Oss Forme bilineari su $\mathbb{C}$

V $\quad \dim V \geq 2$ $\quad b$ forma bilin.

v, w lin. indip. $\quad b(v+\alpha w, v+\alpha w) = b(v,v) + 2\alpha\, b(v,w) + \alpha^2 b(w,w)$

- $b(w,w) = 0$ $\quad w$ isotropo
- $b(w,w) \neq 0$ $\quad \exists\, \alpha \in \mathbb{C}$ t.c. $b(v+\alpha w, v+\alpha w) = 0$.

  $v+\alpha w$ è isotropo

v, w lin. indp. $\Rightarrow \circ^{*}$

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se $b(v,v) < 0$ moltiplicando v per $\dfrac{1}{\sqrt{b(v,v)}}$

si otteneva v' t.c. $b(v', v') = -1$

ma allora $b(iv', iv') = i^2\, b(v', v') = 1$

max dim W t.c. $b_{|W}$ non-deg.

$\exists$ base per cui la matrice è $\left(\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \\ \hline & \begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix} \end{array}\right)$ dim Rad(b)

V di dim 2 (possiamo pensare $V = \mathbb{R}^2$)

b forma bilineare simm. $\mathcal{B}$ base $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ matrice associata

$Q_b(v)$ $\quad v_{\mathcal{B}} = (x, y)$

$\shortparallel$

$b(v,v) = (x,y) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2$ $\qquad \begin{array}{c|cc} & x & y \\ \hline x & a & b \\ y & b & c \end{array}$

Viceversa $\underline{ax^2 + 2bxy + cy^2} \Rightarrow$ forma quadratica $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

Studiamo il luogo dei punti ISOTROPI Z in $\mathbb{P}^1_{\mathbb{C}}$

dim Rad(b) $= 2 -$ rango M:

- ~~dim Rad~~ 2 $\leftrightarrow$ rango $M = 0$ $\leftrightarrow$ forma nulla $Z = \mathbb{P}^1_{\mathbb{C}}$
- 1 rango $M = 1$ $\det M = 0$ $= ac - b^2 = -\Delta/4$ $\quad Z =$ un punto (doppio)
- 0 rango $M = 2$ $\det M \neq 0$ $= -\Delta/4$ $\quad Z =$ due punti distinti

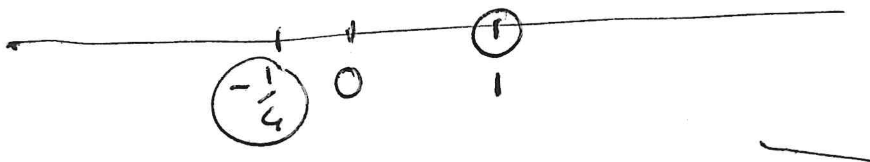
Ex $x^2+3xy-4y^2$ $\qquad \begin{pmatrix} 1 & 3/2 \\ 3/2 & -4 \end{pmatrix}$ $\qquad$ det $\neq 0$

2 punti distinti in $\mathbb{P}^1_{\mathbb{C}}$

$$x = \frac{-3y \pm \sqrt{9y^2+16y^2}}{2}$$

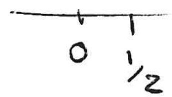
$$x = \frac{-3y+5y}{2} = y \qquad \{(y,y) : y \in \mathbb{C}\} \qquad (1,1)$$

$$x = \frac{-3y-5y}{2} = -4y \qquad \{(-4y,y) : y \in \mathbb{C}\} \qquad (-4,1)$$



$x^2-4xy+4y^2 \qquad \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} = M \qquad$ det $= 0$

$\quad \overset{\shortparallel}{(x-2y)^2} \qquad x = 2y \qquad \{(2y,y) : y \in \mathbb{C}\} \qquad (2,1)$



Rad b $\quad \left(\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 4 & 0 \end{array}\right) \to x = 2y$

$x^2 + 3xy + 4y^2 \qquad M = \begin{pmatrix} 1 & 3/2 \\ 3/2 & 4 \end{pmatrix} \qquad \det \neq 0$ (80)

$$x = \frac{-3y \pm \sqrt{9y^2 - 16y^2}}{2} = \begin{cases} -3y + yi\sqrt{7} \\ -3y - yi\sqrt{7} \end{cases}$$

$$\underset{\overset{\parallel}{(x-iy)^2}}{x^2 - 2ixy - y^2} \qquad \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \qquad \det = 0$$

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su $\mathbb{P}^1_{\mathbb{R}}$

$$Z = \begin{cases} 2 \text{ punti reali} \\ \emptyset \quad (2 \text{ punti complessi coniugati}) \end{cases} \quad \det \neq 0$$

$$\begin{cases} 1 \text{ punto reale (doppio)} \\ \mathbb{P}^1_{\mathbb{R}} \quad \text{se} \quad M = (0) \end{cases} \quad \det M = 0$$

2 punti reali $\iff \det < 0$ autovalori discordi segnatura $(1,0,1)$

$\emptyset \iff \det > 0$ autoval. concordi b definita

no vettori isotropi $\neq 0$

La discussione conclude lo studio di $ax^2+2bxy+cy^2$

Polinomi di 2° grado non omogenei

$\mathbb{A}^1$ $\underbrace{ax^2+2bx+c}$ $\Rightarrow$ $ax^2+2bxy+y^2$

$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$x=\frac{x_1}{x_0}$ $\quad a\frac{x_1^2}{x_0^2}+2b\frac{x_1}{x_0}+c$ $\quad \mathbb{A}^1 \to \mathbb{P}^1$

$ax_1^2+2bx_1x_0+cx_0^2$

$\forall (a,b,c) \Rightarrow ax^2+bx+c \longrightarrow ax_1^2+bx_1x_0+cx_0^2$

$$\begin{array}{ccc} & x_0 & x_1 \\ x_0 & \multicolumn{2}{c}{\multirow{2}{*}{}} \end{array} \begin{pmatrix} 2 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

$(0,1,2)$ $\quad x+2$ $\quad x_1x_0+2x_0^2$

$\underline{x=-2}$ $\quad x_0(x_1+2x_0)$ $\quad \det\neq 0$

$x_0=0$

$x_1=-2x_0$

∞ $\quad (0,x_1)$ $\quad (0,1)$

$-2 \quad 0$ $\quad (x_0,-2x_0)$ $\quad (1,-2)$

$a = b = 0$ $\quad c \neq 0$ $\quad \underline{c x_0^2 = 0}$ $\quad \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \det = 0$ (82)

$\emptyset$ in $\mathbb{A}^1$

1 soluz. doppia $x_0 = 0$

$(0, x_1)$

∞ in $\mathbb{P}^1$

doppio

$\mathbb{P}^1_{\mathbb{R}}$ $\quad \mathbb{P}^1_{\mathbb{C}}$

$\mathbb{A}^1_{\mathbb{R}}$ $\quad \mathbb{A}^1_{\mathbb{C}}$

CONICHE

LUOGO DEI PUNTI DEL PIANO DEFINITI DALL'ANNULLARSI DI UN POLINOMIO DI 2° GRADO

$\mathbb{P}^2_{\mathbb{C}}$ $\quad$ $\mathbb{P}^2_{\mathbb{R}}$

$\mathbb{A}^2_{\mathbb{C}}$ $\quad$ $\mathbb{A}^2_{\mathbb{R}}$

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F \neq \text{polinomio nullo}$$

$$x = \frac{x_1}{x_0} \qquad y = \frac{x_2}{x_0}$$

$$A\frac{x_1^2}{x_0^2} + B\frac{x_1x_2}{x_0^2} + C\frac{x_2^2}{x_0^2} + D\frac{x_1}{x_0} + E\frac{x_2}{x_0} + F$$

moltiplico per x_0^2

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1x_0 + Ex_2x_0 + Fx_0^2$$

$$(x_1, x_2, x_0)\, M \begin{pmatrix} x_1 \\ x_2 \\ x_0 \end{pmatrix}$$

$$\begin{array}{c|c} & x_1 \quad x_2 \quad x_0 \\ \begin{matrix} x_1 \\ x_2 \\ x_0 \end{matrix} & \begin{pmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{pmatrix} \end{array} = M$$

$$\begin{array}{c|c} & x \quad y \quad 1 \\ \begin{matrix} x \\ y \\ 1 \end{matrix} & \begin{pmatrix} & & \bigcirc \\ & & \\ & & \end{pmatrix} \end{array}$$

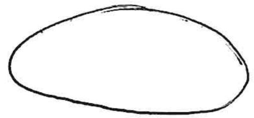
Ex

$$x^2+y^2-1 \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & & -1 \end{pmatrix}$$ circonferenza

$$X^2+Y^2+1 \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ circonferenza (immaginaria)

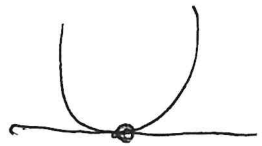
$$x_1^2+x_2^2+x_0^2$$

$$X^2+2y^2-1=0 \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$ ellisse



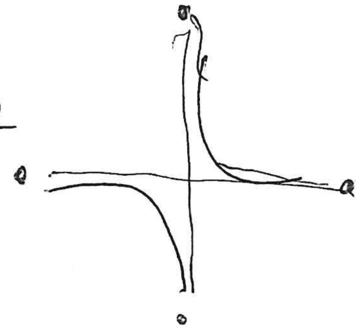
$$y=x^2 \qquad x^2-y$$

$$x_1^2-x_0x_2 \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix}$$ parabola



$$xy=1 \qquad xy-1=0$$

$$x_1x_2-x_0^2 \qquad \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$ iperbole



All'∞

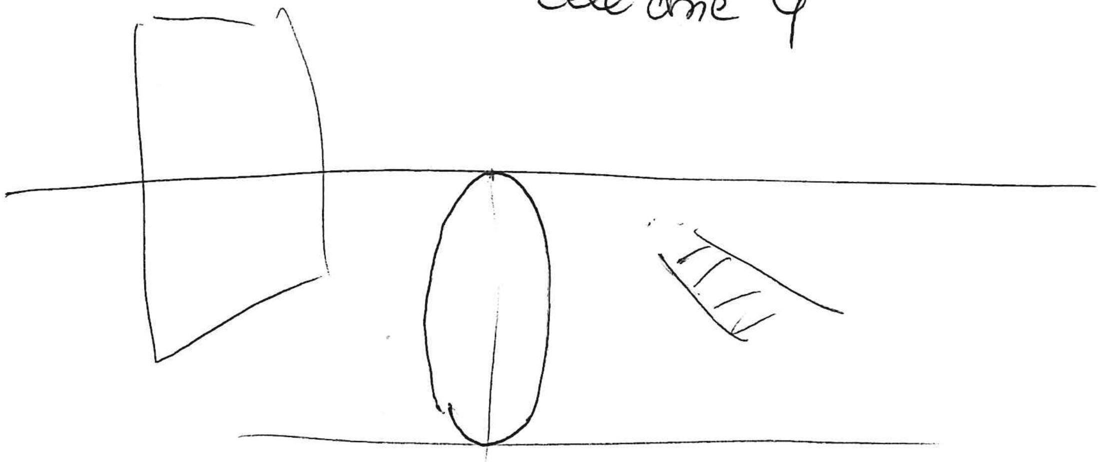
parabola

$$\begin{cases} x_1^2 - x_0 x_2 = 0 \\ x_0 = 0 \end{cases} \qquad \begin{cases} x_1^2 = 0 \\ x_0 = 0 \end{cases} \qquad \begin{cases} x_1 = 0 \\ x_0 = 0 \end{cases}$$

doppio

$$\begin{array}{ccc} x_0 & x_1 & x_2 \\ \end{array}$$
$$(0 \quad 0 \quad 1)$$

punto all'∞ dell'asse y

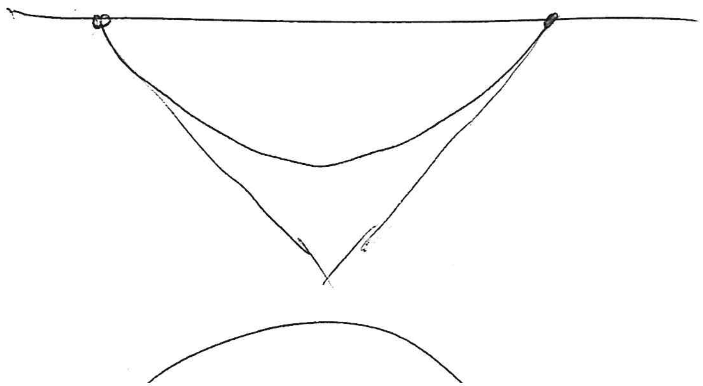


$xy = 1$

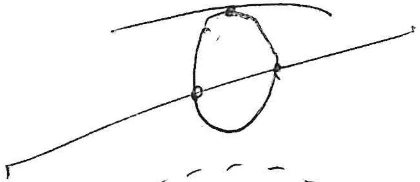
$$\begin{cases} x_1 x_2 - x_0^2 = 0 \\ x_0 = 0 \end{cases} \qquad \begin{cases} x_1 x_2 = 0 \\ x_0 = 0 \end{cases} < \begin{matrix} x_1 = 0 \\ \text{oppure} \\ x_2 = 0 \end{matrix}$$

$$\begin{array}{ccc} x_0 & x_1 & x_2 \\ \end{array}$$
$(0 \quad 0 \quad 1)$ ∞ asse y

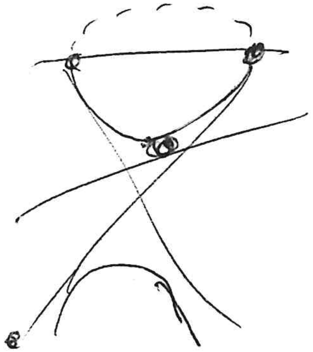
$(0 \quad 1 \quad 0)$ ∞ asse x



parabola



iperbole



ellisse

$x^2+2y^2-1=0$ $\quad \begin{cases} x_1^2+2x_2^2-x_0^2=0 \\ x_0=0 \end{cases}$ $\quad \begin{cases} x_1^2+2x_2^2=0 \\ x_0=0 \end{cases}$ $\quad x_1^2=-2x_2^2$

$x_1=\pm i\sqrt{2}\, x_2$

$(0, i\sqrt{2}, 1)$

$(0, -i\sqrt{2}, 1)$