Correction

① $f: X \to Y$ continuous + surj.

problem prove that X conn. $\Rightarrow$ Y conn.

i.e. Y disconn. $\Rightarrow$ X disconn.

$\exists$ A_1, A_2 open s.t. $A_1 \cup A_2 = Y$, $A_1 \cap A_2 = \emptyset$, $A_1, A_2 \neq \emptyset$

$f^{-1}(A_1), f^{-1}(A_2)$ open (f cont.)

$\Rightarrow$ $f^{-1}(A_1) \cup f^{-1}(A_2) = f^{-1}(A_1 \cup A_2) = X$

$f^{-1}(A_1) \cap f^{-1}(A_2) = f^{-1}(A_1 \cap A_2) = \emptyset$

$\exists y_1 \in A_1, y_2 \in A_2 \overset{surj}{\Rightarrow} \exists x_1 \in X$ s.t. $f(x_1) = y_1$

hence $x_1 \in f^{-1}(A_1) \neq \emptyset$

and similarly $f^{-1}(A_2) \neq \emptyset$

② X pathwise conn $\Rightarrow$ Y pathwise conn

$y_1, y_2 \in Y$ $\quad \exists x_1, x_2 \in X$ s.t. $f(x_1) = y_1$ $f(x_2) = y_2$

$\exists \alpha : [0,1] \to X$ path $\quad \alpha(0) = x_1 \quad \alpha(1) = x_2$

$f \circ \alpha : [0,1] \to Y$ continuous $\quad f\circ\alpha(0) = y_1 \quad f \circ \alpha(1) = y_2$

$f \circ \alpha$ = path from y_1 to y_2

③ Find examples of $f: X \to Y$ (not surj.)

s.t. X conn and Y not-conn

X pathwise conn. and Y not pathwise conn.

both obtained if X pathwise conn. and
Y not conn.

$X =$ x-axis in $\mathbb{R}^2$ $\quad Y = X \cup \{(0,1)\}$
pathwise conn. $\qquad$ not conn.

$f : X \to Y \quad f(x,0) \mapsto (x,0)$ continuous

(4) $g : Y \to X$ cont. $\quad Y$ simply conn.
X not simply conn.
Prove that $g \circ f$ not homotopic to 1_X

(4.a) X pathwise conn.

$$X \xrightarrow{f} Y \xrightarrow{g} X$$

$$\pi(X) \xrightarrow{\pi(f)} \underset{(e)}{\overset{}{\pi(Y)}} \xrightarrow{\pi(g)} \pi(X)$$

If $g \circ f \sim 1_X \Rightarrow \underbrace{\pi(g) \circ \pi(f)}_{\pi(g \circ f)} = 1_{\pi(X)}$

But the composition $\pi(g) \circ \pi(f)$ sends everything to the neutral elem. in $\pi(X)$.
Since $\pi(X) \neq (e)$ $\quad \pi(g) \circ \pi(f)$ cannot be $1_{\pi(X)}$

(4.b) X not pathwise conn.

$g(Y) \subseteq$ one pathwise conn. component X' of X

$g \circ f(X) \subseteq X' \neq X$

If $\exists$ H homotopy between $g \circ f$ and 1_X

$H : X \times [0,1] \to X \quad H(p,0) = g \circ f(p)$
$H(p,1) = 1_X(p) = p$

let $X'' \neq X'$ be another pathwise conn. component in X

$X'' \times [0,1]$ is pathwise conn.

$H(X'' \times [0,1])$ is contained in <u>one</u> pathwise conn. comp. of X

but for $p \in X''$ $\quad H(p,0) = g \circ f(p) \in X'$
$(p,0), (p,1) \in X'' \times [0,1] \quad H(p,1) = 1_X(p) \in X''$ $\Big|$ contradiction

(5) X pathwise conn. $\exists\, p \in X$ s.t.
$X - \{p\}$ has two conn. comp. X_1, X_2
prove that if $x_1 \in X_1$, $x_2 \in X_2$ and
$\alpha : [0,1] \to X$ is a path joining x_1, x_2 then
$\exists\, t \in [0,1]$ s.t. $\alpha(t) = p$

Argue by contradiction: if $\alpha(t) \neq p \ \forall t$
then $\alpha : [0,1] \to X - \{p\}$ and it cannot
join x_1, x_2 since $\alpha[0,1]$ is contained
in <u>one</u> conn. component of $X - \{p\}$

o

c