Feasible parameter set approximation for linear models with bounded uncertain regressors

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Abstract

Nonconvex feasible parameter sets are encountered in set membership identification whenever the regressor vector is affected by bounded uncertainty. This occurs for example when considering standard output error models, or when the available measurements are provided by binary or quantized sensors. In this paper, a unifying framework is proposed to deal with several identification problems involving a nonconvex feasible parameter set and a procedure is proposed for approximating the minimum volume orthotope containing the feasible set. The procedure exploits different relaxations for autoregressive and input parameters, based on the solution of a sequence of linear programming problems. The proposed technique is shown to provide tight bounds in some special cases. Moreover, it is extended to cope with bounds not aligned with the parameter coordinates, in order to obtain polytopic approximations of the feasible set. A number of numerical tests on randomly generated models and data sets demonstrates the accuracy of the computed set approximations.

I. Introduction

The introduction of the set membership paradigm for the characterization of uncertainty in system identification dates back to several decades ago. According to this approach, uncertainties affecting the measured data and the model are unknown-but-bounded. This framework is motivated by the fact that in several contexts a statistical description may lead to unreliable results,

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either for the limited number of measurements available preventing the safe use of asymptotic results or because the source of uncertainty does not *look like* a stochastic system, i.e., the most natural characterization of noises and disturbances can be expressed as hard bounds in an appropriate norm. In this context, the *Feasible Parameter Set* (FPS), defined as the set of model parameters consistent with the constraints represented by the model equations, the available measurements and the noise bounds, represents the fundamental set including all the admissible solutions of the estimation problem. Hence, the structure of this set retains a great importance both for the computation of optimal or suboptimal point estimates and for useful approximation of the entire solution set. It is well known that if uncertainty bounds on measurements are given according to a max or an energy norm, the model is linear in the parameters and the regressor variables are not corrupted by noise, the FPS is a convex polytope or an ellipsoid. For this linear regression model setup, a huge amount of literature is available for describing or approximating the FPS according to different criteria depending on the specific context (see e.g., [1]–[6] and references therein). However, in many realistic contexts one cannot do without considering uncertainty affecting regressors: this is the case for the classical output error (OE) formulation or for errors-in-variables (EiV) identification problems [7]. Moreover, the pervasive diffusion of digital sensors as binary or quantized devices in communications systems, industrial plants or other technological devices has pushed the development of an identification research area where measurements available on the system are binary or quantized variables (see e.g., [8]–[11] and references therein). In this case, the nature of regressor quantization and the characterization of noise in terms of deterministic bounds, make the problem amenable to the set membership approach, although important developments have been made also in the stochastic setting (see e.g. [9], [12]–[14]) or even in a mixed stochastic/deterministic framework [15], [16]. When considering ARX models in this quantized information setting [17], or OE and EiV models in general [18], [19], one is faced with a problem with perturbed regressors. In these cases, the nice property of convexity of the FPS is generally lost, so that almost all of the methods and techniques proposed for the standard linear regression formulation do not apply.

Indeed, bounding or approximating general nonconvex sets is a formidable task and very few results can be found in the literature on set membership identification. References [20], [21] can be recognized as pioneering investigations on the structure of the FPS for ARMAX models, while in [22]–[24] tight bounds for EiV models are derived for the simplified formulation in which regressors are not serially correlated. In [25] an OE model formulation is considered. Approximation of the nonconvex FPS through ellipsoidal and parallelotopic bounds is performed by a recursive procedure consisting in processing one measurement at a time through a two-step procedure. The main problem here is to bound at each time t the parameter set obtained by intersecting the FPS at the preceding time $t-1$ with the admissible parameter set determined by the measurement at time t. Actually, this technique is a direct generalization of the recursive approximation procedures adopted in the linear equation error context (see e.g., [26], [27]). Still in an OE context, in [28] a recursive procedure is proposed for computing an FPS approximation, by decomposing the nonconvex FPS in the union of convex sets. The main limitation of both approaches is that the approximation procedure does not explicitly account for regressor sequential correlation, giving rise to quite conservative results whenever such correlation is present. More recently, in [19], [29] an EiV formulation has been considered in which all the regressor variables are subject to bounded noises. For this general setting, a procedure is provided to derive parameter uncertainty intervals, i.e., bounds on the nonconvex FPS, based on a sequence of problem relaxations leading to linear matrix inequalities and using the sparsity of the resulting relaxed problems. Finally, it is worth mentioning that nonconvex feasible sets are usually encountered in set membership identification of nonlinear systems. Within this context, the interested reader is referred to techniques based on interval analysis [30], [31], or to a number of methods tailored to the specific models at hand (see, e.g., [32]–[34]).

In this paper, a unifying set membership framework is given for dealing with linear models with perturbed regressors including ARX models with quantized measurements and OE models with or without measurement quantization. Although EiV models can be easily embedded in this framework, for the sake of clarity of presentation the paper will explicitly refer to the ARX/OE context. Within this framework an exact description of the FPS set is provided. Since the problem of approximating the FPS turns out in nonconvex optimization problems, a batch procedure for approximating the minimum volume orthotope containing the FPS is constructed. For computing bounds on each model parameter, the proposed algorithm requires the solution of a sequence of linear programming problems obtained by a suitable relaxation of the original problem. The parameter bounding procedure converges to the tight bounds in some special cases, while in general it converges to guaranteed estimates. The numerical results obtained on a considerable number of experiments involving systems of reasonable orders generated randomly, show more than satisfactory reliability of the algorithm. This can be explained by two features of the procedure. First, the relaxations exploit the structure of the original nonconvex optimization problem, e.g., by adopting different algorithms for approximating bounds on the autoregressive and on the input moving average model parameters. Second, the introduction of constraints accounting for parameter bounding through hyperplanes not aligned with the parameter coordinates, contributes to reduce the size of the feasible set of the relaxed linear programs, thus improving the approximation accuracy with respect to the basic relaxed linear programs.

The paper is organized as follows. Section II introduces notation, the unifying framework for dealing with OE and ARX models with quantized measurements and the problem formulation. Section III illustrates the optimization procedure for parameter bounds estimation, given as an iterative sequence of linear programs. Section IV provides improved linear programming relaxations on the basis of non axis-aligned parameter bounds. Section V illustrates some structural features of the FPS for the special case of ARX model identification with binary measurements. Section VI provides several numerical examples showing the achievable accuracy and limitations of the proposed procedure and the related computational burden. Concluding remarks are drawn in Section VII together with a perspective on future work.

II. Problem formulation

Let $\mathbb{R}^N$ denote the N-dimensional Euclidean space. A sequence of real numbers $\{x(t), t = 1, \ldots, N\}$ is identified by a vector $x \in \mathbb{R}^N$. Let q^{-1} be the backward shift operator, i.e., $q^{-1}x(t) = x(t-1)$.

Let us consider the linear time-invariant model

$$x(t) = \sum_{i=1}^{n} a_i\, x(t-i) + \sum_{j=1}^{m} b_j\, u(t-j+1) + d(t) \tag{1}$$

where for each $t = 1, 2, \ldots$, $d(t)$ is an unknown disturbance bounded by a known quantity, i.e. $|d(t)| \leq \delta_t$, $u(t) \in \mathbb{R}$ is a known input signal and $x(t)$ belongs to a known interval

$$x(t) \in [\underline{x}(t),\, \overline{x}(t)]. \tag{2}$$

Parameters a_i and b_j are unknown.

Let $\theta = [a_1, \ldots, a_n, b_1, \ldots, b_m]' \in \mathbb{R}^{n+m}$ denote the parameter vector and let Θ_0 represent the prior information available on such a vector. For a given input-output realization of length N, the feasible parameter set (FPS) is defined as the set of parameters which is compatible with the a priori information and the bounds (2), i.e.

$$\mathcal{F} = \{\theta \in \Theta_0 : \text{ (1)-(2) hold for } t = 1, \ldots, N\}. \tag{3}$$

Notice that, since the signal $x(t)$ is not exactly known, in general the FPS turns out to be a nonconvex set. Hereafter, we denote by $\mathcal{F}_a$ and $\mathcal{F}_b$ the projection of the feasible set $\mathcal{F}$ over the subspaces $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_m\}$, respectively.

It is possible to rewrite (1)-(2) as

$$A(q)\, x(t) = B(q)\, u(t) + d(t) \tag{4}$$

$$x(t) \in [\underline{x}(t),\, \overline{x}(t)] \tag{5}$$

where

$$A(q) = 1 - a_1\, q^{-1} - \ldots - a_n\, q^{-n} \tag{6}$$

and

$$B(q) = b_1 + b_2\, q^{-1} + \ldots + b_m\, q^{-m+1}. \tag{7}$$

Let us state the following technical assumptions which will be enforced throughout the paper.

Assumption 1: The prior information on the parameters a_i and b_j is such that $a_i \in [\underline{a}_i, \overline{a}_i]$, $b_j \in [\underline{b}_j, \overline{b}_j]$, where $\underline{a}_i > -\infty$ and $\overline{a}_i < \infty$, $i = 1, \ldots, n$.

Assumption 2: Bounds $\underline{x}(t)$ and $\overline{x}(t)$ are finite.

Assumption 1 states that a priori finite bounds on parameters a_i must be available. Notice that, conversely, a priori bounds on parameters b_j can be infinite. Assumption 2 requires a finite bound on the signal $x(t)$.

In the following, it is shown how a number of different models in the set membership framework, like OE models, ARX models with quantized measurements, etc., can be cast in the general formulation (4)-(5).

A. Output Error

Let us consider the output error model (see Fig. 1)

$$\begin{aligned} A(q)\, x(t) &= B(q)\, u(t) && (8) \\ y(t) &= x(t) + e(t) && (9) \end{aligned}$$

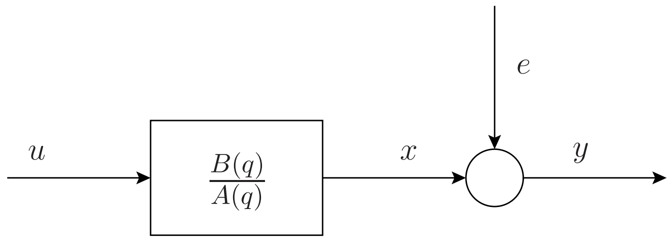

Fig. 1. Structure of the OE model.

where $y(t)$ is the measured model output, $u(t)$ denotes the input signal and $e(t)$ is an additive noise on the system output. The disturbance $e(t)$ is assumed to be unknown but bounded, i.e., $|e(t)| \leq \varepsilon_t,\ t = 1, 2, \ldots$. Notice that the internal signal $x(t)$ is not accessible.

It is easy to show that model (8)-(9) can be rewritten w.l.o.g. in the form (4)-(5). Indeed, by choosing $\delta_t = 0$ (and hence $d(t) = 0$) for $t = 1, 2, \ldots$, one has that (8) coincides with (4), while from (9) and $|e(t)| \leq \varepsilon_t$ one gets

$$x(t) \in [y(t) - \varepsilon_t,\, y(t) + \varepsilon_t] \triangleq [\underline{x}(t),\, \overline{x}(t)]. \tag{10}$$

Notice that for OE models Assumption 2 is always satisfied.

B. Output Error with quantized measurements

Let us define the quantizer operator with P thresholds as

$$\mathcal{Q}(v) = \begin{cases} P & \text{if } z_P < v \leq z_{P+1} \\ P-1 & \text{if } z_{P-1} < v \leq z_P \\ \quad\vdots & \\ 1 & \text{if } z_1 < v \leq z_2 \\ 0 & \text{if } z_0 < v \leq z_1 \end{cases} \tag{11}$$

where $z_1, \ldots, z_P$ denote the P thresholds and $z_0 \triangleq -\infty$, $z_{P+1} \triangleq +\infty$.

Let us now consider an OE model whose outputs are measured by a quantized sensor with P thresholds, as depicted in Fig. 2. Such a model can be written as

$$A(q)\, x(t) = B(q)\, u(t) \tag{12}$$

$$v(t) = x(t) + e(t) \tag{13}$$

$$y(t) = \mathcal{Q}(v(t)). \tag{14}$$

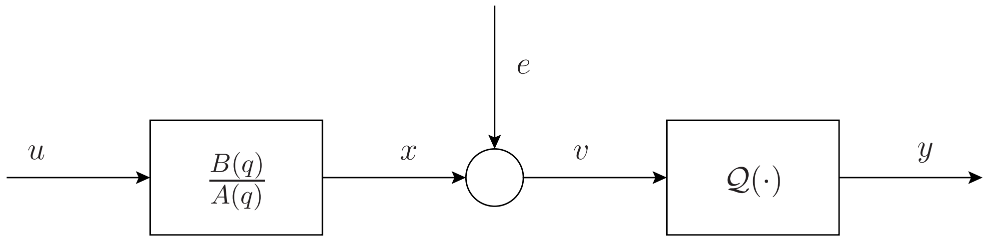

Fig. 2. Structure of the OE model with quantized measurements.

Notice that only signals $u(t)$ and $y(t)$ are available at each time t. By setting $\delta_t = 0$ one has that (12) coincides with (4). Let us now show that (13)-(14) lead to a constraint in the form of (5). From (14), according to (11), one has $v(t) \in [z_{y(t)},\, z_{y(t)+1}]$.[1] By (13), $x(t) = v(t) - e(t)$, and hence

$$x(t) \in [z_{y(t)} - \varepsilon_t,\, z_{y(t)+1} + \varepsilon_t] \triangleq [\underline{x}(t),\, \overline{x}(t)].$$

Notice that, Assumption 2 requires that suitable finite values must be available for z_0 and z_{P+1}: this is not restrictive because finite bounds on the output of the system to be identified are usually known.

C. ARX with quantized measurements

Let us consider an ARX model with quantized output

$$A(q)\, x(t) = B(q)\, u(t) + d(t) \tag{15}$$

$$y(t) = \mathcal{Q}(x(t)) \tag{16}$$

where $x(t)$ is unknown and $|d(t)| \leq \delta_t$ denotes an unknown-but-bounded noise. The structure of this model is depicted in Fig. 3. The knowledge of the quantized output $y(t)$ implies that $x(t)$ must belong to the interval

$$x(t) \in [z_{y(t)},\, z_{y(t)+1}] \triangleq [\underline{x}(t),\, \overline{x}(t)].$$

Thus, (15)-(16) are equivalent to (4)-(5).

Remark 1: The formulation (4)-(5) can cope also with the errors-in-variables identification setting, in which also the input signal $u(t)$ is not known. In the bounded error framework, the

[1]With a slight abuse of notation we will always denote feasible sets by closed intervals.

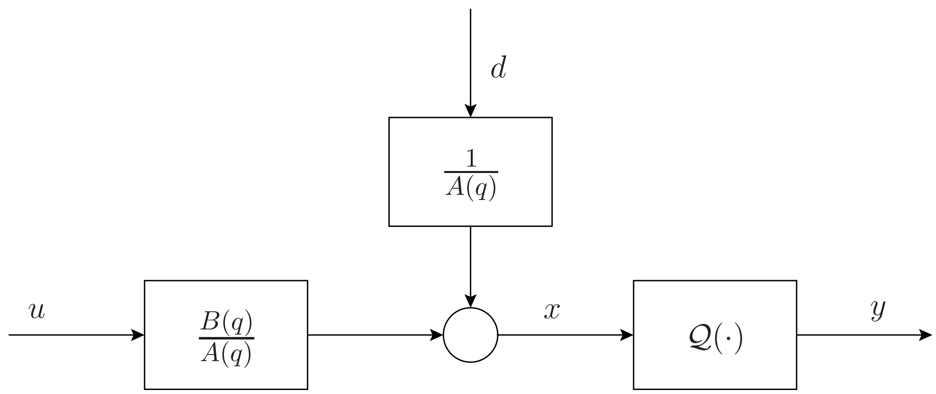

Fig. 3. Structure of the ARX model with quantized measurements.

variables $u(t)$ will belong to an interval $[\underline{u}(t), \overline{u}(t)]$ and thus the vector $x(t)$ can be augmented including the input variables in it, while the parameters b_j can be treated in the same way as the parameters a_i. The presence of output or input quantized measurements can be dealt with as in the OE and ARX models shown above.

III. Feasible set approximation procedure

In this section, an algorithm for the construction of an outer box approximation of the feasible set defined in (3) is presented.

From (1)-(2) and Assumptions 1 and 2, the feasible set $\mathcal{F}$ is defined by the following constraints:

$$\begin{cases} -\delta_t \leq x(t) - \sum_{i=1}^{n} a_i\, x(t-i) - \sum_{j=1}^{m} b_j\, u(t-j+1) \leq \delta_t\,,\ t = r+1, \ldots, N \\ x(t) \in [\underline{x}(t), \overline{x}(t)]\ ,\ t = 1, \ldots, N \\ a_i \in [\underline{a}_i, \overline{a}_i]\ ,\ i = 1, \ldots, n \\ b_j \in [\underline{b}_j, \overline{b}_j]\ ,\ j = 1, \ldots, m \end{cases} \tag{17}$$

where $r = \max\{n, m-1\}$. Notice that the first constraint in (17) is nonlinear in the variables a_i and $x(t-i)$, and so the resulting feasible set $\mathcal{F}$ is in general nonconvex.

Let $\mathcal{B}^* = \{\theta\colon a_i \in [\underline{a}_i^*, \overline{a}_i^*],\ b_j \in [\underline{b}_j^*, \overline{b}_j^*]\}$ be the minimum orthotope containing the set $\mathcal{F}$, i.e., $\mathcal{B}^* = \bigcap_k \mathcal{B}^{(k)}$, where $\mathcal{B}^{(k)}$ represents any set $\mathcal{B}^{(k)} = \{\theta\colon a_i \in [\underline{a}_i^{(k)}, \overline{a}_i^{(k)}],\ b_j \in [\underline{b}_j^{(k)}, \overline{b}_j^{(k)}]\}$ such that $\mathcal{B}^{(k)} \supseteq \mathcal{F}$. Clearly, the exact computation of $\mathcal{B}^*$ is generally intractable, due to nonconvexity of $\mathcal{F}$. The aim is to find an orthotope $\mathcal{B}$ which is an outer approximation of the feasible set, i.e. $\mathcal{B} \supseteq \mathcal{B}^*$, by introducing a suitable convex relaxation of (17).

Thanks to Assumptions 1 and 2, which guarantee finite bounds on a_i and $x(t)$, let us introduce a set of transformed variables α_i, $i = 1, \dots, n$ and $\eta(t)$, $t = 1, \dots, N$

$$\alpha_i = a_i - \underline{a}_i \quad , \quad i = 1, \dots, n \tag{18}$$

$$\overline{\alpha}_i = \overline{a}_i - \underline{a}_i \quad , \quad i = 1, \dots, n \tag{19}$$

$$\eta(t) = x(t) - \underline{x}(t) \quad , \quad t = 1, \dots, N \tag{20}$$

$$\overline{\eta}(t) = \overline{x}(t) - \underline{x}(t) \quad , \quad t = 1, \dots, N. \tag{21}$$

Moreover, let $\mathcal{F}_\alpha = \mathcal{F}_a - \underline{a}$ be the projection of the FPS over the subspace of the new variables α_i.

Let us rewrite (17) as:

$$\begin{cases} -\delta_t \leq \eta(t) + \underline{x}(t) - \sum_{i=1}^{n} (\alpha_i \eta(t-i) + \alpha_i \underline{x}(t-i) + \underline{a}_i \eta(t-i) + \underline{a}_i \underline{x}(t-i)) \\ \qquad - \sum_{j=1}^{m} b_j \, u(t-j+1) \leq \delta_t \ , \ t = r+1, \dots, N \\ \eta(t) \in [0, \overline{\eta}(t)] \ , \ t = 1, \dots, N \\ \alpha_i \in [0, \overline{\alpha}_i] \ , \ i = 1, \dots, n \\ b_j \in [\underline{b}_j, \overline{b}_j] \ , \ j = 1, \dots, m. \end{cases} \tag{22}$$

Different procedures for bounding parameters a_i and b_j are provided in the following.

A. Bounds on autoregressive parameters

It is worthwhile to recall that Assumptions 1 and 2 hold and are instrumental to derive all the results in the section. The proposed procedure firstly aims to finding bounds on the admissible autoregressive parameters a_i. Let us focus on finding lower bounds on a_i, $i = 1, \dots, n$. Similar reasoning can be easily applied for upper bounds.

Let us introduce the following LP problem:

$$
\begin{cases}
\inf \ \alpha_i \\
\text{s.t.:} \\
-\delta_t \le \eta(t) + \underline{x}(t) - \sum_{i=1}^{n} \left(w_i(t-i) + \alpha_i \underline{x}(t-i) + \underline{a}_i \eta(t-i) + \underline{a}_i \underline{x}(t-i)\right) \\
\qquad\qquad - \sum_{j=1}^{m} b_j \, u(t-j+1) \le \delta_t \ , \ t = r+1, \ldots, N \\
\eta(t) \in [0, \, \overline{\eta}(t)] \ , \ t = 1, \ldots, N \\
\alpha_i \in [0, \overline{\alpha}_i] \ , \ i = 1, \ldots, n \\
b_j \in [\underline{b}_j, \overline{b}_j] \ , \ j = 1, \ldots, m \\
0 \le w_i(t) \le \overline{\alpha}_i \eta(t) \ , \ i = 1, \ldots, n \ , \ t = 1, \ldots, N \\
0 \le w_i(t) \le \alpha_i \overline{\eta}(t) \ , \ i = 1, \ldots, n \ , \ t = 1, \ldots, N.
\end{cases}
\tag{23}
$$

where the optimization variables are $\eta(t)$, $w_i(t)$, α_i, b_j, for $t = 1, \ldots, N$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Theorem 1: For any $\theta \in \mathcal{F}$, there exist $w_i(t)$, $i = 1, \ldots, n$, $t = 1, \ldots, N$ such that problem (23) is feasible. Moreover, if $\hat{\alpha}_i$ is the solution of problem (23), one has $\hat{a}_i = \underline{a}_i + \hat{\alpha}_i \le \underline{a}_i^*$.

Proof: Let us define

$$
w_i(t) \ = \ \alpha_i \eta(t). \tag{24}
$$

By introducing the new variables $w_i(t)$ in (22), one has

$$
\begin{cases}
-\delta_t \le \eta(t) + \underline{x}(t) - \sum_{i=1}^{n} \left(w_i(t-i) + \alpha_i \underline{x}(t-i) + \underline{a}_i \eta(t-i) + \underline{a}_i \underline{x}(t-i)\right) \\
\qquad\qquad - \sum_{j=1}^{m} b_j \, u(t-j+1) \le \delta_t \ , \ t = r+1, \ldots, N \\
\eta(t) \in [0, \, \overline{\eta}(t)] \ , \ t = 1, \ldots, N \\
\alpha_i \in [0, \overline{\alpha}_i] \ , \ i = 1, \ldots, n \\
b_j \in [\underline{b}_j, \overline{b}_j] \ , \ j = 1, \ldots, m \\
w_i(t) = \alpha_i \eta(t) \ , \ i = 1, \ldots, n \ , \ t = 1, \ldots, N.
\end{cases}
\tag{25}
$$

By using the bounds on α_i and $\eta(t)$, one has that the variables $w_i(t)$ defined in (24) satisfy the inequalities

$$
0 \le w_i(t) \le \overline{\alpha}_i \eta(t) \tag{26}
$$

$$
0 \le w_i(t) \le \alpha_i \overline{\eta}(t) \tag{27}
$$

for each $i = 1, \ldots, n$, $t = 1, \ldots, N$. By substituting (26)-(27) for the last equality constraints in (25), one obtains the same constraint set as in the LP (23). Hence, the projection of such a constraint set over the subspace $\{\alpha_1, \ldots, \alpha_n\}$ contains the set $\mathcal{F}_\alpha$ and the result immediately follows. ■

As stated in Theorem 1, due to the relaxation on $w_i(t)$, the solution of (23) provides in general a lower bound on $\underline{a}_i^*$ even in the case $n = 1$. Actually, an algorithm able to return the value of $\underline{a}_i^*$ within the desired precision when $n = 1$ can be constructed. Afterwards, the same algorithm will be effectively exploited for the case $n > 1$.

Theorem 1 provides necessary conditions for the feasibility of a given θ, i.e., if (23) is not feasible for a given θ, then $\theta \notin \mathcal{F}$. This is the simple key information which will be exploited in the proposed procedure. For ease of presentation, let us define the following function returning a lower bound on a_i according to (23) and Theorem 1:

$[feas, opt]$=INF_A($i, \underline{a}, \overline{a}, \underline{b}, \overline{b}, u, \delta, \underline{x}, \overline{x}$)

where $feas$ can be `true` or `false` depending on the feasibility of the problem, while opt contains the lower bound (in case of feasible problems).

Algorithm 1 Lower bound on parameter a_i

function $[feas, bound]$=LBOUND_ON_A(i,ϵ_a,$\underline{a}$,$\overline{a}$,$\underline{b}$,$\overline{b}$,u,δ,$\underline{x}$,$\overline{x}$)
  $\underline{q} \leftarrow \underline{a}$; $\overline{q} \leftarrow \overline{a}$; $step \leftarrow (\overline{q}_i - \underline{q}_i)/2$;
  while $(step \geq \epsilon_a/2)$ **do**
    if $(\underline{q}_i + step > \overline{a}_i)$ **then**
      $step \leftarrow \overline{a}_i - \underline{q}_i$;
    end if
    $\overline{q}_i \leftarrow \underline{q}_i + step$;
    $[feas, opt]$=INF_A($i, \underline{q}, \overline{q}, \underline{b}, \overline{b}, u, \delta, \underline{x}, \overline{x}$)
    if $feas$ **then**
      $step \leftarrow step/2$; $\underline{q}_i \leftarrow opt$;
    else
      $\underline{q}_i \leftarrow \overline{q}_i$;
    end if
  end while
  $\underline{a}_i \leftarrow \underline{q}_i$;
  $bound \leftarrow \underline{a}$;
end function

In Algorithm 1 a procedure written in pseudo-code is reported which returns a lower bound $\underline{a}_i$ to $\underline{a}_i^*$. The scalar $step$ denotes the size of the interval on parameter a_i which will be analyzed at the current iteration. Throughout Algorithm 1, $\underline{q} = \underline{a}$, $\overline{q} = \overline{a}$, except for the i-th components $\underline{q}_i$ and $\overline{q}_i$, which denote the current estimated bounds on the lower bound $\underline{a}_i$, i.e., $\underline{a}_i \in [\underline{q}_i, \overline{q}_i]$ at the current iteration. At line 8, the LP (23) is solved with the constraint $\alpha_i \in [0, \overline{q}_i - \underline{q}_i]$. If it is feasible, the lower bound $\underline{q}_i$ is updated with the solution of the LP, while the value of $step$ is halved. If it is unfeasible, $\underline{q}_i$ is set to $\overline{q}_i$. The algorithm terminates when the size of the interval to be analyzed is smaller than a given tolerance ϵ_a.

We can now state the following theorem.

Theorem 2: Algorithm 1 converges in a finite number of iterations.

Proof: In Algorithm 1, every time the LP solved at line 8 is feasible, the variable $step$ is halved. Hence, being the stopping condition $step < \epsilon_a/2$, in order to prove that such a condition is eventually satisfied, it is sufficient to show that infeasibility of the LP cannot occur indefinitely. This directly follows by the fact that the lower bound $\underline{q}_i$ is increased by $step$ every time infeasibility occurs (see lines 12 and 7). ■

The next result concerns tightness of the bound returned by Algorithm 1 in the case $n = 1$.

Theorem 3: Let $n = 1$ and assume that the tolerance ϵ_a in Algorithm 1 is chosen in such a way that the function INF_A returns unfeasible (i.e., $feas =$`false`), whenever $\overline{q}_1 - \underline{q}_1 < \epsilon_a$ and $[\underline{q}_1, \overline{q}_1] \cap [\underline{a}_1^*, \overline{a}_1^*] = \emptyset$. Then, the lower bound $\underline{a}_1$ returned by Algorithm 1 is tight, i.e., $\underline{a}_1^* - \underline{a}_1 \leq \epsilon_a$.

Proof: Whenever $\underline{a}_1^* - \underline{q}_1 > \epsilon_a$, in Algorithm 1 either $\underline{q}_1$ is increased by $step > \epsilon_a$, or the variable $step$ is halved. This will eventually lead either to a value of $\underline{q}_1$ satisfying $\underline{a}_1^* - \underline{q}_1 \leq \epsilon_a$, or to $step \leq \epsilon_a$. In the former case, since $\underline{q}_1$ is not decreasing, Algorithm 1 will return an $\underline{a}_1$ such that $\underline{a}_1^* - \underline{a}_1 \leq \underline{a}_1^* - \underline{q}_1 \leq \epsilon_a$. Otherwise, if $step \leq \epsilon_a$, all the subsequent LPs with the constraint $a_1 \in [\underline{q}_1, \underline{q}_1 + step]$ will be unfeasible as long as $\underline{q}_1 + step < \underline{a}_1^*$. Therefore, feasibility will finally occur only when $\underline{a}_1^* - \underline{q}_1 \leq step \leq \epsilon_a$. Then, $step$ is halved for the last time and $\underline{a}_1$ is set equal to $\underline{q}_1$, thus satisfying $\underline{a}_1^* - \underline{a}_1 \leq \epsilon_a$. ■

Notice that the assumption in Theorem 3 is not restrictive: as ϵ_a approaches zero, the size of the current uncertainty interval on a_1 vanishes, and asymptotically infeasibility can be checked exactly by solving one LP.

Theorem 3 states that for the case $n = 1$, a single run of Algorithm 1 provides tight bounds on a_1. When $n > 1$, Algorithm 1 must be repeated for all a_i, but in this case tightness of bounds are not guaranteed due to the relaxation which involves variables $w_i(t)$ (and consequently parameters

a_i). To this purpose, it is convenient to iterate the previous procedure with the updated bounds until convergence. Again, in general, the computed bounds are not tight, but they are tighter than those obtained by running Algorithm 1 once for each a_i.

B. Bounds on input parameters

Once bounds on a_i are found, it remains to compute bounds on parameters b_j. To find such bounds, let us solve $2m$ LP problems with the same constraints as in (23) and with objective function $\inf\, b_j$ ($\sup\, b_j$ for the upper bound on $\overline{b}_j^*$), for $j = 1, \ldots, m$. In general, the obtained bounds are not tight due to conservatism of the proposed relaxation of the FPS constraints, even when $n = 1$.

Algorithm 2 provides a procedure which permits to improve the bounds on b_j w.r.t. those obtained by solving the $2m$ LPs mentioned above. Moreover, it will be proved that such a procedure provides tight bounds on the input parameters b_j when $n = 1$. For ease of exposition, we concentrate on computing lower bounds on b_j.

Algorithm 2 Lower bound on parameter b_j

function $[bound]$=LBOUND_ON_B($j, \epsilon_a, \epsilon_b, \underline{a}, \overline{a}, \underline{b}, \overline{b}, u, \delta, \underline{x}, \overline{x}$)
    $\underline{q} \leftarrow \underline{b}$; $\overline{q} \leftarrow \overline{b}$; $\widetilde{q} = \overline{q}$;
    while $((\overline{q}_j - \underline{q}_j) \geq \epsilon_b/2)$ **do**
        $\widetilde{q}_j \leftarrow (\overline{q}_j + \underline{q}_j)/2$;
        $[feas, bound\_a]$=LBOUND_ON_A($1, \epsilon_a, \underline{a}, \overline{a}, \underline{q}, \widetilde{q}, u, \delta, \underline{x}, \overline{x}$)
        if $feas$ **then**
            $\overline{q}_j \leftarrow \widetilde{q}_j$;
        else
            $\underline{q}_j \leftarrow \widetilde{q}_j$;
        end if
    end while
    $\underline{b}_j \leftarrow \underline{q}_j$;
    $bound \leftarrow \underline{b}$;
end function

Algorithm 2 follows a logic similar to that of Algorithm 1. Variables $\underline{q}_j$ and $\overline{q}_j$ represent the current estimated bounds on the lower bound $\underline{b}_j$, i.e., $\underline{b}_j \in [\underline{q}_j, \overline{q}_j]$ at the current iteration. The main difference w.r.t. Algorithm 1 is that in Algorithm 2 a bisection procedure is performed on the interval $[\underline{q}_j, \overline{q}_j]$, depending on the feasibility of the procedure LBOUND_ON_A at line 5, which corresponds to Algorithm 1 with the constraint $\underline{b}_j \in [\underline{q}_j, \widetilde{q}_j]$, with $\widetilde{q}_j = \frac{\overline{q}_j + \underline{q}_j}{2}$. If LBOUND_ON_A returns $feas =$`false`, the current lower bound can be increased and one has that $\underline{b}_j \in [\widetilde{q}_j, \overline{q}_j]$; otherwise, the current interval is halved and the algorithm proceeds by testing the interval $[\underline{q}_j, \widetilde{q}_j]$ at the subsequent iteration.

The next result states that the computed bounds on the input parameters are tight when only one autoregressive parameter is involved, i.e. $n = 1$.

Theorem 4: Algorithm 2 converges in a finite number of iterations. Moreover, if $n = 1$, for each $1 \leq j \leq m$, the lower bound $\underline{b}_j$ returned by Algorithm 2 is tight, i.e., $\underline{b}_j^* - \underline{b}_j \leq \epsilon_b$.

Proof: The first statement follows from a reasoning similar to that in the proof of Theorem 2. In order to prove the second statement, recall that when $n = 1$, Theorem 3 guarantees that the procedure LBOUND_ON_A returns tight bounds on the unique autoregressive parameter, irrespectively on the available bounds on the input parameters b_j. This means that if there is no feasible value of b_j in the interval $[\underline{q}_j, \widetilde{q}_j]$, the set $\{\theta \in \mathcal{F} : \ b_j \in [\underline{q}_j, \widetilde{q}_j]\}$ is empty and hence LBOUND_ON_A must return $feas =$`false` when applied to such a set (for a sufficiently small precision ϵ_a). Conversely, when there exists a feasible $b_j \in [\underline{q}_j, \widetilde{q}_j]$, LBOUND_ON_A will return $feas =$`true`. Therefore, a standard bisection allows one to approximate the bounds on each parameter b_j within the desired precision ϵ_b. ■

IV. Polytopic feasible set approximation

In this section, an extension of the bounding procedure reported in Section III aiming at reducing the bounds on parameters a_i is described. Moreover, in addition to finding a tighter box approximation of the feasible set, such an extension can also be effectively used to bound the feasible set through a polytope.

Let us consider the problem of computing the smallest strip, orthogonal to a given direction, containing the FPS. Given a direction vector c, a relaxation of this problem within the subspace of autoregressive parameters a_i turns out to be the computation of

$$\inf \ (\sup) \sum_{i=1}^{n} c_i \alpha_i \tag{28}$$

over the same set of constraints as in (23). Let $\underline{g}$ and $\overline{g}$ be the optimal values of such optimization problems. Then, the strip

$$\underline{g} \leq \sum_{i=1}^{n} c_i \alpha_i \leq \overline{g} \tag{29}$$

is guaranteed to contain the set $\mathcal{F}_\alpha$.

Now, let us assume that one wants to bound the FPS by a polytope composed of S strips orthogonal to the vectors c^k, $k = 1, ..., S$. This can be done recursively in the following way. At the k-th iteration, let $\underline{g}^p$ and $\overline{g}^p$ be constants such that

$$\underline{g}^p \leq \sum_{i=1}^{n} c_i^p \alpha_i \leq \overline{g}^p \quad , \quad p = 1, \dots, k-1. \tag{30}$$

Then, one has to solve the two LPs

$$\begin{cases} \text{inf (sup)} \displaystyle\sum_{i=1}^{n} c_i^k \alpha_i \\ \text{s.t.:} \\ -\delta_t \leq \eta(t) + \underline{x}(t) - \displaystyle\sum_{i=1}^{n} (w_i(t-i) + \alpha_i \underline{x}(t-i) + \underline{a}_i \eta(t-i) + \underline{a}_i \underline{x}(t-i)) \\ \qquad - \displaystyle\sum_{j=1}^{m} b_j \, u(t-j+1) \leq \delta_t \ , \ t = r+1, \dots, N \\ \eta(t) \in [0, \overline{\eta}(t)] \ , \ t = 1, \dots, N \\ \alpha_i \in [0, \overline{\alpha}_i] \ , \ i = 1, \dots, n \\ b_j \in [\underline{b}_j, \overline{b}_j] \ , \ j = 1, \dots, m \\ 0 \leq w_i(t) \leq \overline{\alpha}_i \eta(t) \ , \ i = 1, \dots, n \ , \ t = 1, \dots, N \\ 0 \leq w_i(t) \leq \alpha_i \overline{\eta}(t) \ , \ i = 1, \dots, n \ , \ t = 1, \dots, N \\ \underline{g}^p \leq \displaystyle\sum_{i=1}^{n} c_i^p \alpha_i \leq \overline{g}^p \ , \ p = 1, \dots, k-1. \end{cases} \tag{31}$$

Notice that in (31), the new constraints (30) are added at each step. After k iterations, the feasible set $\mathcal{F}_\alpha$ on parameters α_i can be approximated by the polytope described by the inequalities

$$\underline{g}^p \leq \sum_{i=1}^{n} c_i^p \alpha_i \leq \overline{g}^p \ , \ p = 1, \dots, k. \tag{32}$$

Remark 2: Notice that a less conservative relaxation can be formulated by adding to (31) the constraints

$$\underline{g}^p \eta(t) \leq \sum_{i=1}^{n} c_i^p w_i(t) \leq \overline{g}^p \eta(t) \, , \, p = 1, \dots, k-1, \, t = 1, \dots, N. \tag{33}$$

This approach leads in general to a tighter approximating polytope, but it is computationally feasible only for data sets of limited size. In fact, the constraints set grows by $2(N+1)$ inequalities at each iteration (instead of 2 as in (31)), thus rapidly leading to intractable problems.

The polytopic approximation described in (32) can be remarkably improved by exploiting Algorithm 1. In particular, one can apply the following procedure:

For $k = 1, \dots, S$:

1) tighten the strip orthogonal to c_i^k adopting a procedure analogous to Algorithm 1;
2) update the bounds on $\underline{a}_i$ and $\overline{a}_i$ by applying Algorithm 1, with the function INF_A optimizing over the constraint set (31).

In order to compute polytopic approximations of the feasible input parameters b_j, a similar approach can be taken, following the guidelines provided in Section III-B and the procedure of Algorithm 2.

V. A SPECIAL CASE: ARX MODELS WITH BINARY MEASUREMENTS

In this section, a special instance of the problem described in Section II-C is considered, i.e. ARX model identification with a binary sensor

$$\mathcal{Q}(x(t)) = \begin{cases} 0 & \text{if } x(t) \le z \\ 1 & \text{if } x(t) > z \end{cases} \tag{34}$$

where $z \in \mathbb{R}$ denotes the unique sensor threshold.

In particular, it turns out that the FPS shows some peculiar structural properties.

Theorem 5: For any output sequence $y \in \{0,1\}^N$, any input sequence $u \in \mathbb{R}^N$ and any noise level δ_t, $t = 1, \dots, N$, the set

$$\{\theta \in \Theta_0 : \;\; b_1 = b_2 = \cdots = b_m = 0; \quad \textstyle\sum_{i=1}^n a_i = 1\} \tag{35}$$

is always contained in the feasible set $\mathcal{F}$.

Proof: Let $\epsilon > 0$ and set

$$x(t) = \begin{cases} z + \epsilon & \text{if } y(t) = 1 \\ z - \epsilon & \text{if } y(t) = 0 \end{cases} . \tag{36}$$

Then, for every t, the second constraint in (17) turns out to be

$$\begin{cases} x(t) \in (z, +\infty) & \text{if } y(t) = 1 \\ x(t) \in (-\infty, z] & \text{if } y(t) = 0 \end{cases}$$

which is satisfied by construction, while the first constraint boils down to

$$-\delta_t \le z \pm \epsilon - \sum_{i=1}^{n} a_i \, (z \pm \epsilon) - \sum_{j=1}^{m} b_j \, u(t-j+1) \le \delta_t. \tag{37}$$

By setting $b_1 = \cdots = b_m = 0$ and $a_1 + a_2 + \cdots + a_n = 1$, (37) becomes $-\delta_t \leq \pm\epsilon - \sum_{i=1}^{n} \pm a_i\, \epsilon \leq \delta_t$ which is satisfied for any $\delta_t > 0$, provided that a sufficiently small ϵ is chosen. ■

Remark 3: Theorem 5 shows that, in the case of binary sensors, the hyperplane defined by the equality constraints in (35) is always contained in the FPS, unless this is excluded by suitable a priori information on the system dynamics (i.e., by choosing Θ_0 so that the set (35) is empty). Notice that the constraint $\sum_{i=1}^{n} a_i = 1$ implies that the system transfer function has a pole in 1, which corresponds to the fact that the system can "hold" indefinitely the constant value z even with no input signal, thus letting an arbitrarily small noise generate any binary output sequence y.

Now, let us consider the special case of an ARX(1,1) model $x(t) = a_1 x(t-1) + b_1 u(t) + d(t)$, $|d(t)| \leq \delta$. The following result holds.

Corollary 1: Let $u, y \in \mathbb{R}^N$ be given and $\underline{u} = \min_t u(t)$, $\overline{u} = \max_t u(t)$. Define the set

$$\mathcal{I} = \left\{\theta \in \Theta_0 : \begin{array}{l} 1 - \frac{\delta}{z} \leq a_1 + \frac{\underline{u}}{z} b_1 \leq 1 + \frac{\delta}{z} \\ 1 - \frac{\delta}{z} \leq a_1 + \frac{\overline{u}}{z} b_1 \leq 1 + \frac{\delta}{z} \end{array} \right\}. \tag{38}$$

Then, $\mathcal{I} \subseteq \mathcal{F}$.

Proof: As in the proof of Theorem 5, choose $x(t)$ as in (36). The first constraint in (17) becomes

$$-\delta \leq z \pm \epsilon - a_1(z \pm \epsilon) - b_1 u(t) \leq \delta$$

which is equivalent to

$$1 - \frac{\delta}{z} \leq a_1 + \frac{u(t)}{z} b_1 + g(\epsilon) \leq 1 + \frac{\delta}{z} \tag{39}$$

where $g(\epsilon) = \pm a_1 \frac{\epsilon}{z} \mp \frac{\epsilon}{z}$. For $\epsilon \to 0$, the inequalities (39) represent a strip in the (a_1, b_1)-plane. The intersections of such strips for $t = 1, \ldots, N$ provides the set $\mathcal{I}$ in (38). ■

Corollary 1 highlights that, for a given data set, there is always a set of nonzero measure contained in the FPS (namely, a parallelogram), which does not depend on the output provided by the binary sensor, but only on the extreme values taken by the input signal. Notice that for arbitrarily large input signals and arbitrarily small δ, the set $\mathcal{I}$ boils down to the point $\{\theta : \; b_1 = 0, \; a_1 = 1\}$, in accordance with Theorem 5. The result in Corollary 1 can be easily generalized to ARX models of arbitrary order.

VI. Examples

Example 1: Let us consider the following ARX system of order $n = 1$, $m = 1$

$$\begin{cases} x(t) = a\,x(t-1) + b\,u(t) + d(t) \\ y(t) = \mathcal{Q}(x(t)) \end{cases}$$

where $\mathcal{Q}(\cdot)$ denotes a binary sensor with threshold $z = 1$. Prior information on the system is $a \in [-2;\, 2]$, $|d(t)| \le 0.06$, $|u(t)| \le 1$. The true parameter values are $a = 0.78$, $b = 0.95$. Let $N = 100$ be the length of an identification experiment. The input signal has been uniformly generated in [-1;1]. By applying the proposed technique, one has $a \in [0.6895; 1.0600]$, $b \in [-0.0641; 1.2700]$, which coincides with the minimum orthotope containing the FPS as stated by Theorems 3 and 4. In Figure 4, the true feasible set (obtained by gridding), the computed bounds and the set $\mathcal{I}$ defined in Corollary 1 are reported.

The same system has been simulated by halving the noise level, i.e. $|d(t)| \le 0.03$, the other parameters being the same as in the preceding case. In Figure 5, the new feasible set along with computed bounds and set $\mathcal{I}$ are shown. Notice that in this case the feasible set is not connected. Parameter bounds are $a \in [0.7175; 1.0300]$, $b \in [-0.0320; 1.1496]$. Computation for both cases takes about 13 CPU seconds. [2]

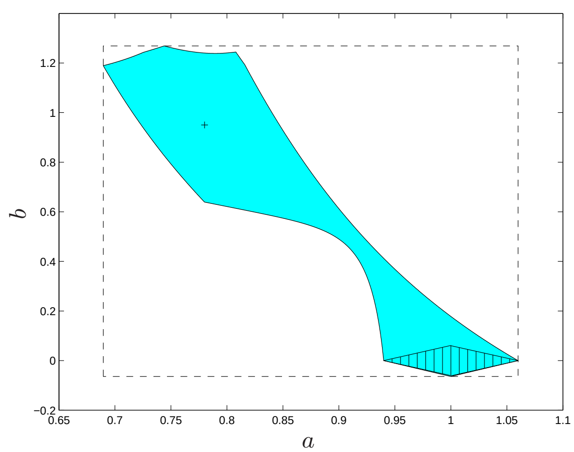

Fig. 4. Example 1. Feasible set (filled region), set $\mathcal{I}$ (striped region) and computed outer box. The true parameter vector is marked by a cross.

[2] Computations are performed under Matlab by using IBM ILOG CPLEX for MATLAB toolbox [35] to solve the LPs, on an Intel Core i7-3770 at 3.40 GHz.

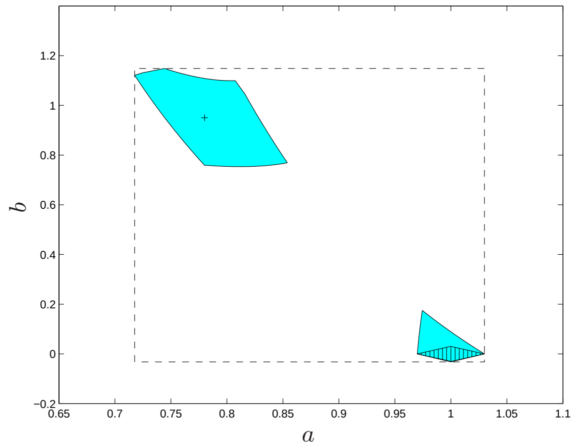

Fig. 5. Example 1. Feasible set (filled region), set $\mathcal{I}$ (striped region) and computed outer box. The true parameter vector is marked by a cross.

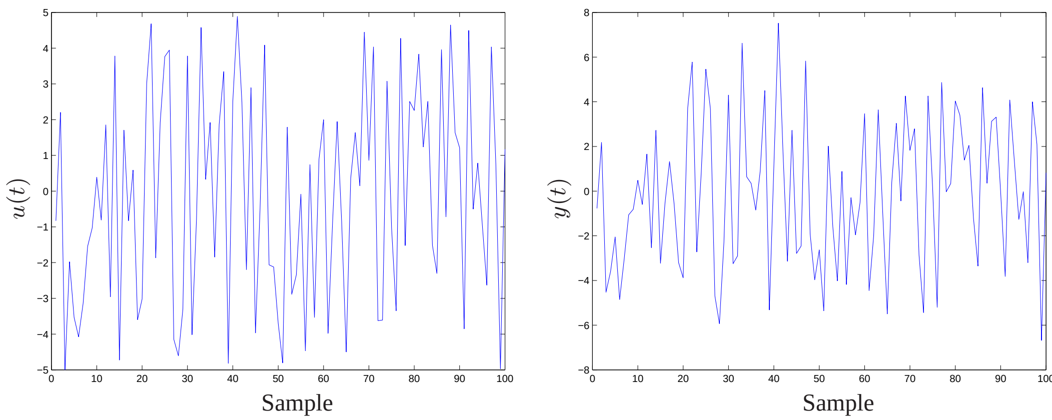

Fig. 6. Example 2. Input signal $u(t)$ and corresponding output $y(t)$, $t = 1, \ldots, 100$.

Example 2: Consider the following OE system of order $n = 2$, $m = 2$

$$\begin{cases} x(t) = -0.56\, x(t-1) - 0.23\, x(t-2) + 1.04\, u(t) + 0.73\, u(t-1) \\ y(t) = x(t) + e(t) \ . \end{cases}$$

We assume to know that the system is stable and that $|e(t)| \leq 0.1$ for all t. Let us apply an input sequence of length $N = 100$ uniformly distributed in [-5;5]. The output noise has been generated with a uniform distribution in [-0.1;0.1]. The input signal $u(t)$ and output $y(t)$ are reported in Fig. 6.

Let us apply the procedures described in Section III with a tolerance $\epsilon_a = \epsilon_b = 10^{-5}$ to obtain an orthotope containing the feasible set. The resulting approximated orthotope and the minimum

TABLE I

EXAMPLE 2. TIGHT AND COMPUTED BOUNDS FOR THE FEASIBLE SET.

	tight bounds		computed bounds	
	min	max	min	max
a_1	-0.56282	-0.55781	-0.56303	-0.55778
a_2	-0.23056	-0.22665	-0.23061	-0.22663
b_1	1.03755	1.04203	1.03747	1.04218
b_2	0.72840	0.73550	0.72817	0.73583

orthotope (computed by gridding) are reported in Table I. The projections $\mathcal{F}_a$ and $\mathcal{F}_b$ of the feasible set along with the computed bounds are reported in Fig. 7.

The total number of solved LPs is 906, for a computation time of 6 seconds.

Example 3: To show the effectiveness of the proposed approach, the bounds obtained by Algorithm 1 are compared with the tight bounds, for a number of different simulations. Two sets of 1000 randomly generated stable Output Error models of order $n = m = 2$ have been simulated by applying input sequences of $N = 100$ samples for two different signal-to-noise ratios. The tolerance ϵ_a used in Algorithm 1 has been set to 10^{-5}. The minimum outer box including the feasible set has been computed by gridding.

Let $\underline{a}_i$, $\overline{a}_i$, $i = 1, 2$, be the computed bounds, while $\underline{a}_i^*$, $\overline{a}_i^*$, $i = 1, 2$, denote the tight bounds. For each simulation, the following indexes have been computed to evaluate the performance of the proposed method:

$$R_1 = \frac{\overline{a}_1 - \underline{a}_1}{\overline{a}_1^* - \underline{a}_1^*} \qquad , \qquad R_2 = \frac{\overline{a}_2 - \underline{a}_2}{\overline{a}_2^* - \underline{a}_2^*}.$$

So, $R_i \geq 1$, $i = 1, 2$, is the ratio between the length of the computed interval on a_i, and the length of the tight interval.

Let us define the signal-to-noise ratio as

$$SNR = \frac{\max\limits_{t=1,\ldots,N} \{x_t\} - \min\limits_{t=1,\ldots,N} \{x_t\}}{\max\limits_{t=1,\ldots,N} \{e_t\} - \min\limits_{t=1,\ldots,N} \{e_t\}}.$$

In Table II, the mean value and standard deviation of R_1,R_2 and of the computation time are reported, for $SNR = 20$ and $SNR = 100$. In both cases, the computation times are below 4 seconds. When $SNR = 100$, the bounds obtained by Algorithm 1 are very close to the tight bounds. For $SNR = 20$, the computed bounds are larger; this is likely due to the fact that the

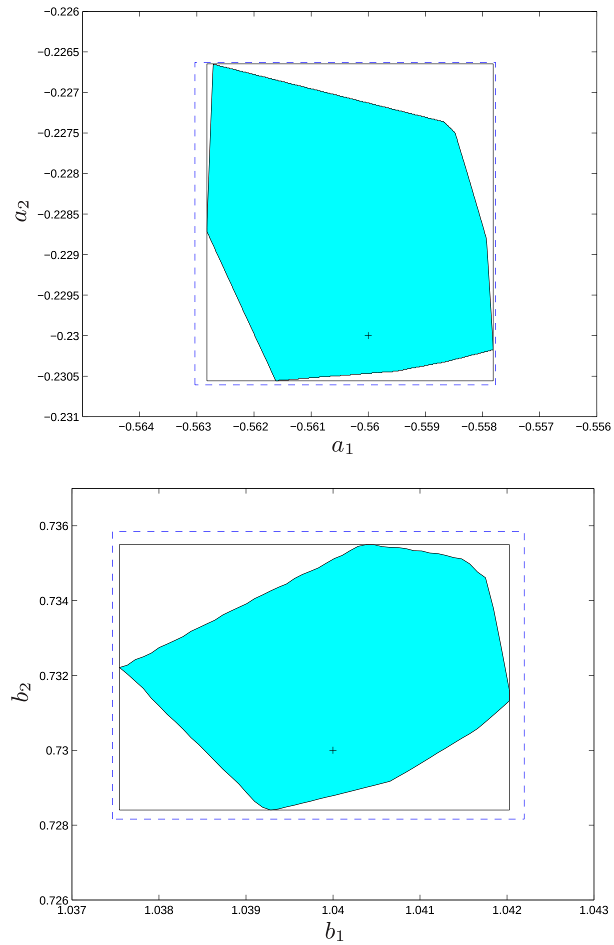

Fig. 7. Example 2. Sets $\mathcal{F}_a$ (top) and $\mathcal{F}_b$ (bottom), true parameter value (cross), minimum outer orthotope (solid) and computed outer orthotope (dashed).

true feasible interval for each parameter is larger, leading to a more conservative relaxation in problem (23).

Notice that this comparison cannot be done for parameters b_j, since computing tight bounds on b_j turns out to be a time demanding task.[3] For the same reason, performance evaluation for systems of order greater than 2 is not feasible.

Example 4: In this example, a number of simulations for different model orders have been performed to show the computation effort required to bound the parameters a_i and b_j, by using the procedures described in Section III. Models have been randomly generated and simulated

[3]Computing bounds on parameters b_j by a gridding technique in Example 2 takes several hours.

TABLE II

EXAMPLE 3. MEAN VALUE AND STANDARD DEVIATION OF R_1, R_2, AND COMPUTATION TIME.

	$SNR = 20$		$SNR = 100$	
	mean	std	mean	std
R_1	1.241	0.280	1.047	0.079
R_2	1.303	0.354	1.060	0.092
Computation time (s)	3.491	2.518	1.896	1.003

TABLE III

EXAMPLE 4. NUMBERS OF SOLVED LPS AND COMPUTATION TIME

$n = m$	# trials	mean # solved LPs	mean computation time
1	100	46.85	0.12 sec
2	100	281.62	1.81 sec
3	100	1441.84	22.65 sec
4	100	2539.38	63.05 sec
5	100	4851.57	203.26 sec
6	100	6795.85	594.33 sec

over a time horizon $N = 100$ with $SNR = 100$.

In Table III, the average number of solved LPs and the average computation times over 100 trials are reported. It is worthwhile to notice that the proposed procedure allows for parameter bound computation in reasonable times even for models with 12 parameters.

Example 5: In this example, the use of a polytopic approximation for bounding the feasible set is reported. It is also shown how this technique may lead to a performance improvement for orthotopic approximation.

Consider the OE model

$$\begin{cases} x(t) = 0.34\,x(t-1) + 0.46\,x(t-2) + 0.25\,u(t) - 0.53\,u(t-1) \\ y(t) = x(t) + e(t)\ . \end{cases}$$

Let us apply a random input sequence of length $N = 500$ and let $SNR = 20$. By applying the procedure described in Section III for finding a box approximation (with a tolerance 10^{-5}), one obtains the bounds on the autoregressive parameters $a_1 \in [0.33393; 0.34409]$, $a_2 \in [0.45597; 0.46466]$. The computation time is less than 1 minute.

Now, let us apply the polytopic approximation algorithm described in Section IV by choosing 200 randomly generated strips in the (a_1, a_2) plane. The computation time turns out to be about

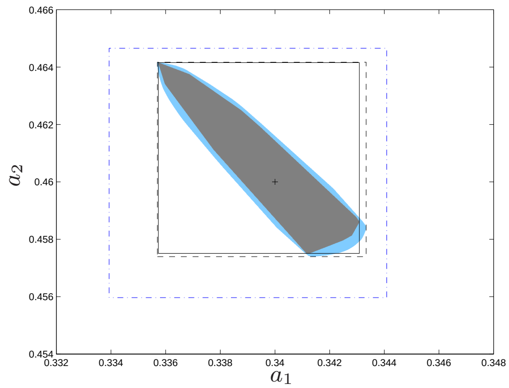

Fig. 8. Example 5. True feasible set (dark) and its polytopic approximation (light) on parameters a_1, a_2. Boxes denote the minimum outer orthotope (solid), final outer orthotope (dashed) and outer orthotope obtained by the procedure of Section III (dashed-dotted).

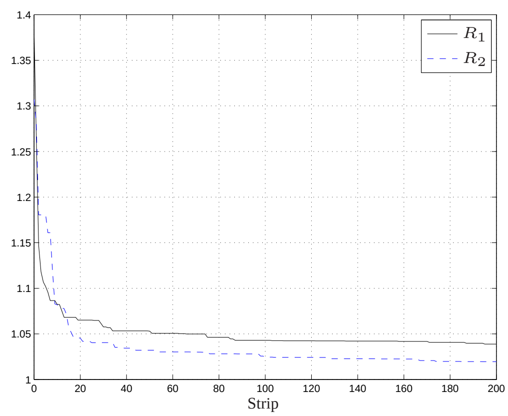

Fig. 9. Example 5. Indexes R_1 and R_2 against the number of approximating strips.

15 minutes. In Fig. 8, the true feasible set and the polytopic approximation are depicted. The same figure shows the minimum outer orthotope, the approximation obtained by applying the procedure of Section III and the orthotope obtained by bounding the polytopic approximation. The numerical bounds obtained are $a_1 \in [0.33570; 0.34333]$, $a_2 \in [0.45739; 0.46417]$.

By comparing indexes R_1 and R_2 defined in Example 3, one has that R_1 reduces from 1.382 to 1.039, while R_2 goes from 1.307 to 1.020. Fig. 9 shows R_1 and R_2 as a function of the processed strips. Notice that the knee of the plot is about 20, meaning that few iterations, i.e. few approximating strips, are sufficient to provide a considerable reduction of the initial bounds.

VII. Conclusions

The approach proposed in the paper has proved to be effective in computing orthotopic and polytopic approximations of nonconvex feasible parameter sets. The main reason lies in the exploitation of the model structure in the construction of the relaxations of the original nonconvex problem, and specifically in treating the autoregressive and input parameters in different ways. The proposed techniques have been successfully tested on a large number of numerical examples with randomly generated models and data sets.

The considered problem is far from being completely solved. As far as the polytopic approximation is considered, a heuristic has been proposed to prevent the explosion of the number of constraints. Nevertheless, alternative relaxations can be devised in order to explicitly trade-off computational complexity and accuracy of the feasible set approximation. Another topic which deserves further investigation is the construction of recursive procedures for computing online approximations of the feasible set. Techniques based on convex relaxations are usually batch in nature and their use within recursive estimation schemes is by no means straightforward. On the other hand, classic recursive techniques for the approximation of convex feasible sets can be adapted to the nonconvex case, but they typically lead to very conservative approximations. Recursive bounding procedures would also be useful to tackle other challenging estimation problems, such as the simultaneous state and parameter estimation for dynamic systems affected by bounded perturbations. Another direction of future research concerns the extension of the proposed techniques to specific families of nonlinear models, such as quantized NARX or nonlinear output error models.

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