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Switching pursuit strategies for multi-pursuer single-evader games

Marco Casini *, Andrea Garulli

Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena, Via Roma 56, 53100 Siena, Italy

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ABSTRACT

This paper introduces a new family of pursuit strategies for multi-pursuer single-evader games in a planar environment. The main idea is to exploit conditions under which capture of the evader in minimum time can be achieved by only two pursuers. The first contribution is to characterize such conditions in terms of the agent positions. Then, new pursuit strategies are proposed in which the multi-pursuer team aims to meet such conditions, switching to a two-pursuer game once they are satisfied. The benefit of this approach is twofold. First, it is shown that naive strategies that are in general unsuccessful can be turned into winning strategies by switching to the appropriate two-pursuer game. Second, the switching mechanism significantly enhances the performance of existing pursuit algorithms, like those based on Voronoi partitions. This is demonstrated by means of extensive numerical simulations.

1. Introduction

Pursuit-evasion games are intensively studied because they allow to tackle a variety of problems involving multi-agent systems. The large number of applications treated in the literature range from mobile robotics [1,2] to optimal robot control [3], space operations [4,5], predator–prey dynamics in biological systems [6], and beyond. Recent years have witnessed a surge in interest towards games involving multiple pursuers [7–9], multiple evaders [10,11], or both [12,13], leading to a broad spectrum of problem formulations and solutions.

Traditionally, pursuit-evasion problems are approached by framing them as differential games [14,15]. Unfortunately, an optimal solution is available only for a limited number of games. An example is that of the minimum-time solution of the two-pursuer one-evader game involving simple-motion players in a two-dimensional environment [16–18]. For the three-pursuer one-evader game, only suboptimal solutions have been derived [19], although it has been established that optimal strategies must involve switching between different linear sub-paths [20,21]. An open-loop minimum-time solution for games with more than three pursuers has been proposed in [22], showing that the agents must move along linear paths. However, a closed-loop solution to this differential game remains elusive. The above observations motivate the investigation of conditions under which the optimal solution of the two-pursuer game can be exploited to devise successful strategies for games involving more than two pursers.

1.1. Related work

Due to the complexity of multi-agent pursuit-evasion games, such problems have been often approached by splitting them in simpler subgames. This has led to the development of solutions based on hierarchical scheduling [23], dynamic pursuit assignment [24], optimal task allocation [25], just to name a few. Considering the scenario of multi-pursuer single-evader game with faster pursuers, several techniques have been proposed which aim at reducing the problem to a two-pursuer game, for which an optimal solution is available. In [26], a pursuit strategy is proposed in which all possible two-pursuer games are compared and the one which yields the smallest capture time is actually selected. The strategy is played continuously in feedback, so that switching may occur between different two-pursuer games (since all such games eventually lead to capture). A similar approach is considered in [27] for reach-avoid games involving pursuers modeled as Dubins cars. It is shown that at most two pursuers are sufficient to defend against a single attacker, thus greatly simplifying the multi-pursuer problem.

A key element of pursuit-evasion games is the information pattern available to the players, which strongly influences both the problem setting and the corresponding solution. For example, in multi-pursuer single-evader games involving simple-motion agents with equal maximum speed, a winning pursuit strategy was proposed in [19] and the game of kind was solved in [28], assuming that the pursuers have access to the velocity of the evader. In the same setting, the pursuit

* Corresponding author.
E-mail addresses: marco.casini@unisi.it (M. Casini), andrea.garulli@unisi.it (A. Garulli).

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task becomes much more difficult when the pursuers know only the current position of the evader. In this case, the solution of the game of kind is provided in [29]. Quite interestingly, such a solution is based on a pursuit strategy which switches progressively to games played by a smaller number of pursuers. However, the strategies adopted in these subgames are not optimal and only guarantee the eventual capture of the evader. The same information pattern is adopted in [7], where the pursuers aim to minimize the area of the evader Voronoi cell. This technique guarantees that the evader is captured in finite time, but it does not optimize over the time required to terminate the game. In [30], the optimal capture time for this class of games is characterized via a set of implicit conditions. However, pursuit techniques that are able to achieve capture in minimum time are not available yet.

1.2. Paper contribution

This paper proposes a new family of pursuit strategies for a multi-pursuer single-evader game, in which agents move in a planar environment with simple motion and equal maximum speed. Both the pursuers and the evader have access only to the agents' positions. The core idea is to derive and leverage conditions under which the minimum-time game boils down to a two-pursuer game. This allows one to design pursuit strategies aiming at enforcing these conditions and transitioning to a two-pursuer game when they are satisfied. Although these switching pursuit strategies are in general suboptimal, they consistently reduce capture time compared to strategies without switching. As a result, they enhance the performance of existing pursuit techniques and enable the development of new effective multi-pursuer strategies.

The main contributions of the paper are as follows. First, conditions under which the minimum-time pursuit strategy boils down to that of a two-pursuer single-evader game are established. In such cases, switching to the two-pursuer game provides an optimal solution to the multi-pursuer one. The second contribution is the definition of new families of switching pursuit strategies, based on the previously derived conditions, that guarantee capture of the evader. Specifically, simple strategies like pure pursuit or fixed-point pursuit, which in general fail to capture the evader, are transformed into successful strategies through a switch to the appropriate two-pursuer game. Furthermore, it is demonstrated through numerical simulations that switching can substantially reduce the capture time, even for strategies that inherently guarantee capture, such as the Voronoi partition-based approach introduced in [7]. On the whole, the proposed switching mechanism provides a powerful framework for the design of a variety of new pursuit strategies in multi-pursuer single-evader games. A preliminary version of this work has been presented in [31]: the present paper contains all the proofs of the results, expands the technical development, and presents a new simulation campaign to thoroughly assess the performance of the proposed technique.

As for the paper organization, the formulation of the multi-pursuer single-evader game is given in Section 2, along with a summary of the two-pursuer game optimal solution. Section 3, after revisiting the solution to the game of kind, provides the conditions under which the minimum-time multi-pursuer game boils down to a two-pursuer one. This paves the way to the introduction of new switching pursuit strategies in Section 4, whose performance is assessed in Section 5 by means of numerical simulations. Concluding remarks and future developments are reported in Section 6.

1.3. Notation and definitions

Given a vector V, its transpose is denoted by V', while $\|V\|$ is its Euclidean norm. For $V, W \in \mathbb{R}^2$, we denote by $\overline{VW}$ the segment with V and W as endpoints. $\mathcal{C}(P,r) = \{Q \in \mathbb{R}^2 : \|Q-P\| \le r\}$ denotes a circle centered in $P \in \mathbb{R}^2$ with radius r. For a closed set $\mathcal{A}$, $\partial\mathcal{A}$ is the boundary of $\mathcal{A}$, while $\mathbb{H}\{\mathcal{A}\}$ denotes the convex hull of $\mathcal{A}$. Given two sets $\mathcal{A},\mathcal{B}$, $\mathcal{A}\backslash\mathcal{B} = \{Q : Q \in \mathcal{A},\ Q \notin \mathcal{B}\}$. The interior of $\mathcal{A}$ is denoted by $\mathtt{int}\{\mathcal{A}\} = \mathcal{A}\backslash\partial\mathcal{A}$.

2. Multi-pursuer single-evader game

A *multi-pursuer single-evader game* (hereafter referred to as MP1EG) is a game in which p pursuers aim at capturing one evader. The positions of the pursuers and the evader at time t are denoted by $P_k(t) \in \mathbb{R}^2, k = 1,\dots,p$, and $E(t) \in \mathbb{R}^2$, respectively. The players move in the plane in simple motion, that is

$$\begin{cases} \dot{E}(t) = v_E(t), \\ \dot{P}_k(t) = v_{P_k}(t), \ k = 1,\dots,p \end{cases}$$

where $v_E(t), v_{P_k}(t) \in \mathbb{R}^2$ are the velocity vectors of the players. We assume that all agents have the same maximum speed, $\|v_E(t)\| \le v_{max}$, $\|v_{P_k}(t)\| \le v_{max}$, $k = 1,\dots,p$, $\forall t \ge 0$. Without loss of generality, it is set $v_{max} = 1$ and the initial game time at $t = 0$. At each time t, $v_E(t)$ and $v_{P_k}(t)$ are computed as functions of the state of the game $\xi(t) = [E'(t)\ P_1'(t)\ \dots\ P_p'(t)]'$. Notice that the agents do not have information on the velocities of the other players. The *strategies* of the agents are closed-loop control laws mapping the current game state into the agent velocity vectors, i.e., $v_E = \phi(\xi)$, $v_{P_k} = \psi_k(\xi)$. Dependence on time is omitted when it is clear from the context.

The aim of the pursuers is to capture the evader, while the evader tries to avoid capture or to protract the game as long as possible. Capture occurs when the distance between at least one pursuer and the evader is equal to the radius of capture $r > 0$. Hereafter, only initial conditions such that

$$\|P_k(0) - E(0)\| > r, \quad k = 1,\dots,p \tag{1}$$

will be considered. A *winning pursuit strategy* is a strategy adopted by the pursuers which guarantees that capture occurs in finite time whatever is the strategy adopted by the evader. The capture time T is such that

$$\|P_k(T) - E(T)\| = r, \ \text{for some } k \in \{1,\dots,p\}. \tag{2}$$

A *minimum time pursuit strategy* is a strategy $\psi = [\psi_1,\dots,\psi_p]$ which is a solution of the min–max problem

$$\min_{\psi} \max_{\phi} \ T. \tag{3}$$

For brevity, we will refer to functions ψ and ϕ solving problem (3) as *optimal strategies*. To the best of our knowledge, (3) is still an open problem for $p \ge 3$. Conversely, for $p = 2$ the problem has been fully solved [16–18]. In the following, the solution of the *two-pursuer one-evader game* (2P1EG) given in [18] is recalled. It will play a key role in the definition of the class of winning strategies for the MP1EG proposed in this work.

2.1. Minimum-time solution of the two-pursuer one-evader game

Consider two pursuers P_1, P_2 trying to capture one evader. Let us define the region

$$\mathcal{D}_{12} = \mathtt{int}\left\{\mathbb{H}\left\{\mathcal{C}(P_1,r) \cup \mathcal{C}(P_2,r)\right\}\right\}. \tag{4}$$

First, a solution to the game of kind is provided [18].

Proposition 1. *There exists a winning pursuit strategy for the 2P1EG if and only if $E \in \mathcal{D}_{12}$.*

Hence, set $\mathcal{D}_{12}$ is the *2-pursuer capture region* associated to P_1 and P_2; if the evader belongs to this region it will be captured by the pursuers in finite time. An example is depicted in Fig. 1. To simplify the treatment, w.l.o.g. we will refer to the reference frame reported in Fig. 1, where the origin is the midpoint of the pursuers, and they lie on the x-axis, i.e., $P_1 = [-d, 0]'$, $P_2 = [d, 0]'$, $E = [x, y]'$. Let $E \in \mathcal{D}_{12}\backslash(\mathcal{C}(P_1,r)\cup\mathcal{C}(P_2,r))$ and define

$$T_{12} = \frac{\kappa r + |y|\sqrt{\kappa^2 - 4x^2(r^2 - y^2)}}{2(r^2 - y^2)} \tag{5}$$

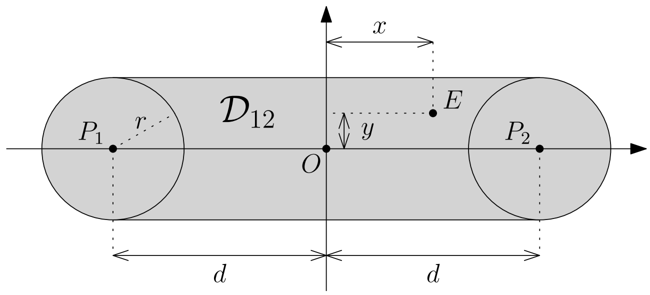

Fig. 1. The 2-pursuer capture region for the 2P1EG.

where $\kappa = d^2 - x^2 + y^2 - r^2$. Now, let us define the point H_{12} satisfying

$$\|E - H_{12}\| = \|P_1 - H_{12}\| - r = \|P_2 - H_{12}\| - r = T_{12}. \tag{6}$$

Within the considered reference frame, one has $H_{12} = [0,\ h_{12}]'$ where

$$h_{12} = \begin{cases} y + \text{sign}(y)\sqrt{T_{12}^2 - x^2} & \text{if} y \neq 0 \\ \pm\sqrt{T_{12}^2 - x^2} & \text{if} y = 0. \end{cases} \tag{7}$$

The following result provides a solution to the minimum-time 2P1EG [18].

Proposition 2. *Let $p = 2$ and $E \in \mathcal{D}_{12}$. The strategies solving problem (3) require each player to travel at maximum speed along a straight line to the point H_{12} defined by (6). Then, the evader will be captured in H_{12}, at time T_{12} given by (5).*

It is worth remarking that the strategies provided by Proposition 2 are optimal in closed-loop sense. In particular, this means that if the pursuers adopt the optimal strategy and the evader does not, the evader will be captured in a time smaller than T_{12} in (5).

3. Switching conditions in multi-pursuer games

In this section, the solution of the game of kind for the MP1EG is first recalled. Then, a condition is derived under which the MP1EG boils down to a 2P1EG.

For a pair of pursuers P_i, P_j, let $\mathcal{D}_{ij}$ be the corresponding 2-pursuer capture region, given by (4). If $E \in \mathcal{D}_{ij}$, let us denote by ψ^{ij} the minimum-time pursuit strategy for the 2P1EG played by P_i and P_j, and by ϕ^{ij} the corresponding optimal evader strategy. Moreover, we denote by T_{ij} the time needed by P_i and P_j to capture E when the agents P_i, P_j and E play the optimal 2P1EG strategies. The capture time T_{ij} and capture point H_{ij} are computed according to (5)–(6), for each pair i, j such that $E \in \mathcal{D}_{ij}$.

Let us define the convex hull of the pursuer locations

$$\mathcal{P} = \mathbb{H}\left\{P_1, P_2, \dots, P_p\right\} \tag{8}$$

and introduce the *multi-pursuer capture region*, defined as

$$\mathcal{M} = \mathtt{int}\left\{\mathbb{H}\left\{\bigcup_{i=1}^{p} C(P_i, r)\right\}\right\}. \tag{9}$$

The solution of the game of kind for the MP1EG proposed in [29] is recalled next.

Proposition 3. *If $E \in \mathcal{M}$ in (9), then there exists a winning pursuit strategy. On the contrary, if $E \notin \mathcal{M}$, the evader can avoid capture indefinitely.*

Fig. 2 shows an example of the set $\mathcal{M}$, in a problem with three pursuers. In the figure, the evader is outside $\mathcal{M}$, and hence it can escape by going straight in the direction v, which is orthogonal to the dashed hyperplane separating the evader from $\mathcal{M}$.

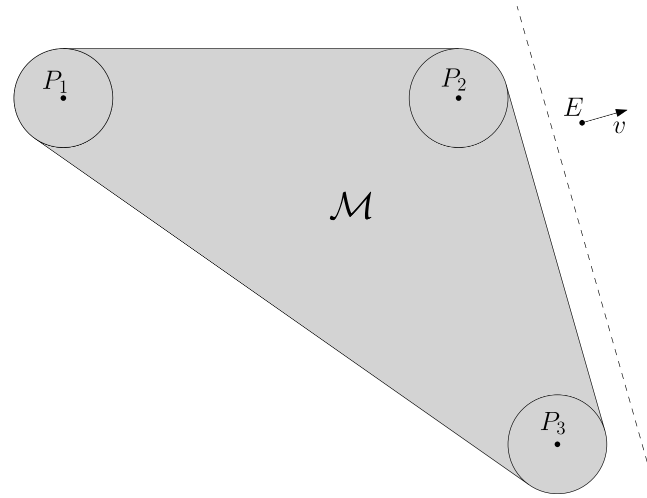

Fig. 2. Example of multi-pursuer capture region $\mathcal{M}$ for $p = 3$ (shaded). Since $E \notin \mathcal{M}$, the evader can avoid capture by moving along the direction v.

The next theorem reports conditions under which the MP1EG reduces to the 2P1EG, in the sense that only two pursuers can provide capture in minimum time, irrespectively of the strategy played by the other pursuers.

Theorem 1. *Assume that at time t, $E \in \mathcal{D}_{ij}$ for some $i, j \in \{1, 2, \dots, p\}$, and*

$$\|H_{ij} - P_k\| \geq T_{ij} + r, \quad k = 1, \dots, p. \tag{10}$$

Then, from time t onwards, the optimal MP1EG strategies for P_i, P_j, E are ψ^{ij} and ϕ^{ij}, respectively, and capture occurs at time $t + T_{ij}$. The strategies adopted by the other pursuers are irrelevant to the game duration.

Proof. W.l.o.g. assume $t = 0$. If P_i, P_j and E play their optimal strategies for the 2P1EG, capture will occur at the location H_{ij} at time T_{ij}. So, $\|H_{ij} - P_i\| = \|H_{ij} - P_j\| = T_{ij} + r$. Let P_k denote a generic pursuer not involved in the considered 2P1EG, i.e., $k \neq i, j$. We prove that, if condition (10) holds, any strategy adopted by P_k cannot reduce the capture time below T_{ij}. This is equivalent to show that P_k cannot capture the evader during its path to H_{ij}. According to Proposition 2, let $v_E = \frac{H_{ij} - E(0)}{\|H_{ij} - E(0)\|}$ denote the velocity vector of the evader. Since E travels towards H_{ij} following a straight path, one has $E(\tau) = E(0) + \tau v_E$, $0 \leq \tau \leq T_{ij}$. Since $\|H_{ij} - E(0)\| = T_{ij}$, then $C(E(0), r) \subset C(H_{ij}, T_{ij} + r)$ and $C(E(\tau), r) \subset C(H_{ij}, T_{ij} + r - \tau)$, for all τ such that $0 \leq \tau \leq T_{ij}$ (see Fig. 3).

Consider the case $\|H_{ij} - P_k(0)\| > T_{ij} + r$, where strict inequality is assumed in (10). It follows that $P_k(0) \notin C(H_{ij}, T_{ij} + r)$ and then for any possible trajectory of P_k one has $P_k(\tau) \notin C(H_{ij}, T_{ij} + r - \tau)$, $0 \leq \tau \leq T_{ij}$. Hence, $P_k(\tau) \notin C(E(\tau), r)$, for all $0 \leq \tau \leq T_{ij}$ and then P_k cannot capture E before T_{ij}.

It remains to consider the case when condition (10) holds with equality, i.e., $\|H_{ij} - P_k\| = T_{ij} + r$. Let $Q = E(0) - r v_E$. To avoid capture at time 0, it must be $P_k(0) \neq Q$. By going straight towards H_{ij}, the kth pursuer will always lie on the border of $C(H_{ij}, T_{ij} + r - \tau)$, thus leading again to $P_k(\tau) \notin C(E(\tau), r)$ for $0 \leq \tau < T_{ij}$. When $\tau = T_{ij}$ one will have $\|P_k(T_{ij}) - H_{ij}\| = r$, meaning that P_i, P_j and P_k will capture the evader simultaneously at time T_{ij}. $\square$

Notice that, if P_i, P_j and E play their optimal 2P1EG strategies, one has $\|H_{ij} - P_i\| = \|H_{ij} - P_j\| = T_{ij} + r$ and so (10) can be rewritten as

$$\|H_{ij} - P_k\| \geq \|H_{ij} - P_i\|, \quad k = 1, \dots, p. \tag{11}$$

Such a condition states that if there is no pursuer closer to H_{ij} than P_i (and P_j), then (10) holds and the optimal MP1EG strategies for P_i, P_j, E

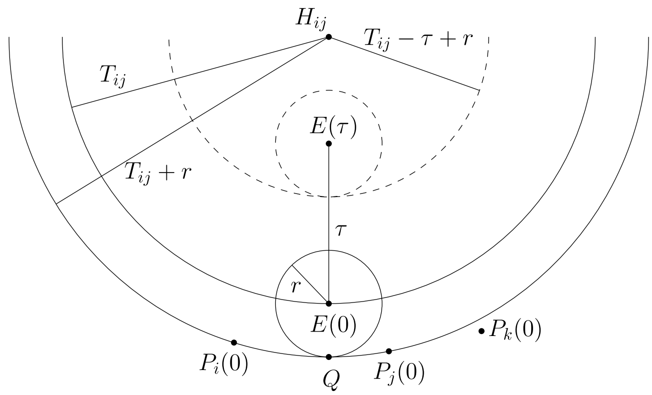

Fig. 3. Sketch of the proof of Theorem 1.

are ψ^{ij} and ϕ^{ij}. This means that the other pursuers cannot improve the capture time even if they go straight to H_{ij}. As a direct consequence of Theorem 1, if there exist two distinct pairs of pursuers (P_i, P_j) and (P_h, P_l) satisfying the conditions of Theorem 1, then $T_{ij} = T_{hl}$.

Theorem 1 states that if condition (10) or (11) is satisfied by a pair of pursuers P_i, P_j at a certain time t, the strategies of these two pursuers must switch to ψ^{ij} in order to be optimal from time t onwards, irrespectively of what the other pursuers will do. Similarly, the evader must switch to strategy ϕ^{ij} in order to maximize its survival time. This fact will be exploited in Section 4 to devise new families of winning pursuit strategies for the MP1EG. The next result provides a further insight on the way pursuers cooperate to capture the evader after condition (10) has occurred.

Theorem 2. *Assume that condition (10) occurs at time $t > 0$, and let P_i, P_j denote the pursuers related to the corresponding 2P1EG. If $E(t) \notin \overline{P_i(t)P_j(t)}$ then $\exists k \neq i, j$ such that $H_{ij}(t)$ is the centroid of $P_i(t), P_j(t), P_k(t)$.*

Proof. Since $E(t) \notin \overline{P_i(t)P_j(t)}$, $H_{ij}(t)$ is uniquely defined due to (7). By (11) one has

$$\|H_{ij}(t) - P_k(t)\| \geq \|H_{ij}(t) - P_i(t)\| = \|H_{ij}(t) - P_j(t)\|, \forall k \neq i, j. \tag{12}$$

Let $\varepsilon > 0$ and set $\tau = t - \varepsilon$. Being $\mathcal{D}_{ij}$ an open set (see (4)), it is possible to choose ε sufficiently small so that $E(\tau) \in \mathcal{D}_{ij}(\tau)$, with $H_{ij}(\tau)$ defined according to (6). On the other hand, since at time τ condition (10) does not hold, one has that there exists $k \neq i, j$ such that

$$\|H_{ij}(\tau) - P_k(\tau)\| < \|H_{ij}(\tau) - P_i(\tau)\| = \|H_{ij}(\tau) - P_j(\tau)\|. \tag{13}$$

By the arbitrariness of ε and the continuity of the agents' trajectories, (12) and (13) imply that

$$\|H_{ij}(t) - P_k(t)\| = \|H_{ij}(t) - P_i(t)\| = \|H_{ij}(t) - P_j(t)\|$$

and hence $H_{ij}(t)$ is the centroid of $P_i(t), P_j(t), P_k(t)$. □

As a consequence of Theorem 2, if the evader is not aligned with pursuers P_i and P_j when condition (10) is verified, there are at least three pursuers that will end up capturing the evader simultaneously by heading towards their centroid $H_{ij}(t)$.

4. Switching pursuit strategies

In this section, the concept of *switching pursuit strategy* is introduced. The aim of such strategies is to guarantee that condition (10) holds at some finite time, and hence capture is assured by switching to the appropriate 2P1EG strategies.

Definition 1. A pursuit strategy is a *potential switching strategy* if there exists a pair of pursuers P_i, P_j for which condition (10) holds at some finite time, for any possible strategy of the evader. A potential switching strategy is referred to as *switching strategy* if P_i, P_j play the optimal 2P1EG strategy as soon as condition (10) holds.

By the previous definition, a switching strategy leads to the evader capture in finite time, and so it is a winning strategy. In fact, from the switching time onwards the pursuers will play the optimal 2P1EG strategy, guaranteeing the evader capture.

In general, since condition (10) depends on the evader position, it is not easy to design pursuit strategies that eventually lead to its satisfaction. Therefore, it is useful to derive sufficient conditions under which (10) holds, that the pursuers can enforce whatever is the strategy played by the evader. Hereafter, two such conditions are presented.

Let us define the union of all the 2-pursuer capture regions as

$$\widehat{\mathcal{D}} = \bigcup_{i,j \in \{1,\dots,p\}} \mathcal{D}_{ij}. \tag{14}$$

An example of set $\widehat{\mathcal{D}}$ is shown in Fig. 4-(a), while the corresponding set $\widehat{\mathcal{D}} \backslash \mathtt{int}\,\{\mathcal{P}\}$ is depicted in Fig. 4-(b). The next theorem provides a sufficient condition for switching to 2P1EG.

Theorem 3. *If $E \in \widehat{\mathcal{D}} \backslash \mathtt{int}\,\{\mathcal{P}\}$, then there exists a pair of pursuers P_i, P_j for which condition (10) holds.*

Proof. See Appendix. □

Remark 1.

Theorem 3 can be alternatively formulated stating that if $E \in \mathcal{M} \backslash \mathtt{int}\,\{\mathcal{P}\}$, there exists a pair of pursuers for which condition (10) holds. In fact, if $E \in \mathcal{M} \backslash \mathtt{int}\,\{\mathcal{P}\}$, there are necessarily two pursuers P_i, P_j, such that $E \in \mathcal{D}_{ij}$, which implies $E \in \widehat{\mathcal{D}} \backslash \mathtt{int}\,\{\mathcal{P}\}$.

Notice that Theorem 3 guarantees the satisfaction of (10) if $E \in \partial\mathcal{P}$. Moreover, Theorem 3 suggests that if the pursuers play a strategy which steadily reduces the size (in some sense) of $\mathcal{P}$ over time, then (10) is eventually assured. The following result guarantees the satisfaction of (10) whenever the pursuers are sufficiently close to each other.

Theorem 4. *Let us consider a pursuit strategy such that*

$$\max_{i,j=1,\dots,p} \|P_i(\bar{t}) - P_j(\bar{t})\| \leq \sqrt{3}r \tag{15}$$

at a certain time $\bar{t} > 0$. Then, there exist a time $\hat{t} < \bar{t}$ and a pair of pursuers P_i, P_j for which condition (10) holds at $\hat{t}$. Hence, the considered pursuit strategy is a potential switching strategy.

Proof. See Appendix. □

Notice that condition (15) in Theorem 4 is much simpler than (10), since it does not depend on the evader position. Therefore, it can be effectively employed in designing families of winning switching strategies, even starting from naive pursuit strategies that would not be successful without switching. A first family consists of pursuers P_k going straight towards a fixed point $M \in \mathbb{R}^2$, with speed $\|v_{P_k}(t)\| \geq \varepsilon$, for some $0 < \varepsilon \leq 1$, and stopping once they reach M. It is referred to as *Fixed-Point Pursuit Strategy (FPPS)*. The following result holds.

Theorem 5. *Any FPPS is a potential switching strategy.*

Proof. Let

$$d = \max_{k \in \{1,\dots,p\}} \|P_k(0) - M\|.$$

Then, at time $\tau = \max\left\{\left(d - \frac{\sqrt{3}}{2}r\right)/\varepsilon, 0\right\}$ one has $\|P_k(\tau) - M\| \leq \frac{\sqrt{3}}{2}r$, $k = 1, \dots, p$, and hence $\|P_i(\tau) - P_j(\tau)\| \leq \|P_i(\tau) - M\| + \|M - P_j(\tau)\| \leq \sqrt{3}r$, for any pair of pursuers P_i, P_j. Thus, the result follows by Theorem 4. □

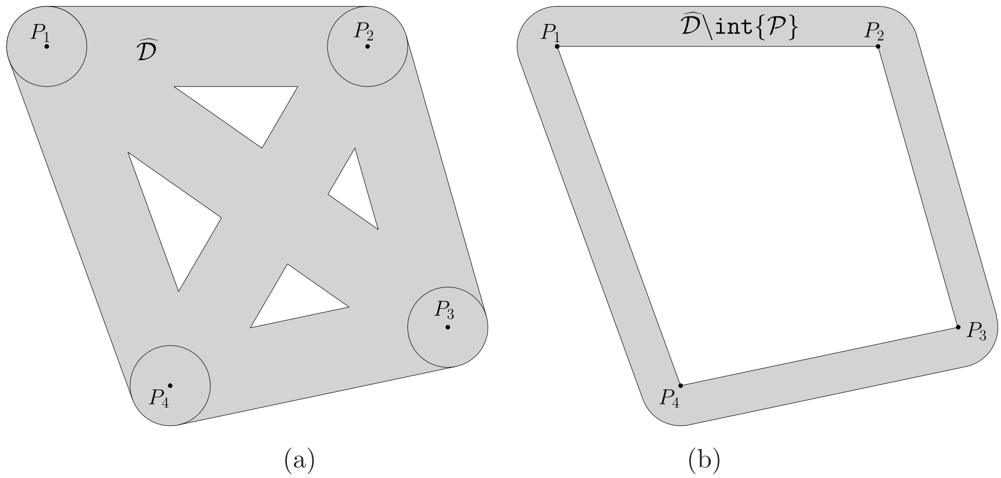

Fig. 4. (a) Set $\widehat{D}$ denoting the union of all the 2-pursuer capture regions; (b) set $\widehat{D}\backslash\mathtt{int}\{\mathcal{P}\}$.

An example of FPPS is the *centroid* strategy, in which all pursuers (or a subset of them, including the evader in their convex hull) move with the same speed towards the center of the minimum-radius circle which includes them. Notice that such a strategy is consistent with the fact that under certain conditions, capture occurs in the centroid of a subset of pursuers, as stated by Theorem 2.

Another simple pursuit strategy is the so-called *pure pursuit*. In this strategy, all pursuers always point towards the evader at maximum speed, i.e.

$$v_{P_k}(t) = \frac{E(t) - P_k(t)}{\|E(t) - P_k(t)\|}, \quad k = 1, \dots, p. \tag{16}$$

It is well known that pure pursuit does not guarantee capture of the evader when all agents have the same maximum speed. However, it becomes a winning strategy if switching to 2P1EG is adopted.

Theorem 6. *The pure pursuit strategy is a potential switching strategy.*

Proof. Let $\delta_k(t) = \|E(t) - P_k(t)\|$. Then, by using (16),

$$\frac{1}{2}\frac{d\delta_k^2(t)}{dt} = (E(t) - P_k(t))'(\dot{E}(t) - \dot{P}_k(t)) = (E(t) - P_k(t))'\dot{E}(t) - \delta_k(t).$$

Being $\|\dot{E}(t)\| = \|v_E(t)\| \le 1$, one has $\frac{d\delta_k^2(t)}{dt} \le 0$ and in particular $\frac{d\delta_k^2(t)}{dt} = 0$ if and only if $v_E(t) = v_{P_k}(t)$ given by (16). Hence, $\lim_{t\to+\infty} \delta_k(t) = \bar{\delta}_k \ge 0$ and $\lim_{t\to+\infty} \dot{\delta}_k(t) = 0$, which in turns leads to $\lim_{t\to+\infty} v_{P_k}(t) = \lim_{t\to+\infty} v_E(t)$, $\forall k = 1, \dots, p$. Since, according to (16), all pursuers point towards the evader, this means that all pursuers and the evader tend to be asymptotically aligned, with the evader lying outside the segment containing the pursuers. Therefore, $\lim_{t\to+\infty} E(t) \notin \mathcal{P}$. By continuity of the agents' trajectories, $E(t)$ must have crossed $\partial\mathcal{P}$ at some time instant and hence the conclusion follows by Theorem 3. □

Theorems 5 and 6 are just examples of winning pursuit strategies based on switching; it is apparent that many other switching strategies can be devised by exploiting the geometric conditions in Theorems 3 and 4.

Remark 2. It is worth stressing that in all the proposed strategies, switching from MP1EG to 2P1EG occurs only once during the game, provided that after the switching the two pursuers and the evader involved in the 2P1EG play the corresponding optimal strategies defined by Proposition 2. This means that the switching strategies are not prone to chattering, as it may happen in other types of games (see, e.g., [26]). Clearly, if the evader does not play its optimal strategy, another switching to a different 2P1EG may occur later, with the effect of further reducing the capture time. In fact, by definition of optimal strategy, once the first switching has occurred, the optimal 2P1EG strategy of the evader maximizes the evader survival time.

5. Numerical simulations

In order to show the benefits of using a switching strategy, several numerical simulations are reported in this section. Three pursuit strategies are considered: pure-pursuit (PPS), centroid-based (CS), and the strategy proposed in [7], referred to as Voronoi-based strategy (VS). Notice that VS is a winning strategy, provided that when the game starts, the evader lies in the interior of the convex hull of the pursuers (so that the evader Voronoi cell is not empty, see [7] for details). The prefix S- is used to denote a given pursuit strategy when pursuers switch to 2P1EG as soon as condition (10) holds. For instance, S-PPS denotes the switching pure-pursuit strategy.

In all simulations, the maximum speed of the players is set to 1 and the capture radius is $r = 1$. In the figure illustrating the agents' trajectories, pursuers are drawn in blue and the evader in red. Initial conditions are marked with a square, while the final positions of the agents are denoted by a dot. A dashed circle with radius r is drawn around the capture position of the evader.

Example 1. Let us consider a four-pursuer one-evader game, where pursuers are initially located at the vertices of a square with side length 20 centered at the origin, i.e., $P_1(0) = [-10, -10]'$, $P_2(0) = [-10, 10]'$, $P_3(0) = [10, -10]'$, $P_4(0) = [10, 10]'$. The evader starts from $E(0) = [0, 5]'$, and it goes upwards with maximum speed during each game.

If the pursuers move according to PPS and CS, the evader can escape, and so such pursuit strategies are not winning in this scenario. On the contrary, if pursuers switch when (10) occur, the evader is captured at time $t = 22.66$ and $t = 33.91$ for S-PPS and S-CS, respectively. The trajectories traveled by the players in such cases are depicted in Figs. 5 and 6.

When pursuers play the original version of the Voronoi-based strategy, they are able to capture the evader at time $t = 20.02$, as depicted in Fig. 7. When playing the corresponding switching version, capture time is reduced to $t = 16.61$. So, in this example, the capture time needed by VS is about 20% greater than that needed by S-VS. The agents' trajectories are shown in Fig. 8, where it can be observed that in this case the capture point coincides with the centroid of all the four pursuers, after condition (10) has occurred.

Example 2. The aim of this example is to evaluate the benefit of using S-VS in place of VS. To this purpose, a simulation campaign with different number of pursuers located in different places is performed. In all simulations, the evader position is initially set to the origin. Pursuers start randomly inside a circle of radius R_p centered at the origin. Only initial positions such that the evader belongs to $\mathtt{int}\{\mathcal{P}\}$ are considered, so that VS always achieves capture of the evader.

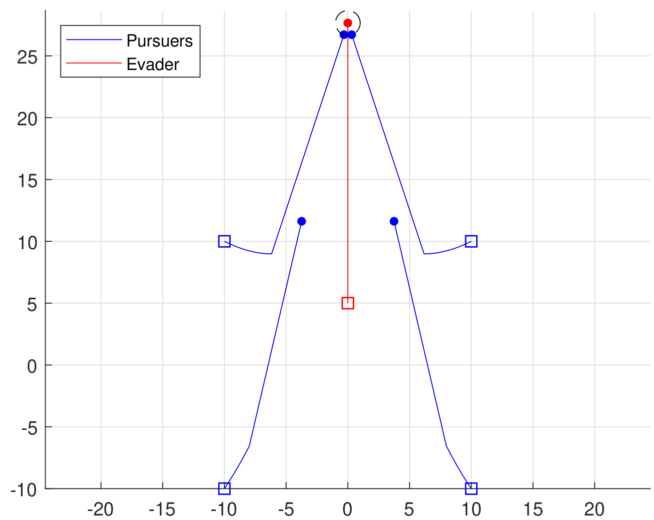

Fig. 5. Example 1. The evader goes upwards while pursuers play S-PPS. Capture occurs at $t = 22.66$.

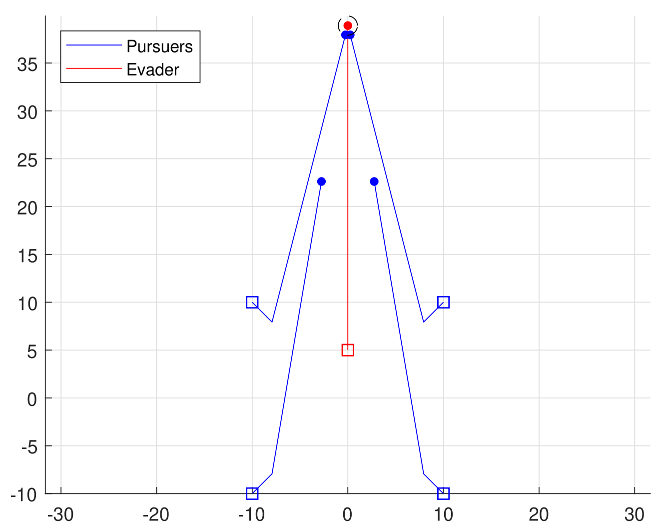

Fig. 6. Example 1. The evader goes upwards while pursuers play S-CS. Capture occurs at $t = 33.91$.

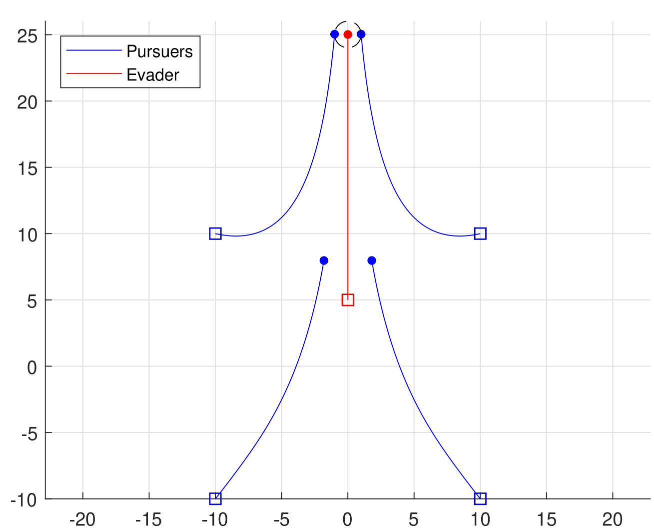

Fig. 7. Example 1. The evader goes upwards while pursuers play VS. Capture occurs at $t = 20.02$.

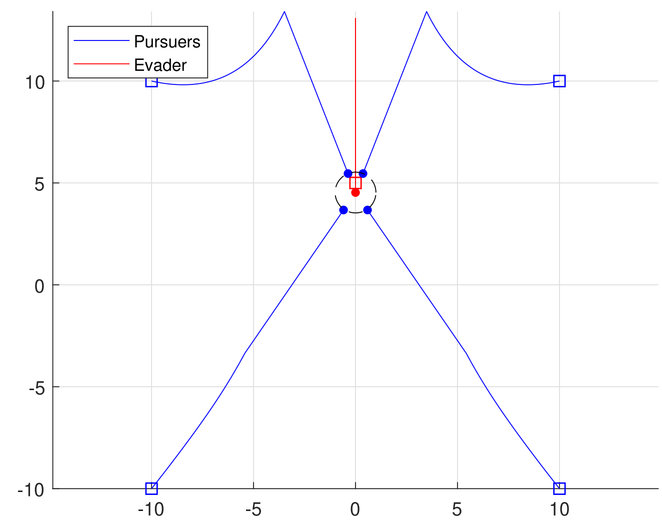

Fig. 8. Example 1. The evader goes upwards while pursuers play S-VS. Capture occurs at $t = 16.61$.

For a generic game, let us denote by T_{VS} and $T_{S\text{-}VS}$ the capture time when pursuers play VS and S-VS, respectively. The convenience of adopting the switching strategy is measured through the ratio of the mentioned capture times, that is $\rho = \frac{T_{VS}}{T_{S\text{-}VS}}$. The evader strategy consists in pointing towards the farthest vertex of its own Voronoi cell at each time.

The number p of pursuer ranges from 3 to 10, while the radius of the circle of the initial pursers' position varies as $R_p \in \{10, 30, 50\}$. For all possible combinations of p and R_p, 100 games are played. The corresponding boxplots of ρ are reported in Fig. 9. To increase readability, the upper limit of the ρ-axis is set to 2. However, it is worthwhile stressing that for several games much larger values have been observed. For instance, for $p = 3$ and $R_p = 50$ the maximum ratio is $\rho = 12.6$. Notice that in games with $\rho = 1$ the switching condition (10) never occurs, so that $T_{VS} = T_{S\text{-}VS}$. It can be observed that there are few games in which $\rho < 1$. This is due to the fact that the evader plays a predefined strategy that, in general, does not correspond to the optimal one in a min–max sense, which is unknown. Therefore, when the pursuers play VS, neither the pursuers nor the evader are playing their optimal strategy. In such a case, there is no guarantee that the capture time be greater than the one obtained when the pursuers play optimally after the switching. Nevertheless, the numerical tests show that $\rho \geq 1$ in most cases. On the whole, the simulation results demonstrate that playing the switching strategy leads to significantly smaller capture times than adopting the original Voronoi-based strategy.

6. Conclusions

A new family of switching pursuit strategies for a multi-pursuer single-evader game has been introduced. It is based on the key observation that the game can be reduced to a two-pursuer single-evader setting, for which a minimum-time solution is available. Conditions for transitioning between these two games have been derived, enabling the definition of new winning strategies and enhancing the performance of existing ones. Future work will focus on developing pursuit strategies that achieve the switching condition while minimizing the overall capture time or other relevant performance indexes. Moreover, the approach will be extended to handle multi-pursuer multi-evader games. The use of switching pursuit strategies in scenarios involving a superior evader will also be considered.

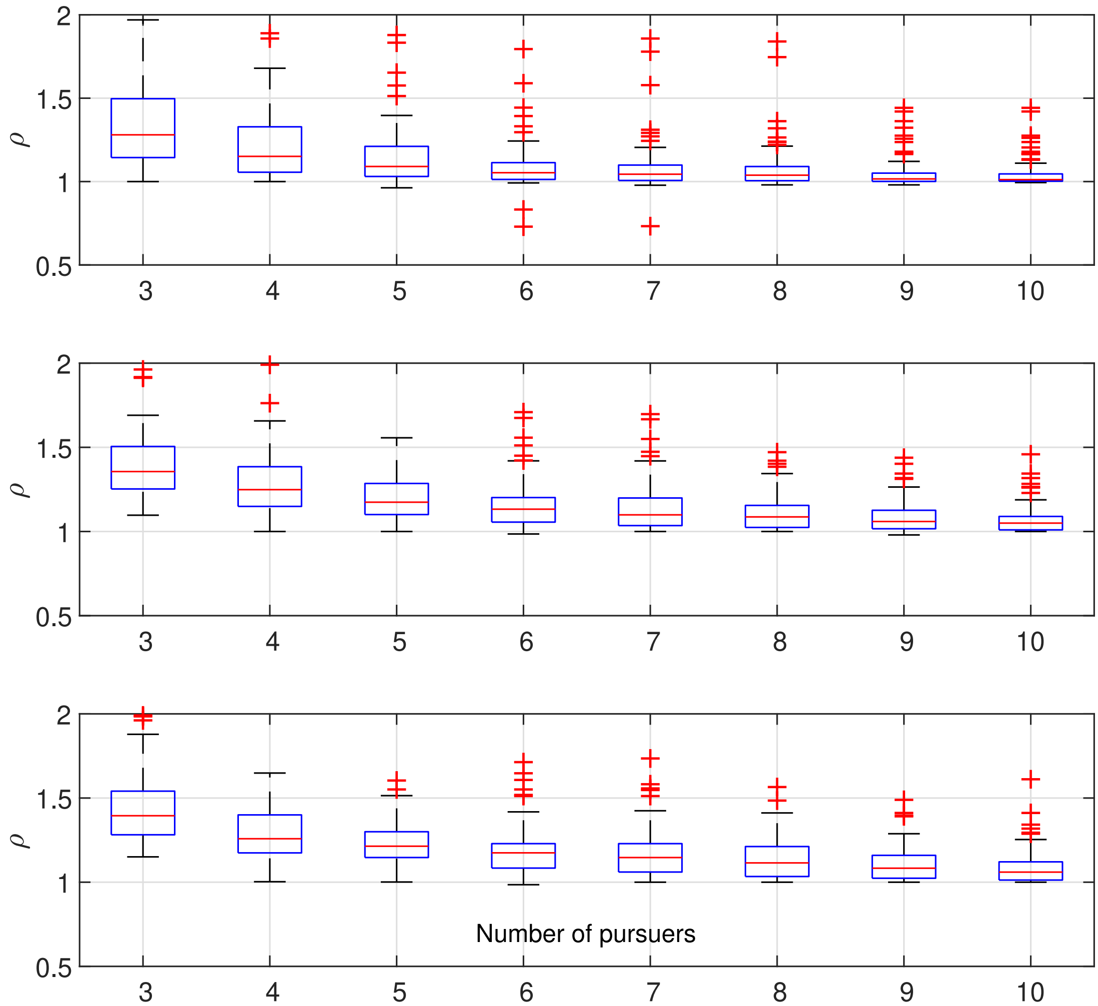

Fig. 9. Example 2. Boxplot of the performance index ρ for $R_p = 10$ (top), $R_p = 30$ (middle) and $R_p = 50$ (bottom).

CRediT authorship contribution statement

Marco Casini: Writing – original draft, Software, Methodology, Conceptualization. **Andrea Garulli:** Writing – review & editing, Validation, Methodology, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

Proof of Theorem 3

In order to prove Theorem 3, the following lemmas are introduced.

Lemma 1. *Let P_1, P_2 and E be such that $E \in \mathcal{D}_{12} \backslash (C(P_1, r) \cup C(P_2, r))$, and set $P_3 = P_2 + \alpha(P_2 - P_1)$, $\alpha > 0$. Then, $T_{13} > T_{12}$.*

Proof. W.l.o.g., let us refer to the reference frame introduced in Section 2.1, see Fig. 10. Hence, from $E = [x, y]' \in \mathcal{D}_{12} \backslash (C(P_1, r) \cup C(P_2, r))$, one has $-d < x < d$ and $(x \pm d)^2 + y^2 - r^2 > 0$. Moreover, since $E \in \mathcal{D}_{12} \backslash (C(P_1, r) \cup C(P_2, r))$, one has also $E \in \mathcal{D}_{13} \backslash (C(P_1, r) \cup C(P_3, r))$. Therefore, the capture time T_{13} of pursuers P_1 and P_3 can be computed by using (5), in which d and x are replaced by $d + \alpha d$ and $x - \alpha d$, respectively. One gets

$$T_{13} = \frac{\kappa_\alpha r + |y| \sqrt{\kappa_\alpha^2 - 4(x - \alpha d)^2 (r^2 - y^2)}}{2(r^2 - y^2)} \tag{17}$$

in which $\kappa_\alpha = \kappa + 2\alpha d(d + x)$. In order to prove $T_{13} > T_{12}$, it is sufficient to show that T_{13} in (17) is a strictly increasing function of α, for $\alpha > 0$. First, one has

$$\frac{d\kappa_\alpha}{d\alpha} = 2d(d + x) > 0.$$

Then, it remains to show that $\kappa_\alpha^2 - 4(x - \alpha d)^2 (r^2 - y^2)$ is strictly increasing in α. Through some straightforward manipulations, one obtains

$$\frac{d}{d\alpha} \left\{\kappa_\alpha^2 - 4(x - \alpha d)^2 (r^2 - y^2)\right\} = 4\left\{d(d - x) + 2d^2\alpha\right\} \left\{(x + d)^2 + y^2 - r^2\right\}$$

which is positive for all $\alpha > 0$. □

Lemma 2. *Consider P_1, P_2 and E such that $E \in \mathcal{D}_{12} \backslash (C(P_1, r) \cup C(P_2, r))$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the lines containing the segments $\overline{P_1 E}$ and $\overline{P_2 E}$, respectively. Then, let Q_1 and Q_2 be the intersections of $\mathcal{L}_1$ and $\mathcal{L}_2$ with the circumference $\partial C(H_{12}^+, T_{12} + r)$, opposite to P_1 and P_2, respectively. Define the region $\mathcal{W}_{12}$ as the portion of the circle $C(H_{12}^+, T_{12} + r)$ bounded by the segments $\overline{Q_1 E}$, $\overline{Q_2 E}$ and the minor arc of circumference $\widehat{Q_2 Q_1}$. Then, if a pursuer $P_3 \in \mathcal{W}_{12}$, then either $E \in \mathcal{D}_{13}$ or $E \in \mathcal{D}_{23}$.*

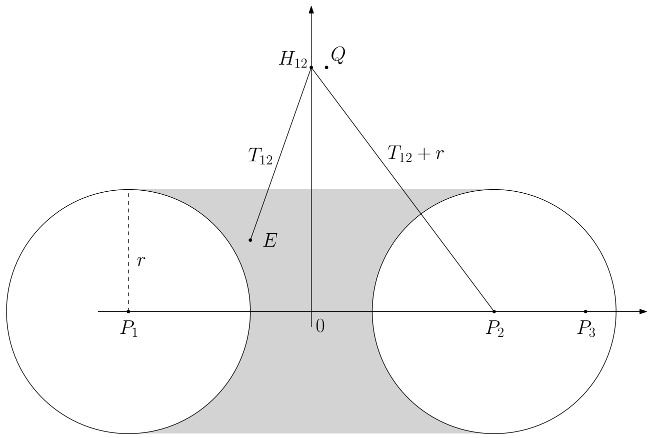

Fig. 10. Sketch of the proof of Lemma 1. The shaded region denotes $\mathcal{D}_{12}\backslash(\mathcal{C}(P_1,r)\cup\mathcal{C}(P_2,r))$.

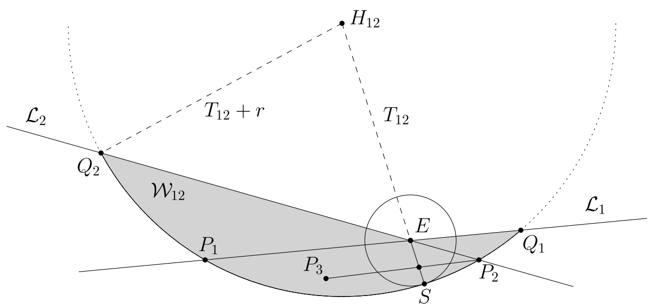

Fig. 11. Sketch of the proof of Lemma 2.

Proof. Let us consider the region $\mathcal{W}_{12}$ depicted in Fig. 11. Let $S = \partial\mathcal{C}(E,r) \cap \partial\mathcal{C}(H_{12}^+, T_{12}+r)$ be the point of tangency between the two circumferences. Notice that the segment $\overline{ES}$ divides $\mathcal{W}_{12}$ in two disjoint regions. Thus, either $\overline{P_1P_3}$ or $\overline{P_2P_3}$ will intersect $\overline{ES}$. Let us assume that $\overline{P_2P_3} \cap \overline{ES} \neq \emptyset$. Then, the minimum distance between E and $\overline{P_2P_3}$ is less than r, and hence $E \in \mathcal{D}_{23}$. A similar argument can be adopted if $\overline{P_1P_3} \cap \overline{ES} \neq \emptyset$. □

Now, we can proceed to prove Theorem 3. Let P_i, P_j be the pair of pursuers such that $E \in \mathcal{D}_{ij}$ and $T_{ij} \leq T_{hl}$, for all pairs $h,l \in \{1,\dots,p\}$ for which $E \in \mathcal{D}_{hl}$. We show that for this pursuer pair, condition (11), and hence (10), holds. Condition (11) states that $\|P_k - H_{ij}\| \geq \|P_i - H_{ij}\| = T_{ij}+r,\ \ k=1,\dots,p$. By contradiction, assume that there exists a pursuer P_z which satisfies $\|P_z - H_{ij}\| < T_{ij}+r$, i.e., $P_z \in \mathtt{int}\left\{\mathcal{C}(H_{ij}, T_{ij}+r)\right\}$. Let us define $\mathcal{L}_{iE}$ and $\mathcal{L}_{jE}$ as the lines passing through P_i, E and P_j, E, respectively. Since $P_z \in \mathcal{P}$ and by assumption $E \notin \mathtt{int}\{\mathcal{P}\}$, then P_z must belong to the region $\mathcal{W}_{ij}$, shown in Fig. 12. By Lemma 2, either $E \in \mathcal{D}_{iz}$ or $E \in \mathcal{D}_{jz}$. W.l.o.g., assume $E \in \mathcal{D}_{jz}$. Let $\mathcal{L}_{jz}$ be the line crossing P_j and P_z. It holds $\mathcal{L}_{jz} \cap \mathcal{C}(H_{ij}, T_{ij}+r) = \{P_j, P_q\}$, where P_q denotes a fictitious pursuer. Then, by Lemma 1 one has $T_{jz} < T_{jq} = T_{ij}$. Since by assumption $T_{ij} \leq T_{hl}$ for all $h,l \in \{1,\dots,p\}$, a contradiction occurs. □

Proof of Theorem 4

According to Theorem 3 and Remark 1, if $E(0) \in \mathcal{M}(0)\backslash\mathtt{int}\{\mathcal{P}(0)\}$, condition (10) already holds at time $t=0$. So, let us analyze the case $E(0) \in \mathtt{int}\{\mathcal{P}(0)\}$. Let us first show that (15) implies

$$\mathcal{P}(\bar{t}) \subset \bigcup_{i=1}^{p} \mathcal{C}(P_i(\bar{t}), r). \tag{18}$$

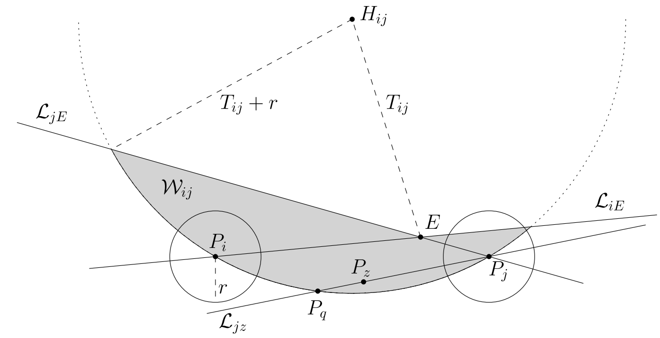

Fig. 12. Sketch of the proof of Theorem 3. The shaded area $\mathcal{W}_{ij}$ denotes the admissible region of P_z.

If $p=3$ and the distance between each pursuer pair is exactly equal to $\sqrt{3}r$, then $\mathcal{P}$ is an equilateral triangle and (18) follows from simple geometric arguments. Clearly, if the pursuers are closer to each other, (18) still holds. In the generic case of p pursuers, $\mathcal{P}$ can be partitioned in triangles and the previous reasoning can be repeated for each triangle. Therefore, (18) is satisfied for a generic polytope $\mathcal{P}$. This means that $E(\bar{t}) \notin \mathcal{P}(\bar{t})$, otherwise capture had already occurred. Since $E(0) \in \mathtt{int}\{\mathcal{P}(0)\}$, by continuity of the agents' trajectories there exists $\hat{t} < \bar{t}$ such that $E(\hat{t}) \in \partial\mathcal{P}(\hat{t})$ and then, by Theorem 3, condition (10) holds at time $\hat{t}$. □

Data availability

Data will be made available on request.

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