X, Y path. conn.

- $X \xrightarrow{f} Y$ continuous

$$\pi(X) \xrightarrow{\pi(f)} \pi(Y)$$

$[\alpha] \in \pi(X)$ $\qquad \pi(f)[\alpha] = [f \circ \alpha]$

$\pi(id)$ $\qquad X \xrightarrow{id} X$ $\qquad \boxed{\pi(id) = id_{\pi(X)}}$

$$\pi(X) \ni [\alpha] \rightsquigarrow [id \circ \alpha] = [\alpha]$$

$X \xrightarrow{f} Y \xrightarrow{g} Z$ $\qquad \boxed{\pi(g \circ f) = \pi(g) \circ \pi(f)}$

$$\pi(X) \xrightarrow{\pi(f)} \pi(Y) \xrightarrow{\pi(g)} \pi(Z)$$

$$\pi(g \circ f)[\alpha] = [(g \circ f) \circ \alpha]$$

$$(\pi(g) \circ \pi(f))[\alpha] = \pi(g)(\pi(f)[\alpha])$$

$$= \pi(g)[f \circ \alpha] = [g \circ (f \circ \alpha)]$$

<u>Corl</u> If X, Y homeom. then $\pi(X)$, $\pi(Y)$ isomorphic. $\pi(f)^{-1} = \pi(f^{-1})$

<u>Pf</u> $f: X \to Y$ $\qquad f^{-1}: Y \to X$

$\pi(f): \pi(X) \to \pi(Y)$ $\qquad \pi(f^{-1}): \pi(Y) \longrightarrow \pi(X)$

$$\pi(f^{-1}) \circ \pi(f) = \pi(f^{-1} \circ f) = \pi(id_X) = id_{\pi(X)}$$

$$\pi(f) \circ \pi(f^{-1}) = id_{\pi(Y)}$$

Prop

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \quad f \sim g$$

$$\pi(X) \underset{\pi(g)}{\overset{\pi(f)}{\rightrightarrows}} \pi(Y).$$ Then $\pi(f) = \pi(g)$.

$\forall \alpha$

$$\left.\begin{array}{l} \pi(f)[\alpha] = [f \circ \alpha] \\ \pi(g)[\alpha] = [g \circ \alpha] \end{array}\right] = \quad \begin{array}{l} f \sim g \Rightarrow \\ \Leftarrow f\circ\alpha \sim g \circ \alpha \end{array}$$

Coral If $X \sim Y$ then $\pi(X), \pi(Y)$ are isomorphic.

pf $X \sim Y \quad \exists\, f: X \to Y \quad g: Y \to X$

s.t. $g \circ f \sim id_X \quad f \circ g \sim id_Y$

$\pi(f): \pi(X) \to \pi(Y) \quad \pi(g): \pi(Y) \to \pi(X)$

$\pi(g) \circ \pi(f) = \pi(g \circ f) = \pi(id_X) = id_{\pi(X)}$

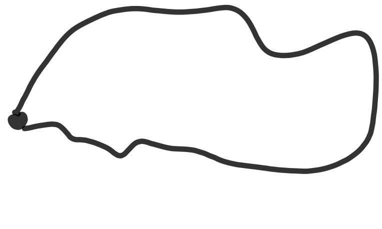
$\pi(f) \circ \pi(g) = id_{\pi(Y)}$

If $X = \{0\}$ $\pi(X) = (e)$ $[e] = [\text{constant loop}]$

As a consequence, since $\mathbb{R}^2 \sim \{0\}$

then $\pi(\mathbb{R}^2) = (e)$

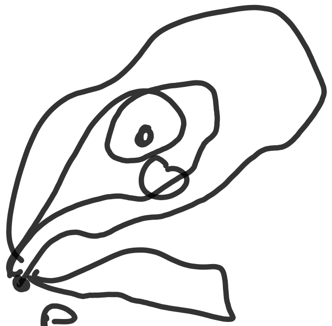
all loops are homotopic $\circ$



In general $\pi(\mathbb{R}^n) = (e)$

because $\mathbb{R}^n \sim \{0\}$

FACT $X = \mathbb{R}^2 - \{0\}$ then $\pi(X) \neq (e)$



(observe that, on the contrary $\pi(\mathbb{R}^3 - \{0\}) = (e)$)

p

In particular $\mathbb{R}^3 - \{0\}$ cannot be homotopy equivalent to $\mathbb{R}^2 - \{0\}$

Notice that $\mathbb{R}^3 \sim \mathbb{R}^2$

$f : \mathbb{R}^2 \to \mathbb{R}^3 \quad f(x,y) = (x,y,0)$

$g : \mathbb{R}^3 \to \mathbb{R}^2 \quad f(x,y,z) = (x,y)$

$g \circ f = id_{\mathbb{R}^2} \qquad f \circ g(x,y,z) = f(x,y) = (x,y,0) \neq id_{\mathbb{R}^3}$

$f \circ g \sim id_{\mathbb{R}^3}$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, shift left, "f \circ g"] \arrow[r, shift right, "id"'] & \mathbb{R}^3 \end{tikzcd}$$

need $H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ homotopy $f \circ g \sim id_{\mathbb{R}^3}$

$H(x,y,z,t) = (x,y,zt)$

$t=0 \quad H(x,y,z,0) = (x,y,0) = f \circ g(x,y,z)$

$t=1 \quad H(x,y,z,1) = (x,y,z) = id(x,y,z)$

- All spaces $\mathbb{R}^n$ are homotopic

Are $\mathbb{R}^2$, $\mathbb{R}^3$ homeomorphic? $\boxed{\text{NO}}$

proof Assume $\exists$ a homeom.

$f: \mathbb{R}^2 \to \mathbb{R}^3$ $\qquad f^{-1}: \mathbb{R}^3 \to \mathbb{R}^2$

$f(0) = p \in \mathbb{R}^3$

$f|_{\mathbb{R}^2 - \{0\}} : \mathbb{R}^2 - \{0\} \to \mathbb{R}^3 - \{p\}$

$f^{-1}|_{\mathbb{R}^3 - \{p\}} : \mathbb{R}^3 - \{p\} \to \mathbb{R}^2 - \{0\}$

homeom.

if $\mathbb{R}^2$, $\mathbb{R}^3$ homeom. $\Rightarrow$ $\mathbb{R}^2 - \{0\}$, $\mathbb{R}^3 - \{p\}$ are homeom.

impossible since $\pi_1(\mathbb{R}^2 - \{0\}) \neq \pi_1(\mathbb{R}^3 - \{p\})$

Rem With the same trick one proves that $\mathbb{R}$, $\mathbb{R}^2$ are not homeom.

$\mathbb{R} - \{0\}$ disconn. while $\mathbb{R}^2 - \{p\}$ conn.

Prop $\pi(X\times Y) = \pi(X)\times\pi(Y)$

$$\begin{tikzcd} & X\times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

$P_0 \in X \quad Q_0 \in Y$

$(P_0, Q_0) \in X\times Y$

$$\pi(X, P_0) \times \pi(Y, Q_0) \xleftarrow{\pi(\pi_1)\times\pi(\pi_2)} \pi(X\times Y, (P_0,Q_0))$$

$$\begin{array}{ccccc} \pi(X,P_0) & \times & \pi(Y,Q_0) & \longrightarrow & \pi(X\times Y, (P_0,Q_0)) \\ \Psi & & \Psi & & \Psi \\ [\alpha] & & [\beta] & & [\alpha\times\beta] \end{array}$$

$\alpha : [0,1] \to X$

$\beta : [0,1] \to Y$

$\alpha\times\beta(t) = (\alpha(t), \beta(t)) \in X\times Y.$

Since $f\sim f'$ $g\sim g' \Rightarrow f\times g \sim f'\times g'$ ↑ well defined

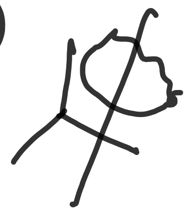
(exercise)

Ex If $\pi(X) = G \Rightarrow$

$$\pi(X\times\mathbb{R}) = \pi(X)\times\pi(\mathbb{R}) = G\times(e) = G$$

$$\pi(\mathbb{R}^3 - \{\text{line}\}) = \pi((\mathbb{R}^2 - \{\text{point}\})\times\mathbb{R})$$

$$= \pi(\mathbb{R}^2 - \{\text{point}\}) \neq (e)$$

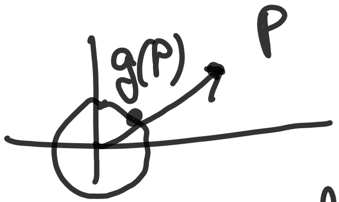


Ex $\mathbb{R}^2 - \{0\} \sim S^1$ so $\pi(S^1) =$
$= \pi(\mathbb{R}^2 - \{0\}) \neq (e)$

$f : S^1 \to \mathbb{R}^2 - \{0\}$ $\qquad$ $g : \mathbb{R}^2 - \{0\} \to S^1$

$S^1 = \{(x,y) ; x^2 + y^2 = 1\}$

$f =$ inclusion $\qquad$ $g(P) = \dfrac{1}{\|P\|} P$



$g \circ f = id_{S^1}$

$f \circ g = g \sim id_{\mathbb{R}^2 - \{0\}}$

$$H(P,t) = \left(1 - \left(1 - \frac{1}{\|P\|}\right)t\right)P$$

$\pi_1(S^1)$ ——

Ex $\pi(S^1 \times \mathbb{R}) \neq (e)$

$\pi(S^1 \times S^1) \neq (e)$

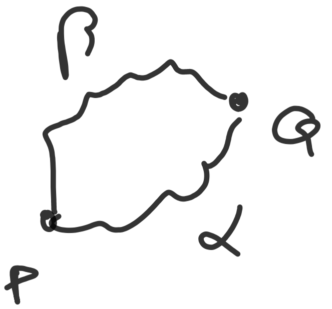
$\quad \| $

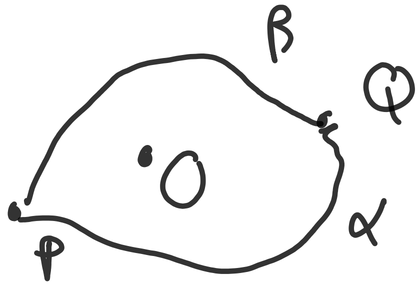
$\pi(S^1) \times \pi(S^1)$

Def X (pathwise connected) is SIMPLY CONNECTED $\Leftrightarrow \pi(X) = (e)$

each $\mathbb{R}^n$ is simply conn,
$\mathbb{R}^2 \setminus \{0\}$, S^1 ~~are~~ not simply conn.

Thm If X is simply conn. and α, β paths with same initial and final pts $\Rightarrow \alpha \sim \beta$

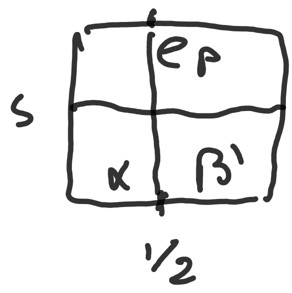




pf $\beta' = \beta(1-t)$, $\alpha\beta'$ is a loop
X simply conn. $\alpha\beta' \sim e_P$

$\alpha : [0,1] \to \mathbb{R}^2$
$\beta : [0,1] \to \mathbb{R}^2$

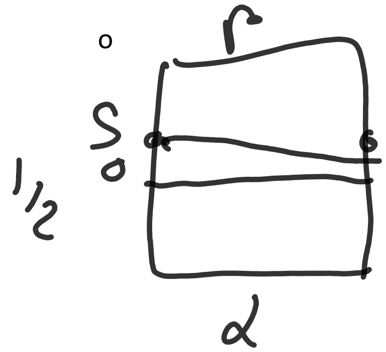
$H : [0,1] \times [0,1] \to \mathbb{R}^2$



$[0,1] =$ slice for $t = 1/2$

$H|_{slice} : [0,1] \to \mathbb{R}^2$ path from Q to P

$\bar{H}$ homotopy $\alpha \sim \beta$ $\quad \bar{H} : [0,1] \times [0,1] \to \mathbb{R}^2$



$\bar{H}|_{[0,1] \times \{s_0\}} = H|_{\text{slice over } t = s_0}$ path

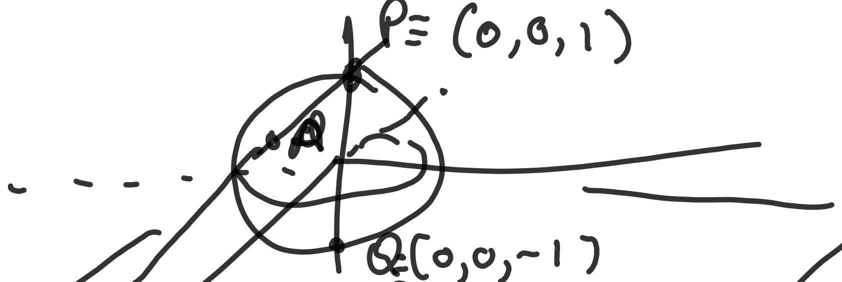
Prop If $X = X_1 \cup X_2$
- X_1, X_2 open and simply conn.
- $X_1 \cap X_2$ pathwise conn.

$\Rightarrow \pi(X) = (e)$

Prop $\Rightarrow$ **Ex** $\pi(S^2) = (e)$

stereographic projection

$S^2 \subseteq \mathbb{R}^3 \qquad S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$



$\to$ tg plane at Q

$\| \quad z = -1$

stereog. proj from P $\varphi : S^2 \setminus \{P\} \to \mathbb{R}^2$

φ is a homeom. $S^2 - \{P\} \longleftrightarrow \mathbb{R}^2$

$\pi(S^2 - \{P\}) = (e)$ $\quad X_1 = S^2 - \{P\}$

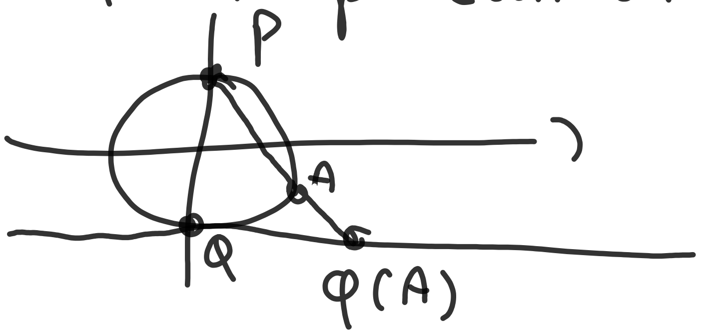
$\pi(S^2 - \{Q\}) = (e)$ $\quad X_2 = S^2 - \{Q\}$

$X_1 \cap X_2 = S^2 - \{P, Q\}$ pathwise conn.

prop $\Rightarrow \pi(S^2) = (e)$

similarly $\pi(S^n) = (e)$ (simply conn.)
$n > 1$

$n = 1$ Prop cannot apply



$S^1 - \{P\} \longleftrightarrow \mathbb{R}$ $\quad X_1 = S^1 - \{P\}$ simply

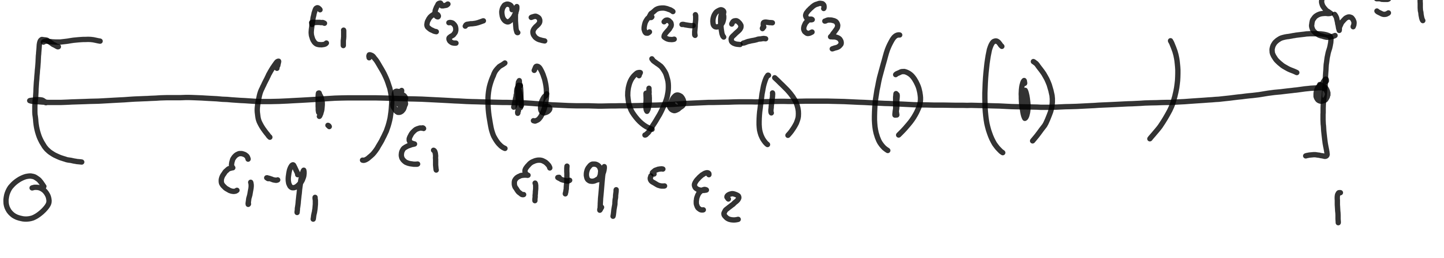
$S^1 - \{Q\} \longleftrightarrow \mathbb{R}$ $\quad X_2 = S^1 - \{Q\}$ conn.

$X_1 \cap X_2 = S^1 - \{P, Q\}$ not connected

Lemma $[0,1]$ $\{U_i\}$ open cover

$\exists\ 0 < t_1 < t_2 \cdots < t_n = 1$ $\quad [0,1]$ is compact

s.t. $[t_j, t_{j+1}] \subseteq U_j$



$0 \in U_1 \Rightarrow \exists\ \varepsilon_1$ s.t. $[0, \varepsilon_1) \subseteq U_1$

$\varepsilon_1 \in U_2 \Rightarrow \exists\ q_1$ s.t. $(\varepsilon_1 - q_1, \underbrace{\varepsilon_1 + q_1}_{= \varepsilon_2}) \subseteq U_2$

. . . . .

take $t_1 \in (\varepsilon_1 - q_1, \varepsilon_1)$ $\qquad [0, t_1] \subseteq U_1$

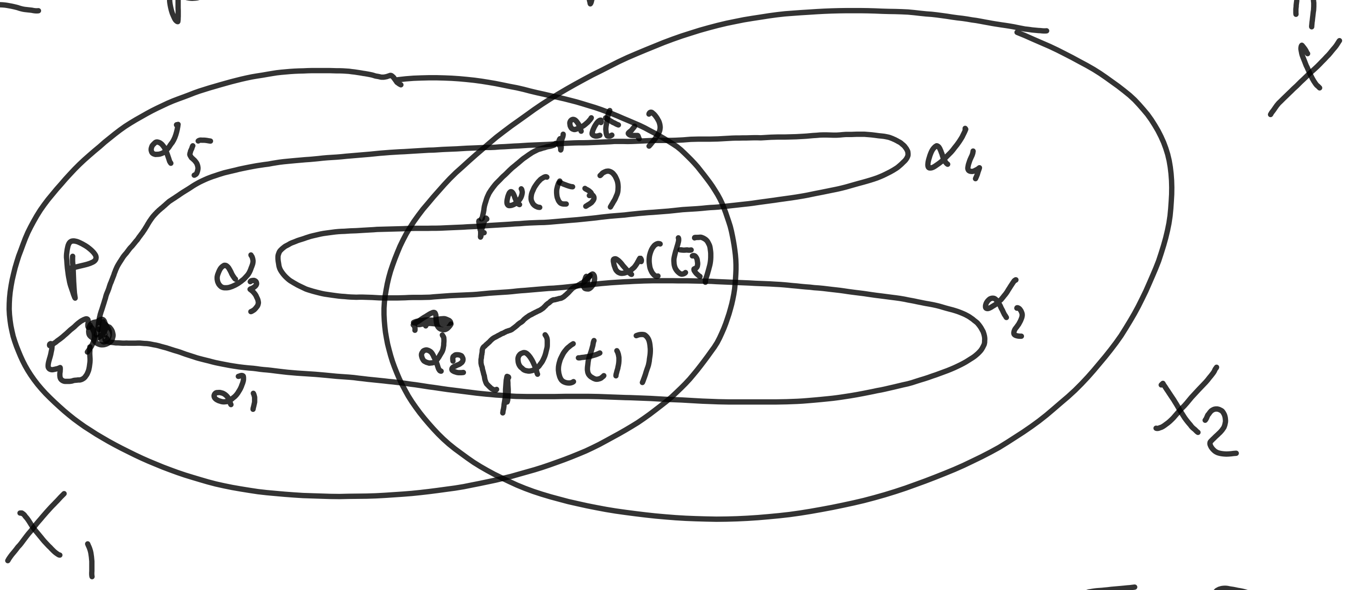
$t_2 \in (\varepsilon_2 - q_2, \varepsilon_1 + q_1)$ $\qquad [t_1, t_2] \subseteq U_2$

$\vdots$ $\qquad \vdots$

$t_n = 1$

---

pf of the Prop. $\qquad \alpha : [0,1] \to \underbrace{X_1 \cup X_2}_{= X}$



$\alpha^{-1}(X_1) \quad \alpha^{-1}(X_2)$ open in $[0,1]$

open covering of $[0,1]$

Apply Lemma $\qquad 0 [\quad]_{t_1} [\quad]_{t_2} \quad]_1$

$[0, t_1] \in \alpha^{-1}(X_1) \qquad \alpha[0, t_1] \subseteq X_1$

$[t_1, t_2] \in \alpha^{-1}(X_2) \qquad \alpha[t_1, t_2] \subseteq X_2 \qquad \mapsto \alpha(t_1) \in X_1 \cap X_2$

$\vdots \qquad \alpha[t_2, t_3] \subseteq X_1$

$\vdots$

$\alpha_1 = \alpha|_{[0,t_1]}$ $\alpha_2 = \alpha|_{[t_1, t_2]}$ . . . .

c $\bar{\alpha}_2$ path from $\alpha(t_1)$ $\alpha(t_2)$

$\bar{\alpha}_4$ " $\alpha(t_3)$ $\alpha(t_4)$

α_2 path in X_2 joining $\alpha(t_1), \alpha(t_2)$

$\bar{\alpha}_2$ " " "

since X_2 simply conn $\Rightarrow$ $\alpha_2 \sim \bar{\alpha}_2$

similarly $\bar{\alpha}_4 \sim \alpha_4$

$\alpha \sim \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \sim \alpha_1 \bar{\alpha}_2 \alpha_3 \bar{\alpha}_4 \alpha_5 \sim e_p$

loop in X_1 simply conn.