpha & \cos\alpha \\ \cos\alpha & -\operatorname{sen}\alpha \end{pmatrix} = -1$$

$\mathrm{id}: (V,b) \to (V,b)$ isometria

Prop $f: (V,b) \to (W,b')$ $\quad g: (W,b') \to (U,b'')$ isometrie

Allora $g \circ f : (V,b) \to (U,b'')$ è isometria

dim

$\forall\, v,w \qquad b(v,w) \overset{?}{=} b''\big((g\circ f)(v), (g\circ f)(w)\big)$

$$b''\big((g\circ f)(v), (g\circ f)(w)\big) = b''\big(g(\underline{f(v)}), g(\underline{f(w)})\big) \xRightarrow{g \text{ è isometria}} b'(f(v), f(w)) \xRightarrow{f \text{ isometria}} b(v,w)$$

Corol Il prodotto di matrici ortogonali è ortogonale

Ex A, B $n \times n$ ortogonali

AB $(AB)^{-1} = B^{-1}A^{-1} = B^t A^t = (AB)^t$

$\Rightarrow$ AB è ortogonale

$\{$ matrici quadrate $n \times n \}$ prodotto righe per colonne

$\cup$

$\{$ matrici quadrate $n \times n$ det $\neq 0 \}$ GRUPPO

$\parallel$

GL_n

$\cup$

$O_n = \{$ matrici ortogonali $\} \supseteq SO_n = \{$ matrici ortogonali speciali det $= 1 \}$

SOTTOGRUPPO

$GL(n) \supseteq O(n)$ matrici ortogonali $\supseteq \{$ matrici ortog. det $= 1$ sottogruppo $\} = SO(n)$ gruppo ortogonale speciale

M ortog. $\Rightarrow$ $\det M = \pm 1$

$GL(n) \supseteq U(n) = \{$ matrici $\det^2 = 1 \} \supseteq SU(n) = \{$ matrici di det $1 \}$

$$SO(n) = SU(n) \cap O(n)$$

$\mathbb{R}^2$ $\quad$ b def. positiva $\quad$ $\mathcal{B}$ = base ortonorm (v_1, v_2)

base ortonormale (e_1, e_2) versori

$f: \mathbb{R}^2 \to \mathbb{R}^2 \qquad f(e_1) = v_1 \qquad f(e_2) = v_2$

isometria $\quad (\mathbb{R}^2, \cdot) \to (\mathbb{R}^2, b)$

$\forall v, w$

$b(f(v), f(w))$

$v = (a,b) = ae_1 + be_2 \qquad f(v) = a f(e_1) + b f(e_2) = av_1 + bv_2$

$w = (c,d) = ce_1 + de_2 \qquad f(w) = \qquad = cv_1 + dv_2$

$b(f(v), f(w)) = b(av_1 + bv_2, cv_1 + dv_2) = ac\, b(v_1, v_1) + (ad + bc)\, b(v_1, v_2) + bd\, b(v_2, v_2)$

$= ac + bd = (a,b) \cdot (c, d) = v \cdot w$

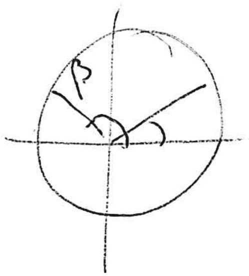
ISOMETRIA

$\underline{\beta = -\alpha}$

$$\operatorname{sen}\beta = -\operatorname{sen}\alpha \qquad \cos\beta = \cos\alpha$$

$$\begin{pmatrix} \cos\alpha & -\operatorname{sen}\alpha \\ \operatorname{sen}\alpha & \cos\alpha \end{pmatrix} \quad \text{rotazione di } \alpha$$

$\underline{\beta = 180^\circ - \alpha}$



$$\operatorname{sen}\beta = \operatorname{sen}\alpha \qquad \cos\beta = -\cos\alpha$$

$$\begin{pmatrix} \cos\alpha & \operatorname{sen}\alpha \\ \operatorname{sen}\alpha & -\cos\alpha \end{pmatrix}$$

$$\begin{pmatrix} \cos\alpha & \operatorname{sen}\alpha \\ \operatorname{sen}\alpha & -\cos\alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \operatorname{sen}\alpha & \cos\alpha \\ -\cos\alpha & \operatorname{sen}\alpha \end{pmatrix} = \cancel{\begin{pmatrix} \operatorname{sen}\beta & -\cos\beta \\ \cos\beta & \end{pmatrix}}$$

$$\gamma = 90^\circ - \alpha$$

$$\gamma' = -\gamma$$

$$\| $$

$$\begin{pmatrix} \cos\gamma & \operatorname{sen}\gamma \\ -\operatorname{sen}\gamma & \cos\gamma \end{pmatrix} = \begin{pmatrix} \cos\gamma' & -\operatorname{sen}\gamma' \\ \operatorname{sen}\gamma' & \cos\gamma' \end{pmatrix}$$

rotazione di γ'

$\mathbb{R}^2$ $\qquad$ $\mathcal{B} = (e_1, e_2)$ versori

CLASSIFICAZIONE delle ISOMETRIE f

$$M_f^{\mathcal{B}\mathcal{B}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ ortogonale}$$

$(a, c) \cdot (a, c) = 1$ $\qquad$ $(b, d) \cdot (b, d) = 1$

$(a, C) \cdot (b, d) = 0$

$a^2 + \cancel{b}c^2 = 1$ $\qquad$ $a^2 \leq 1$ $\qquad$ $-1 \leq a \leq 1$ $\qquad$ $a = \cos\alpha$ $\qquad$ $\cancel{b}c = \operatorname{sen}\alpha$

$b^2 + d^2 = 1$ $\qquad$ $b = \operatorname{sen}\beta$ $\qquad$ $d = \cos\beta$

$$\begin{pmatrix} \cos\alpha & \operatorname{sen}\beta \\ \operatorname{sen}\alpha & \cos\beta \end{pmatrix}$$

$$\underbrace{\cos\alpha \operatorname{sen}\beta + \operatorname{sen}\alpha \cos\beta}_{\operatorname{sen}(\alpha+\beta)} = 0$$

$\alpha + \beta = 0$ $\qquad$ $\alpha + \beta = 180^\circ$

$\beta = -\alpha$ $\qquad$ $\beta = 180^\circ - \alpha$

$$\begin{pmatrix} \cos\alpha & \operatorname{sen}\alpha \\ \operatorname{sen}\alpha & -\cos\alpha \end{pmatrix} f \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \text{rotazione}$$

$$\begin{pmatrix} \cos\alpha & \operatorname{sen}\alpha \\ \operatorname{sen}\alpha & -\cos\alpha \end{pmatrix} = \text{Rotazione} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \text{rotazione} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

f = rotazione o simmetria

$$\begin{pmatrix} \cos\alpha & -\operatorname{sen}\alpha & 0 \\ \operatorname{sen}\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ rotazione intorno a e_3

simmetria resp. a un piano $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

simm. rsp. a retta $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$