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libro Geometria 1 Sernesi
ed. Boringhieri

Geometria

24/25

V spazio vettoriale su K

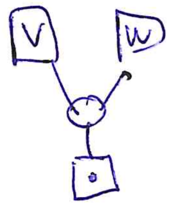
$\in$

v, w $\qquad$ $v+w$

$\alpha \in K$ $\qquad$ αv

$v \, w$?

$V = \mathbb{R}^n$



Prodotto scalare

$V \times V \to \mathbb{R}$

$V \qquad b: V \times V \to K \qquad \forall v,w \quad b(v,w) \in K$

Def Si chiama FORMA BILINEARE ogni $b: V \times V \to K$ t.c.

1) $b(v_1+v_2, w) = b(v_1, w) + b(v_2, w)$

2) $b(v, w_1+w_2) = b(v, w_1) + b(v, w_2)$

3) $b(\alpha v, w) = \alpha b(v,w) = b(v, \alpha w)$

$\forall v, v_1, v_2, w, w_1, w_2 \in V$

$\forall \alpha \in K$

NB $\forall w \in V$ ottengo $V \to K \quad v \rightsquigarrow b(v,w)$ è lineare

$\forall v \in V$ " $V \to K \quad w \rightsquigarrow b(v,w)$

Ex 1 $\cdot : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \qquad \cdot((a,b),(c,d)) = ac+bd$

$(a,b)\cdot(c,d)$

$\cdot((a,b)+(a',b'),(c,d)) \stackrel{?}{=} \cdot((a,b),(c,d)) + \cdot((a',b'),(c,d))$

$\shortparallel \qquad\qquad\qquad\qquad \shortparallel$

$\cdot((a+a', b+b'),(c,d)) \qquad = \qquad (ac+bd)+(a'c+b'd)$

$\shortparallel$

$(a+a')c+(b+b')d$

$b : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ $\qquad b((\bar a,\bar b),(\bar c,\bar d)) = 2ac - ad + 3bd$

è una forma bilineare

$$b(\alpha(\bar a,\bar b),(\bar c,\bar d)) \overset{?}{=} \alpha\, b((\bar a,\bar b),(\bar c,\bar d))$$

$$\shortparallel \qquad\qquad\qquad \shortparallel$$

$$b((\alpha\bar a,\alpha\bar b),(\bar c,\bar d)) \qquad \alpha(2\bar a\bar c - \bar a\bar d + 3\bar b\bar d)$$

$$\shortparallel \qquad\qquad\qquad \shortparallel$$

$$2\alpha\bar a\bar c + \alpha\bar a\bar d + 3\alpha\bar b\bar d = 2\alpha\bar a\bar c - \alpha\bar a\bar d + 3\alpha\bar b\bar d$$

$\Downarrow$

Commut. $\quad \forall v,w \quad b(v,w) = b(w,v)$

Diremo che b è SIMMETRICA se $\uparrow$ $\forall v,w$

$b((1,1),(2,3)) =$

$2\cdot 2 + 1\cdot 3 + 3\cdot 3 = 16$

$b((2,3),(1,1)) =$

$2\cdot 2\cdot 1 + 2\cdot 1 + 3\cdot 3 = 15$

non commut.

Prop $\quad b(0,w) = b(0v,w) = 0\, b(v,w) = 0$

$\quad b(v,0) \quad - - - - \quad = 0$

Il viceversa non è detto

Es $\quad (1,0)\cdot(0,1) = 0$

$\mathcal{B} = (v_1,\dots,v_n)$ ogni vettore si scrive in modo unico $\quad v = \alpha_1 v_1 + \dots + \alpha_n v_n$

$v \rightsquigarrow (\alpha_1, \dots \alpha_n)$ coord. $v_{\mathcal{B}}$

Ex $\quad \dim V = 2 \quad \mathcal{B} = (v_1, v_2)$ base $\quad b: V \times V \to K$

$v, w \in V$

$$b(v,w) = b(\alpha_1 v_1 + \alpha_2 v_2, \beta_1 v_1 + \beta_2 v_2) \overset{①}{=} b(\alpha_1 v_1, \beta_1 v_1 + \beta_2 v_2) + b(\alpha_2 v_2, \beta_1 v_1 + \beta_2 v_2)$$

$v = \alpha_1 v_1 + \alpha_2 v_2$

$w = \beta_1 v_1 + \beta_2 v_2$

$$\overset{②}{=} b(\alpha_1 v_1, \beta_1 v_1) + b(\alpha_1 v_1, \beta_2 v_2) + b(\alpha_2 v_2, \beta_1 v_1) + b(\alpha_2 v_2, \beta_2 v_2)$$

$$\overset{③}{=} \alpha_1\beta_1 \underbrace{b(v_1,v_1)} + \alpha_1\beta_2 \underbrace{b(v_1,v_2)} + \alpha_2\beta_1 \underbrace{b(v_2,v_1)} + \alpha_2\beta_2 \underbrace{b(v_2,v_2)}$$

$V \qquad \mathcal{B} = (v_1 \dots v_n)$

<u>Ex</u> $\mathbb{R}^n \qquad \mathcal{B}$ = base dei versori = base canonica

$M = I \qquad b((a_1 \dots a_n), (b_1 \dots b_n)) = (a_1 \dots a_n) \cdot (b_1 \dots b_n)$

$(a_1, a_2) = a_1(1,0) + a_2(0,1) \qquad \mathbb{R}^2 \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathcal{B}$ = canonica

$$b((1,0)(1,0)) = 0 = b((0\,1)(0\,1)) = 0$$
$$b((1\,0)(0\,1)) = 1 = b((0\,1)(1,0)) = 1$$

$$b((\alpha,\beta)(\gamma,\delta)) = \alpha\delta + \beta\gamma$$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$b((\alpha,\beta)(\gamma,\delta)) = \alpha\delta - \beta\gamma = \det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$V \qquad \mathcal{B} = (v_1 \dots v_n) \qquad b$ forma bilineare

b simmetrica $\Longleftrightarrow$ $M_b^{\mathcal{B}} = (m_{ij})$

$$m_{ij} = b(v_i, v_j) = b(v_j, v_i) = m_{ji} \Rightarrow M_b^{\mathcal{B}} \text{ è simmetrica}$$

$$b(v,w) = \sum \alpha_i \beta_j \, b(v_i, v_j) = \sum \beta_j \alpha_i \, b(v_j, v_i) = b(w,v)$$

$v = \alpha_1 v_1 + \dots + \alpha_n v_n$

$w = \beta_1 v_1 + \dots + \beta_n v_n$

Prop $\quad b(v,w) = \underbrace{(\alpha_1 \dots \alpha_n)}_{v_{\mathcal{B}}^t} M_b^{\mathcal{B}} \underbrace{\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}}_{w_{\mathcal{B}}} = v_{\mathcal{B}}^t M_b^{\mathcal{B}} w_{\mathcal{B}}$

$$b(v,w) = \underbrace{v_{\mathcal{B}}^t}_{1\times n}\ \underbrace{M_b^{\mathcal{B}}}_{n\times n}\ \underbrace{w_{\mathcal{B}}}_{n\times 1}$$

$$\|$$

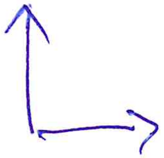
$$\left(b(v,w)\right)^t = \left(v_{\mathcal{B}}^t M_b^{\mathcal{B}} w_{\mathcal{B}}\right)^t = \underbrace{w_{\mathcal{B}}^t}\ \underbrace{\left(M_b^{\mathcal{B}}\right)^t}_{= M_b^{\mathcal{B}}}\ \underbrace{\left(v_{\mathcal{B}}^t\right)^t}_{= v_{\mathcal{B}}}$$

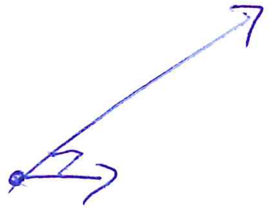
$$= w_{\mathcal{B}}^t M_b^{\mathcal{B}} v_{\mathcal{B}} = b(w,v)$$

se $M_b^{\mathcal{B}}$ simm. $\left(M_b^{\mathcal{B}}\right)^t = M_b^{\mathcal{B}}$

$\Updownarrow$

b simm.





$$\begin{pmatrix} 1 & \boxed{0} \\ 0 & 1 \end{pmatrix}$$

V $\quad \dim V = n$ $\quad \{\text{forme bilineari su } V\}$ spazio vett. di dim. n^2

$\mathcal{B}$

$$\Updownarrow$$

$$\{\text{matrici } n \times n\}$$

$$(b+b')(v,w) = b(v,w) + b'(v,w)$$

$$(\alpha b)(v,w) = \alpha\big(b(v,w)\big)$$

Oss $b(\alpha_1 v_1 + \dots + \alpha_n v_n, w) = \alpha_1 b(v_1, w) + \dots + \alpha_n b(v_n, w)$

$b(v, \beta_1 w_1 + \dots + \beta_n w_n) = \beta_1 b(v, w_1) + \dots + \beta_n b(v, w_n)$

V $\mathcal{B} = (v_1 \dots v_n)$ base

$v = \alpha_1 v_1 + \dots + \alpha_n v_n$

$w = \beta_1 v_1 + \dots + \beta_n v_n$

$\to (\alpha_1, \dots, \alpha_n) = v_{\mathcal{B}}$

$$b(v, w) = \sum_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \alpha_i \beta_j \underline{b(v_i, v_j)}$$

$b(v_i, v_j)$

$M^{\mathcal{B}}_{\mathcal{B}} = (m_{ij})$ $m_{ij} = b(v_i, v_j)$

Prop Dati V e $\mathcal{B} = (v_1 \dots v_n)$ base

Comunque scelti gli scalari $b(v_i, v_j)$ $\forall i, j$

$\exists!$ forma bilineare b su V che soddisfa

$\forall$ $v = \alpha_1 v_1 + \dots + \alpha_n v_n$
$w = \beta_1 v_1 + \dots + \beta_n v_n$ pongo $b(v,w) = \sum \alpha_i \beta_j b(v_i, v_j)$

$v' = \alpha'_1 v_1 + \dots + \alpha'_n v_n$

$$\underline{b(v+v', w)} = \sum (\alpha_i + \alpha'_i) \beta_j \, b(v_i, v_j) = \sum (\alpha_i \beta_j + \alpha'_i \beta_j) \, b(v_i, v_j)$$

$$= \sum \left[\alpha_i \beta_j b(v_i, v_j) + \alpha'_i \beta_j b(v_i, v_j) \right] = \sum \alpha_i \beta_j b(v_i, v_j) + \sum \alpha'_i \beta_j b(v_i, v_j)$$

$$= \underline{b(v,w) + b(v', w)}$$

$$v + v' = (\alpha_1 v_1 + \dots + \alpha_n v_n) + (\alpha'_1 v_1 + \dots + \alpha'_n v_n) = (\alpha_1 + \alpha'_1) v_1 + \dots + (\alpha_n + \alpha'_n) v_n$$

$$a v = a (\alpha_1 v_1 + \dots + \alpha_n v_n) = (a\alpha_1) v_1 + \dots + (a \alpha_n) v_n$$

$V \qquad \mathcal{B}=(v_1,\dots,v_n) \qquad M = M_b^{\mathcal{B}}$

$$M = I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \qquad b(v_i,v_j) = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$v = \alpha_1 v_1 + \dots + \alpha_n v_n$

$w = \beta_1 v_1 + \dots + \beta_n v_n$

$$b(v,w) = \sum \alpha_i \beta_j \, b(v_i,v_j)$$

$$= \sum \alpha_i \beta_i \, b(v_i,v_i) = \sum \alpha_i \beta_i$$

$$= (\alpha_1,\dots,\alpha_n)\cdot(\beta_1 \dots \beta_n) = \underline{\underline{v_{\mathcal{B}} \cdot w_{\mathcal{B}}}}$$

$M = (0)$

$b(v_i,v_j)=0$ sempre

$$b(v,w) = \sum \alpha_i \beta_j \, b(v_i,v_j) = 0 \quad \forall v,w$$

<u>forma bilineare nulla</u>

Def V $\quad b$ = forma bilin.

Diremo che v, w sono ortogonali secondo b

$$v \perp_b w \iff b(v,w)=0$$

Inoltre $\forall v \in V$ si pone l'ortogonale di v secondo b

$$v^{\perp_b} = \{ w : \ b(v,w)=0 \ \text{ cioè } \ v \perp_b w \}$$

NB In generale se $v \perp_b w$ non è detto che $w \perp_b v$

$\mathbb{R}^2$ $\quad \mathcal{B} = \text{canonica} = \{v_1, v_2\}$ $\quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = M_b^{\mathcal{B}}$ $\quad b(v_2, v_1) = 0$, $\; v_2 \perp_b v_1$ $\quad b(v_1, v_2) = 1$, $\; v_1 \not\perp_b v_2$

Prop Se b è simm. allora $v \perp_b w \iff w \perp_b v$

V $\quad b$

$\exists\, v \quad b(v,v)=0$? $\quad v \perp_b v$

certo se $v=0$ $\quad b(0,0)=0$

$v \neq 0$

Ex $\quad \mathbb{R}^2 \quad b = \cdot \quad v=(a,b) \quad \underset{\underset{b(v,v)}{\shortparallel}}{v\cdot v} = a^2+b^2$ se $v\neq 0$ $\quad v\cdot v \neq 0$

$\mathbb{C}^2 \quad b=\cdot \quad v=(a,b) \quad v\cdot v = a^2+b^2=0$

esempio $v=(1,i)$

$v\cdot v = 1\cdot 1 + i\cdot i = 1+i^2 = 0$

Ex $\quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad b((a,b)\,(c,d)) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$b((a,b)\,(a,b)) = \det\begin{pmatrix} a & b \\ a & b \end{pmatrix} = 0$

Def Diremo che v è ISOTROPO risp. b se $b(v,v)=0$ cioè $v \perp_b v$.

$M_b^{\mathcal{B}}$ $\begin{pmatrix} 0 & \\ & \end{pmatrix}$ $b(v_1,v_1)=0$

se sulla diag. di $M_b^{\mathcal{B}}$ ci sono 0 allora $\exists$ vettori isotropi non nulli

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Prop v $\quad v^{\perp_b} = \{w : b(v,w)=0\}$ è un sottospazio

dim

(1) $w_1, w_2 \in v^{\perp_b}$

$\Downarrow$

$b(v,w_1)=0=b(v,w_2)$

$w_1+w \overset{?}{\in} v^{\perp_b}$

$$b(v, w_1+w_2) \overset{?}{=} 0$$
$$\| \qquad\qquad \|$$
$$b(v,w_1)+b(v,w_2)$$
$$\underset{0}{\|} \qquad \underset{0}{\|}$$

(2) $w_1 \in v^{\perp_b}$

$\alpha \in K$

$\alpha w_1 \overset{?}{\in} v^{\perp_b}$

$b(v,\alpha w_1) \overset{?}{=} 0$

$$b(v,\alpha w_1) = \alpha \overset{\overset{0}{\|}}{b(v,w_1)} = 0$$

Ex $V = \mathbb{R}^2$ $\quad \mathcal{B} = (v_1, v_2)$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$b(v_1, v_1) = 1 \qquad b(v_2, v_2) = -1 \qquad b(v_1, v_2) = b(v_2, v_1) = 0$

$v_1 + v_2$

$$b(v_1+v_2, v_1+v_2) = b(v_1, v_1+v_2) + b(v_2, v_1+v_2)$$
$$= b(v_1,v_1) + b(v_1,v_2) + b(v_2,v_1) + b(v_2,v_2)$$
$$= \underset{1}{b(v_1,v_1)} + 2\underset{0}{b(v_1,v_2)} + \underset{-1}{b(v_2,v_2)} = 0$$

$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$V^{\perp_b}$

Oss v è isotropo $\iff$ $v \in v^{\perp_b}$

$V \quad b \qquad W$ sottospazio

$$W^{\perp_b} = \{ w : \forall v \in W \quad b(v,w)=0 \quad \text{cioè} \quad v \perp_b w \}$$

Prop Se $W = L(v_1, \dots, v_k)$ allora $W^{\perp_b} = v_1^{\perp_b} \cap v_2^{\perp_b} \cap \dots \cap v_k^{\perp_b}$

in particolare $W^{\perp_b}$ è un sottosp.

dim

Sia $u \in W^{\perp_b}$ $\quad \forall v \in W \quad b(v,u)=0$ in particolare

$$b(v_1, u) = 0 \quad \dots \quad b(v_k, u) = 0$$
$$\Downarrow \qquad\qquad\qquad \Downarrow$$
$$u \in v_1^{\perp_b} \qquad u \in v_k^{\perp} \Rightarrow u \in v_1^{\perp_b} \cap \dots \cap v_k^{\perp_b}$$

$$\Rightarrow \quad W^{\perp_b} \subseteq v_1^{\perp_b} \cap \dots \cap v_k^{\perp_b}$$

Viceversa $u \in v_1^{\perp_b} \cap \dots \cap v_k^{\perp_b}$ cioè $b(v_1,u)=0 \quad \dots \quad b(v_k,u)=0$

$\forall v \in W \quad v = \alpha_1 v_1 + \dots + \alpha_k v_k$

$$b(v,u) = b(\alpha_1 v_1 + \dots + \alpha_k v_k, u) = \alpha_1 \underset{0}{\underset{\parallel}{b(v_1,u)}} + \dots + \alpha_k \underset{0}{\underset{\parallel}{b(v_k,u)}} = 0 \quad \Rightarrow u \in W^{\perp_b}$$

N.B. $\quad V^{\perp_b} = L(v)^{\perp_b}$

Ex $\quad \mathbb{R}^2 \quad \mathcal{B} =$ base canon. $\quad M_b^{\mathcal{B}} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

$(1,1)^{\perp_b} = \{ w = (c,d) \ : \ b((1,1)\,(c,d)) = 0 \}$

$$(1,1)\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = (3,6)\begin{pmatrix} c \\ d \end{pmatrix} = 3c+6d = 0$$

$$c = -2d$$

$= \{(-2d,d) : d \in \mathbb{R}\} = L(-2,1)$

$(0,0)^{\perp_b} = \mathbb{R}^2$

$(2,-1)^{\perp_b} = \{(c,d) : b((2,-1),(c,d)) = 0 \}$ $\qquad (2,-1)^{\perp} = \mathbb{R}^2$

$$(2,-1)\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = (0,0)\begin{pmatrix} c \\ d \end{pmatrix}$$

$\Uparrow$

$0 = 0$

Oss $v^{\perp_b}$ — dim $n-1$ $\quad n = \dim V$

— dim n cioè $v^{\perp_b} = V$

Def Radicale di V risp. b $=$ Rad(b)

$= \{v : \forall w \quad b(v,w) = 0\}$ cioè $\{v : v^{\perp_b} = V\}$

$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = M_b^{\mathcal{B}}$ $\qquad (2,-1) \in \mathrm{Rad}(b)$

$(a,b) \in \mathrm{Rad}(b) \Leftrightarrow \forall (c,d) \quad b((a,b),(c,d)) = 0$

$$(a,b)\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = (a+2b, 2a+4b)\begin{pmatrix} c \\ d \end{pmatrix} = 0$$

cioè $(a+2b)c + (2a+4b)d = 0 \quad \forall (c,d)$

$$\Rightarrow \begin{cases} a+2b = 0 \\ 2a+4b = 0 \end{cases} \qquad \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right)$$

$M_b^{\mathcal{B}}$ $\qquad v \in \mathrm{Rad}(b)$

$$v_{\mathcal{B}}^t \, M_b^{\mathcal{B}} \, w_{\mathcal{B}} = 0 \quad \forall w$$

$$\Rightarrow \quad v_{\mathcal{B}}^t M_b^{\mathcal{B}} = \text{vettore nullo} = 0$$

$$\left(v_{\mathcal{B}}^t M_b^{\mathcal{B}}\right)^t = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\|$$

$$\left(M_b^{\mathcal{B}}\right)^t v_{\mathcal{B}}$$

$\Rightarrow$ $v_{\mathcal{B}}$ è soluz. del sist. lin. omog. che ha $\left(M_b^{\mathcal{B}}\right)^t$ per matrice.

<u>Prop</u> $\quad \mathrm{Rad}(b) \ni 0$ inoltre

$\mathrm{Rad}(b) \ni$ vettori non nulli $\iff \left(\left(M_b^{\mathcal{B}}\right)^t \middle| \, 0 \right)$ ha soluz. non nulle

$$\iff \text{rango} \left(M_b^{\mathcal{B}}\right)^t < n \iff \det \left(M_b^{\mathcal{B}}\right)^t = 0$$

$$\det \left(M_b^{\mathcal{B}}\right)^t = \det \left(M_b^{\mathcal{B}}\right)$$

<u>Def</u> Diremo che una forma bilin. b è DEGENERE se $Rad(b) \ni$ vettori non nulli

Ciò avviene quando $\det M_b^{\mathcal{B}} = 0$.