A two-pursuer one-evader game with equal speed and finite capture radius

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Abstract

In this paper, a two-pursuer one-evader game in the plane is considered. All the agents have simple motion and the same speed. As opposed to the game with superior pursuers, capture can occur in finite time only by defining a nonzero capture radius and for a subset of initial game states. Such a set is characterized and the full game solution is provided. In particular, the value function of the game and explicit expressions of the closed-loop optimal strategies of all the agents are derived. The results are validated via numerical simulations, comparing the optimal control actions with alternative strategies for both the evader and the pursuers.

Keywords: Pursuit-evasion games, autonomous agents, cooperative control, differential games

1 Introduction

Pursuit-evasion problems are dynamic games in which two players, or groups of players, pursue conflicting objectives. The study of this class of problems has a long history and its mathematical foundations rely on the theory of differential games [1–3]. A survey of the different settings considered in the literature can be found in [4], while a taxonomy of applications in the mobile robotics field is reported in [5].

Multi-pursuer games usually involve a team of cooperating agents which attempt to capture one or more evaders. The literature on the subject can be classified in terms of the assumptions concerning the motion models and the environment, see, e.g., [6–10] and references therein. Both decentralized solutions [11, 12] and centralized ones [13–16] have been proposed for multi-pursuer one-evader games; the advantages of the latter have been discussed in [17]. Problems in which the evader is faster (superior) than the pursuers have also been addressed [18–20]. Notwithstanding this rich production, a number of pursuit-evasion games have not been solved yet, even in quite simple settings. For example, a full closed-loop solution of the three-pursuer one-evader game is still lacking, although optimal open-loop strategies have been recently proposed [21, 22].

Within the vast research activity on multi-pursuer games, a special attention has been devoted to games involving two pursuers and one evader. The original minimum time game, involving agents with simple motion on a plane, was formulated in [1] under the name "the two cutters and the fugitive ship", and a solution based on geometric arguments was provided. In this version of the game, the two pursuers are faster than the evader and capture occurs when at least one pursuer reaches exactly the position of the evader. The problem has recently received renewed attention: optimality of the solution proposed by Isaacs has been formally proved in [23], while a complete solution of the game of kind is presented in [24]. In [25], the same game is considered but the objective of the pursuers is relaxed to achieving a predefined finite distance from the evader (capture radius). Evading strategies maximizing capture time have been analyzed in [26], for different behaviors of the pursuers. The case of a faster evader is treated in [27], where control strategies allowing the evader to safely cross the line connecting the two pursuers are derived. Several other scenarios have been addressed in the literature, including different motion models, environmental constraints or the use of relay strategies. For a fairly complete review, the interested reader is referred to [26].

In this paper, an alternative version of the two-pursuer one-evader game is considered. All the agents have simple motion and the same velocity. As a consequence, capture can be achieved only within a finite capture radius and for a bounded set of initial conditions (capture set). In this respect, the problem is substantially different from that involving superior pursuers, whose solution cannot be adapted to cope with the equal speed game. The same setting has been considered in [28], where the capture set and the optimal strategies of the agents are obtained by applying geometric arguments. The contribution of this paper is to derive a complete game-theoretic solution of the problem. First, a novel characterization of the game states for which the pursuers may capture the evader (game of kind) is provided. Then, explicit expressions of the optimal open-loop strategies of all the agents are derived. The main result consists in computing the value function of the game, which provides a solution to the Isaacs equation. This allows one to show that the above control actions constitute a saddle-point of the game, and hence they are also closed-loop optimal strategies. The results are confirmed by numerical simulations, comparing the derived optimal controls with alternative strategies for both the evader and the pursuers.

The rest of the paper is organized as follows. The dynamic game is formulated in Section 2. The optimal player strategies are presented in Section 3 and they are validated via numerical tests in Section 4. Some concluding remarks are provided in Section 5.

2 Problem formulation

Let $E(t) = [x_e(t)\ y_e(t)]' \in \mathbb{R}^2$ and $P_i(t) = [x_i(t)\ y_i(t)]' \in \mathbb{R}^2,\ i = 1,2$, denote the evader and pursuer locations at time t. The players motion model is

$$\begin{cases} \dot{E}(t) = e(t)\,, \\ \dot{P}_i(t) = w_i(t)\,,\ \ i = 1,2, \end{cases} \tag{1}$$

where $e(t) \in \mathbb{R}^2$ and $w_i(t) \in \mathbb{R}^2$, $i = 1,2$, are the players velocity vectors. It is assumed that the pursuers and the evader have the same speed. Without loss of generality, one can set $\|e(t)\| = 1$ and $\|w_i(t)\| = 1$. Thus, system (1) can be written as

$$\begin{cases} \dot{x}_e(t) = \cos\theta(t)\,, \\ \dot{y}_e(t) = \sin\theta(t)\,, \\ \dot{x}_i(t) = \cos\psi_i(t)\,, \\ \dot{y}_i(t) = \sin\psi_i(t)\,,\ \ i = 1,2, \end{cases} \tag{2}$$

where the heading angles $\theta(t)$ and $\psi_i(t)$ are the evader and pursuer control inputs, respectively. In the following, the state of the game will be denoted by $\xi(t) = [E'(t)\ P_1'(t)\ P_2'(t)]'$.

The pursuers aim at capturing the evader: this occurs when the distance of at least one pursuer from the evader is equal to the radius of capture $r > 0$. Hereafter, only initial conditions such that $\|E(0) - P_i(0)\| > r$, for $i = 1,2$, will be considered. The objective of the pursuers is to achieve capture in minimum time, while the evader tries to delay capture as much as possible. Hence, the game can be formulated as

$$\min_{\psi_1,\psi_2}\ \max_{\theta} \int_0^{t_f} dt \tag{3}$$

where t_f is such that $\|E(t_f) - P_i(t_f)\| = r$ for either $i = 1$ or $i = 2$.

In order to solve problem (3), it is useful to introduce a suitable reference system to describe the initial game state (hereafter, explicit dependence on time will be omitted unless when necessary). Let the origin be the midpoint between the position of the two pursuers, $\frac{P_2+P_1}{2}$, and the x-axis be aligned with the direction $P_2 - P_1$. In this reference frame, denote the pursuer positions by $P_1 = [-d\ 0]'$ and $P_2 = [d\ 0]'$. Moreover, let $E = [x\ y]'$ be the position of the evader (see Fig. 1). It is easy to check that the relationships with the game state ξ are given by

$$d = \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}{2}\,, \tag{4}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{x_2 - x_1}{2d} & \dfrac{y_2 - y_1}{2d} \\ -\dfrac{y_2 - y_1}{2d} & \dfrac{x_2 - x_1}{2d} \end{pmatrix} \begin{pmatrix} x_e - \dfrac{x_1 + x_2}{2} \\ y_e - \dfrac{y_1 + y_2}{2} \end{pmatrix}\,. \tag{5}$$

Notice that in the new reference system the game is symmetric with respect to both axes; for this reason, only the case $x \geq 0$, $y \geq 0$ can be considered. In the following, we will refer to the initial game state in terms of the quantities d, x and y. It is stressed that this reference frame is used only to simplify the mathematical expressions appearing in the main results: the same formulas in terms of the full system state ξ can be easily recovered by applying the transformations (4)-(5) at every time t.

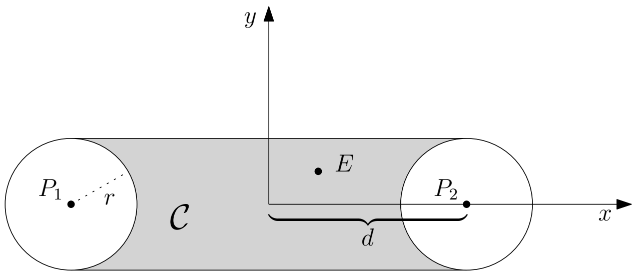

Fig. 1 Reference frame defined by (4)-(5) and set $\mathcal{C}$ in (7) (gray region). The two circles represent the initial capture areas of the pursuers.

3 Optimal player strategies

Due to the finite capture radius, the admissible evader positions for which the game is well defined belong to the set

$$\mathcal{E} = \{E : \ (x \pm d)^2 + y^2 > r^2\}. \tag{6}$$

We first characterize the set of states for which the game does not admit a finite-time solution, and hence the evader is able to escape indefinitely.

Theorem 1 *Game (3) does not have finite-time solution if $E \notin \mathcal{C}$, where*

$$\mathcal{C} = \{(x, y) \in \mathcal{E} : \ |x| < d\,, \ |y| < r\}\,. \tag{7}$$

Proof Let $y \geq r$ and assume the evader chooses the constant input $\theta(t) = \frac{\pi}{2}$, for $t \geq 0$. Then, one has $E(t) = [x \; y+t]'$. In order to capture the evader at time t_f, pursuer P_2 must reach a distance r from E. The shortest path for P_2 to do so is clearly a straight line towards the evader position at time t_f. Since $P_2(t) = [d \; 0]'$, this leads to

$$(d-x)^2 + (y+t_f)^2 = (t_f + r)^2 \,, \tag{8}$$

which is equivalent to

$$2(r-y)t_f = (d-x)^2 + y^2 - r^2 \,. \tag{9}$$

If $y \geq r$, equation (9) does not admit a positive solution in t_f. Therefore, P_2 cannot capture the evader in finite time. The same argument applies to P_1. Therefore, for $y \geq r$, the evader can indefinitely run away from the pursuers by heading orthogonally to the initial line of the pursuers.
Now, let $0 < y < r$ and $x \geq d$, such that $(x-d)^2 + y^2 > r^2$. Assume the evader chooses the constant input $\theta(t) = \tan^{-1}(\frac{y}{x-d})$, for $t \geq 0$. Then, one has

$$E(t) = \left[x + \frac{x-d}{\sqrt{(x-d)^2+y^2}}\, t \qquad y + \frac{y}{\sqrt{(x-d)^2+y^2}}\, t \right]' .$$

Clearly, P_2 can never reach E, that is moving straight in the direction $E(0) - P_2(0)$. Let us check if P_1 can a reach a distance r from E, at some time t_f. The best way to do so for P_1 is to travel a straight line towards the evader, which gives the relationship

$$\left(x + d + \frac{x-d}{\sqrt{(x-d)^2+y^2}}\, t_f \right)^2 + \left(y + \frac{y}{\sqrt{(x-d)^2+y^2}}\, t_f \right)^2 = (t_f + r)^2 \,, \tag{10}$$

which, after some manipulations, becomes

$$2\left\{ r\sqrt{(x-d)^2+y^2} - y^2 - x^2 + d^2 \right\} t_f = \left[(x+d)^2 + y^2 - r^2 \right] \sqrt{(x-d)^2+y^2}. \tag{11}$$

From $(x+d)^2 + y^2 > r^2$, one has that the right hand side in (11) is positive. On the other hand, from $(x-d)^2 + y^2 > r^2$ and $x \geq d$ one gets

$$r\sqrt{(x-d)^2+y^2} - y^2 - x^2 + d^2 < (x-d)^2 - x^2 + d^2 = 2d(d-x) \leq 0 \,.$$

Therefore, equation (11) does not admit a positive solution in t_f and capture cannot occur. □

It is worth observing that the set $\mathcal{C}$ in (7) vanishes when the capture radius r is equal to zero. In fact, if exact capture is required, the evader can always delay it indefinitely by just heading orthogonally to the line connecting the pursuers, as it is evident from (9). Therefore, a finite capture radius is necessary for the pursuers to capture the evader, in the equal speed game. In the subsequent development, it will be shown that game (3) indeed admits a finite-time solution whenever $E \in \mathcal{C}$, and hence $\mathcal{C}$ turns out to be the so-called *capture set* of the game, providing a solution to the game of kind.

Let us denote by $\text{ang}(v_x, v_y)$ the four-quadrant inverse tangent of the vector $v = [v_x \; v_y]'$, i.e., $\text{ang}(v_x, v_y)$ is the angle of vector v with the x-axis, in the range $[-\pi, \pi]$. The next result gives an explicit expression of the open-loop optimal strategies of the agents when $E \in \mathcal{C}$, and the corresponding capture time.

Theorem 2 *Let $E \in \mathcal{C}$. Assume the evader control action is given by*

$$\theta(t) = \theta^* = \text{ang}(-x, y_{ef}^* - y)\,, \tag{12}$$

while the pursuer inputs are

$$\psi_1(t) = \psi_1^* = \text{ang}(d, y_{ef}^*)\,, \quad \psi_2(t) = \psi_2^* = \text{ang}(-d, y_{ef}^*) \tag{13}$$

where

$$y_{ef}^* = \begin{cases} \dfrac{x^2 + y^2 + r^2 - d^2 + 2rt_f^*}{2y}\,, & \text{if } y \neq 0 \\ \sqrt{(t_f^*)^2 - x^2}\,, & \text{if } y = 0. \end{cases} \tag{14}$$

Then, the game terminates at time

$$t_f^* = \frac{\kappa r + |y|\sqrt{\kappa^2 - 4x^2(r^2 - y^2)}}{2(r^2 - y^2)}\,, \tag{15}$$

where $\kappa = d^2 - x^2 + y^2 - r^2$. Moreover, every other pursuer or evader open-loop control action leads to a capture time which is larger than t_f^*.

Proof According to [2, Theorem 2], one has to show that the evader and pursuer controls (12) and (13), respectively maximize and minimize the Hamiltonian function of the problem, which is given by

$$\begin{aligned} H(\xi, \lambda, \theta, \psi) \;=\; & \lambda_{ex} \cos\theta + \lambda_{ey} \sin\theta + \lambda_{1x} \cos\psi_1 \\ & + \lambda_{1y} \sin\psi_1 + \lambda_{2x} \cos\psi_2 + \lambda_{2y} \sin\psi_2 + 1 \end{aligned} \tag{16}$$

where $\lambda = [\lambda_{ex}\ \lambda_{ey}\ \lambda_{1x}\ \lambda_{1y}\ \lambda_{2x}\ \lambda_{2y}]'$ is the costate vector. Maximization of (16) with respect to θ leads to

$$\cos\theta^* = \frac{\lambda_{ex}}{\sqrt{\lambda_{ex}^2 + \lambda_{ey}^2}}\,, \qquad \sin\theta^* = \frac{\lambda_{ey}}{\sqrt{\lambda_{ex}^2 + \lambda_{ey}^2}} \tag{17}$$

while minimization with respect to ψ_1 and ψ_2 gives

$$\cos\psi_i^* = -\frac{\lambda_{ix}}{\sqrt{\lambda_{ix}^2 + \lambda_{iy}^2}}\,, \qquad \sin\psi_i^* = -\frac{\lambda_{iy}}{\sqrt{\lambda_{ix}^2 + \lambda_{iy}^2}} \qquad i = 1, 2. \tag{18}$$

Since the Hamiltonian does not depend explicitly on the state ξ, one has

$$\dot{\lambda}'(t) = -\frac{\partial}{\partial \xi} H(\xi, \lambda, \theta, \psi) = 0$$

which means that the costate variables are all constant. Then, from (17)-(18), one has that also the optimal control actions $\theta(t)$ and $\psi_i(t)$ are constant, i.e., the optimal paths traveled by the pursuers and the evader are all straight lines. One consequence of this fact is that, in order to travel the shortest path, the pursuers must necessarily head towards the final position of the evader. Moreover, it is easy to verify that the pursuers must capture the evader simultaneously. Indeed, assume by contradiction that only one pursuer (say P_1) captures the evader at time $\bar{t}$. Since the pursuers travel the same distance, this must necessarily occur in the half-plane closer to P_1. But in this case, the evader may choose an alternative target point, in the same half-plane, allowing it to avoid capture at time $\bar{t}$ (see Fig. 2).

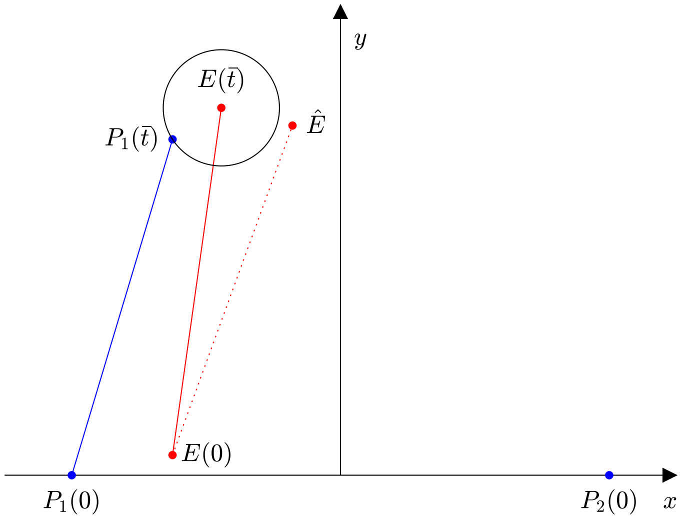

Fig. 2 If pursuer P_1 captures the evader in the half-plane closer to $P_1(0)$, then the evader may travel the same distance towards $\hat{E}$ (dotted path) without being captured at time $\bar{t}$. Therefore, the pursuers must capture the evader simultaneously, which necessarily occurs when the evader is on the perpendicular bisector of the segment connecting $P_1(0)$ and $P_2(0)$.

From the above observations, it can be concluded that the distance between the initial position of each pursuer and the point where capture occurs must be $t_f + r$, i.e.

$$\|P_i(0) - E(t_f)\| = t_f + r\,, \quad i = 1, 2. \tag{19}$$

Denoting the final position of the evader by $E(t_f) = [x_{ef}\ y_{ef}]'$, equations (19) can be rewritten as

$$\sqrt{(x_{ef} + d)^2 + y_{ef}^2} = t_f + r, \tag{20}$$

$$\sqrt{(x_{ef} - d)^2 + y_{ef}^2} = t_f + r, \tag{21}$$

from which one obtains $x_{ef} = 0$, i.e., capture occurs when the evader is on the perpendicular bisector of the segment connecting $P_1(0)$ and $P_2(0)$. Since also the evader moves along a straight path, one has $\|E(t_f) - E(0)\| = t_f$, i.e.,

$$\sqrt{(x_{ef} - x)^2 + (y_{ef} - y)^2} = t_f. \tag{22}$$

By substituting $x_{ef} = 0$ in (20)-(22), one has

$$d^2 + y_{ef}^2 = (t_f + r)^2 \tag{23}$$

$$x^2 + (y_{ef} - y)^2 = t_f^2, \tag{24}$$

from which one gets, for $y \neq 0$,

$$y_{ef} = \frac{x^2 + y^2 + r^2 - d^2 + 2rt_f}{2y}\,. \tag{25}$$

Substituting (25) into (23), after some straightforward manipulations one has

$$(r^2 - y^2)t_f^2 - \kappa r t_f + \frac{\kappa^2 + 4x^2y^2}{4} = 0\,, \tag{26}$$

where $\kappa = d^2 - x^2 + y^2 - r^2$. By exploiting the fact that the admissible initial evader positions satisfy $(x-d)^2 + y^2 > r^2$, and the assumptions $|x| < d$ and $|y| < r$, it can be shown that $\kappa > 0$ and the equation (26) always admits two real positive solutions. Clearly, the optimal capture time t_f^* corresponds to the largest one, which is equal to (15).
In the case $y = 0$, from (24) one obtains

$$y_{ef} = \sqrt{t_f^2 - x^2} \tag{27}$$

and then from (23), $t_f^* = \frac{d^2 - x^2 - r^2}{2r}$, which corresponds to (15) with $y = 0$. Substituting the obtained expressions of t_f^* into (25) and (27), one gets (14). Since the optimal strategy for all players is to point towards $E(t_f) = [0 \; y_{ef}]'$, the expressions of the control actions (12)-(13) follow directly. □

In the case $y = 0$, the solution provided by Theorem 2 is not unique: indeed, the players may choose the angles opposite to those in (12)-(13), and obtain the same capture time t_f^* (in such a case the game evolves in the halfplane $y < 0$, instead of $y > 0$). This situation is known as a *game dilemma* and $y = 0$ is referred to as a *dispersal surface* [1, Chapter 6]. Nevertheless, thanks to the finite capture radius, as soon as the players make either of the two equivalent choices, the symmetry of the game is broken and one gets $y \neq 0$, for which the solution provided by Theorem 2 is unique.

It can be observed that the control actions (12)-(13) in Theorem 2 are the same as those obtained in [28] by applying geometric arguments leading to the characterization of the evader safe region. It is also worth stressing that Theorem 2 guarantees that if the agents adopt the control actions (12)-(13) from the initial time onwards, capture occurs within time t_f^* given by (15). This provides an *open-loop* representation of the optimal control strategies (see the thorough discussions in [2] and [21]). Now, the objective is to show that the same control actions, when computed as a function of the current state $\xi(t)$[1], are also optimal *closed-loop* strategies and constitute a saddle point for game (3). To this aim, one has to find the value function $V(\xi)$ of the game and show that the control actions are the solutions of the Isaacs equation (see [2, Theorem 1])

$$\min_{\psi_1,\psi_2} \max_{\theta} \frac{\partial V(\xi)}{\partial \xi} f(\xi,\theta,\psi) + 1 = 0 \; , \tag{28}$$

where $f(\xi,\theta,\psi)$ is the vector field of system (2).

Let us consider the candidate value function

$$V(\xi) = \alpha(\xi) t_1(\xi) + (1 - \alpha(\xi)) t_2(\xi) \tag{29}$$

where

$$t_i(\xi) = \frac{(x_i - x_e)^2 + (y_i - y_e)^2 - r^2}{2\delta_i(\xi)} \; , \quad i = 1,2 \tag{30}$$

[1]Notice that this can be done by considering $d(t)$, $x(t)$ and $y(t)$ in (4)-(5) as functions of $\xi(t)$, for all $t \geq 0$.

and

$$\alpha(\xi) = \frac{\eta_2(\xi)\delta_1(\xi)}{\eta_2(\xi)\delta_1(\xi) - \eta_1(\xi)\delta_2(\xi)} \tag{31}$$

with

$$\eta_i(\xi) = (x_i - x_e)\sin\theta^* - (y_i - y_e)\cos\theta^* \tag{32}$$
$$\delta_i(\xi) = r + (x_i - x_e)\cos\theta^* + (y_i - y_e)\sin\theta^* \tag{33}$$

for $i = 1, 2$. In (29)-(33), the optimal evader control action θ^* is given by (12), and is expressed in terms of ξ by using the substitutions (4)-(5). The following result provides the complete solution of game (3).

Theorem 3 *Let $\theta^*(\xi)$, $\psi^*(\xi)$ be closed-loop control strategies, defined respectively by* (12) *and* (13)*, with the substitutions* (4)*-*(5)*. Then, $V(\xi)$ in* (29) *is the value function of game* (3) *and the control strategies $\theta^*(\xi)$, $\psi^*(\xi)$ provide a saddle point for the game.*

Proof According to [2, Theorem 1], we have to show that: (i) $V(\xi)$ is continuously differentiable in the capture set $\mathcal{C}$; (ii) $V(\xi)$ satisfies the Isaacs equation (28).
(i) From the definition of $V(\xi)$, it is sufficient to prove that its denominator does not vanish in $\mathcal{C}$. To this aim, we show that $\delta_i(\xi) > 0$, $i = 1, 2$, and $\eta_2(\xi)\delta_1(\xi) - \eta_1(\xi)\delta_2(\xi) > 0$ for all ξ such that $|x| < d$ and $|y| < r$. First, notice that $\delta_i(\xi)$ and $\eta_i(\xi)$ are invariant with respect to rotations and translations of the reference frame. Hence, in the reference system defined by (4)-(5), one has $x_i - x_e = \mp d - x$ and $y_i - y_e = -y$. Hence,

$$\delta_i = r + (\mp d - x)\cos\theta^* - y\sin\theta^*.$$

From (12) and (24), one can write

$$\cos\theta^* = \frac{-x}{\sqrt{x^2 + (y^*_{ef} - y)^2}} = \frac{-x}{t^*_f}, \qquad \sin\theta^* = \frac{y^*_{ef} - y}{\sqrt{x^2 + (y^*_{ef} - y)^2}} = \frac{y^*_{ef} - y}{t^*_f}.$$

which leads to

$$\delta_i = r + \frac{\mp dx + x^2 - yy^*_{ef} + y^2}{t^*_f}.$$

By substituting the expression of y^*_{ef} in (14) one gets

$$\begin{aligned}\delta_i &= r + \frac{\mp dx + \frac{1}{2}(x^2 + y^2 + d^2 - r^2) - rt^*_f}{t^*_f}\\ &= \frac{(x \mp d)^2 + y^2 - r^2}{2t^*_f} > 0\end{aligned}$$

where the last inequality comes from (6).
By using a similar procedure, after some straightforward manipulations one gets

$$\begin{aligned}\eta_2\delta_1 - \eta_1\delta_2 = {} & r\{(x_2 - x_1)\sin\theta^* - (y_2 - y_1)\cos\theta^*\}\\ & +(x_2 - x_e)(y_1 - y_e) - (x_1 - x_e)(y_2 - y_e)\end{aligned}$$

which, in the reference system defined by (4)-(5), becomes

$$\eta_2\delta_1 - \eta_1\delta_2 = 2d\left\{\frac{r(y_{ef}^* - y)}{t_f^*} - y\right\}.$$

For $y = 0$, the expression is clearly positive. For $y \neq 0$, by using once again y_{ef}^* in (14), one obtains

$$\begin{aligned}\eta_2\delta_1 - \eta_1\delta_2 &= \frac{2d}{t_f^*}\left\{r\frac{x^2 - y^2 + r^2 - d^2 + 2rt_f^*}{2y} - yt_f^*\right\} \\ &= \frac{2d}{t_f^*}\left\{\frac{2r^2t_f^* - r\kappa}{2y} - yt_f^*\right\} \\ &= \frac{d}{yt_f^*}\left\{2(r^2 - y^2)t_f^* - r\kappa\right\}.\end{aligned}$$

Finally, substituting the expression of t_f^* in (15), one has

$$\eta_2\delta_1 - \eta_1\delta_2 = \frac{d}{t_f^*}\operatorname{sgn}(y)\sqrt{\kappa^2 - 4x^2(r^2 - y^2)} > 0.$$

(ii) Now let us prove that $V(\xi)$ satisfies (28). From the proof of Theorem 2, the values θ^* and ψ^* optimizing (28) are given by (12)-(13), and therefore it is sufficient to check that

$$\frac{\partial V(\xi)}{\partial \xi} f(\xi, \theta^*, \psi^*) + 1 = 0\,. \tag{34}$$

Let us first show that $t_i(\xi)$ in (30) represents the time required by pursuer i to capture the evader, from the current game state ξ, if the agents adopt the optimal actions θ^* and ψ^*. By using the same arguments as in the proof of Theorem 2, if pursuer i captures the evader at time t_i, one must have

$$x_i + (t_i + r)\cos\psi_i^* = x_e + t_i\cos\theta^* \tag{35}$$

$$y_i + (t_i + r)\sin\psi_i^* = y_e + t_i\sin\theta^* \tag{36}$$

for $i = 1, 2$. By eliminating ψ_i^* and solving for t_i, from (35)-(36) one gets (30). Since the pursuers capture the evader simultaneously, one has $t_1(\xi) = t_2(\xi)$, which leads to

$$\left[(x_1 - x_e)^2 + (y_1 - y_e)^2 - r^2\right]\delta_2(\xi) = \left[(x_2 - x_e)^2 + (y_2 - y_e)^2 - r^2\right]\delta_1(\xi). \tag{37}$$

Hereafter, in order to streamline the treatment, we adopt the following shorthand notations: $\Delta_{xi} = x_i - x_e$, $\Delta_{yi} = y_i - x_e$, $r_i^2 = \Delta_{xi}^2 + \Delta_{yi}^2$, $c^* = \cos\theta^*$, $s^* = \sin\theta^*$.

Let us differentiate $V(\xi)$ with respect to each state variable. From (29) one has

$$\frac{\partial V(\xi)}{\partial \xi_i} = \alpha(\xi)\frac{\partial t_1(\xi)}{\partial \xi_i} + (1 - \alpha(\xi))\frac{\partial t_2(\xi)}{\partial \xi_i} + \frac{\partial V(\xi)}{\partial \theta^*}\frac{\partial \theta^*(\xi)}{\partial \xi_i} \tag{38}$$

where it has been taken into account that $t_1(\xi) = t_2(\xi)$ and that $V(\xi)$ depends on ξ also through the evader control action θ^*. Let us show that $\frac{\partial V(\xi)}{\partial \theta^*} = 0$. Exploiting once again $t_1(\xi) = t_2(\xi)$, one has

$$\begin{aligned}
\frac{\partial V(\xi)}{\partial \theta^*} &= \alpha(\xi)\frac{\partial t_1(\xi)}{\partial \theta^*} + (1-\alpha(\xi))\frac{\partial t_2(\xi)}{\partial \theta^*} \\
&= \alpha(\xi)(r_1^2 - r^2)\frac{\Delta_{x1}s^* - \Delta_{y1}c^*}{2\delta_1^2(\xi)} + (1-\alpha(\xi))(r_2^2 - r^2)\frac{\Delta_{x2}s^* - \Delta_{y2}c^*}{2\delta_2^2(\xi)} \\
&= \frac{\eta_2(\xi)\delta_1(\xi)}{\eta_2(\xi)\delta_1(\xi) - \eta_1(\xi)\delta_2(\xi)}\frac{\eta_1(\xi)}{2\delta_1^2(\xi)}(r_1^2 - r^2) + \\
&\quad - \frac{\eta_1(\xi)\delta_2(\xi)}{\eta_2(\xi)\delta_1(\xi) - \eta_1(\xi)\delta_2(\xi)}\frac{\eta_2(\xi)}{2\delta_2^2(\xi)}(r_2^2 - r^2) \\
&= \frac{\eta_1(\xi)\eta_2(\xi)}{2\delta_1(\xi)\delta_2(\xi)[\eta_2(\xi)\delta_1(\xi) - \eta_1(\xi)\delta_2(\xi)]}[(r_1^2 - r^2)\delta_2(\xi) - (r_2^2 - r^2)\delta_1(\xi)] \\
&= 0
\end{aligned}$$

where the last equality follows from (37).

By differentiating $t_1(\xi)$ in (30) with respect to each state variable one obtains

$$\frac{\partial t_1(\xi)}{\partial x_e} = -\frac{\partial t_1(\xi)}{\partial x_1} = \frac{(r_1^2 - r^2)c^* - 2\Delta_{x1}\delta_1(\xi)}{2\delta_1^2(\xi)}\ , \tag{39}$$

$$\frac{\partial t_1(\xi)}{\partial y_e} = -\frac{\partial t_1(\xi)}{\partial y_1} = \frac{(r_1^2 - r^2)s^* - 2\Delta_{y1}\delta_1(\xi)}{2\delta_1^2(\xi)}\ , \tag{40}$$

$$\frac{\partial t_1(\xi)}{\partial x_2} = \frac{\partial t_1(\xi)}{\partial y_2} = 0\ . \tag{41}$$

On the other hand, by exploiting (35)-(36), the vector field $f(\xi, \theta^*, \psi^*)$ can be written as

$$f_1(\xi, \theta^*, \psi^*) = c^* \qquad f_2(\xi, \theta^*, \psi^*) = s^*\ , \tag{42}$$

$$f_3(\xi, \theta^*, \psi^*) = \frac{t_1(\xi)c^* - \Delta_{x1}}{t_1(\xi) + r} \qquad f_4(\xi, \theta^*, \psi^*) = \frac{t_1(\xi)s^* - \Delta_{y1}}{t_1(\xi) + r}\ , \tag{43}$$

$$f_5(\xi, \theta^*, \psi^*) = \frac{t_2(\xi)c^* - \Delta_{x2}}{t_2(\xi) + r} \qquad f_6(\xi, \theta^*, \psi^*) = \frac{t_2(\xi)s^* - \Delta_{y2}}{t_2(\xi) + r}\ . \tag{44}$$

By using (39)-(41) and (42)-(44), one gets

$$\begin{aligned}
&\sum_{i=1}^{6}\frac{\partial t_1(\xi)}{\partial \xi_i}f_i(\xi, \theta^*, \psi^*) = \\
&= \frac{(r_1^2 - r^2)c^* - 2\Delta_{x1}\delta_1(\xi)}{2\delta_1^2(\xi)}\left(c^* - \frac{t_1(\xi)c^* - \Delta_{x1}}{t_1(\xi) + r}\right) + \\
&\quad \frac{(r_1^2 - r^2)s^* - 2\Delta_{y1}\delta_1(\xi)}{2\delta_1^2(\xi)}\left(s^* - \frac{t_1(\xi)s^* - \Delta_{y1}}{t_1(\xi) + r}\right) \\
&= \frac{(r_1^2 - r^2)\delta_1(\xi) - 2r\delta_1(\xi)(\Delta_{x1}c^* + \Delta_{y1}s^*) - 2r_1^2\delta_1(\xi)}{2\delta_1^2(\xi)(t_1(\xi) + r)} \\
&= -\frac{r_1^2 + r^2 + 2r(\Delta_{x1}c^* + \Delta_{y1}s^*)}{2\delta_1(\xi)(t_1(\xi) + r)} \\
&= -\frac{r_1^2 + r^2 + 2r(\Delta_{x1}c^* + \Delta_{y1}s^*)}{r_1^2 - r^2 + 2r\left[r + \Delta_{x1}c^* + \Delta_{y1}s^*\right]} \\
&= -1
\end{aligned} \tag{45}$$

in which the definitions of $t_1(\xi)$ in (30) and $\delta_1(\xi)$ in (33) have been exploited. By following the same reasoning, one can show also that

$$\sum_{i=1}^{6} \frac{\partial t_2(\xi)}{\partial \xi_i} f_i(\xi, \theta^*, \psi^*) = -1 \ . \tag{46}$$

Finally, substituting (38) into (34) and using (45)-(46), one gets

$$\begin{aligned} &\frac{\partial V(\xi)}{\partial \xi} f(\xi, \theta^*, \psi^*) + 1 = \\ &= \alpha(\xi) \sum_{i=1}^{6} \frac{\partial t_1(\xi)}{\partial \xi_i} f_i(\xi, \theta^*, \psi^*) + (1 - \alpha(\xi)) \sum_{i=1}^{6} \frac{\partial t_2(\xi)}{\partial \xi_i} f_i(\xi, \theta^*, \psi^*) + 1 \\ &= \alpha(\xi)(-1) + (1 - \alpha(\xi))(-1) + 1 = 0 \end{aligned} \tag{47}$$

which concludes the proof. □

Remark 1 The proof of Theorem 3 is based on a reasoning similar to that adopted in [23, Theorem 1]. In particular, the expression of the value function (29)-(33) is inspired by that of the value function in [23]. However, the results are substantially different. Indeed, the value function in [23] concerns a game in which the pursuers are faster than the evader and its expression diverges as the velocity of the pursuers approaches that of the evader. Moreover, the game with superior pursuers always terminates with the evader capture, while the equal-speed game admits a finite-time solution only for the initial conditions defined by the capture set $\mathcal{C}$ in Theorem 1. Finally, it is worth observing that Theorems 2 and 3 provide explicit expressions of the closed-loop control strategies and of the time-to-capture (i.e., of the optimal game solutions), while the corresponding solution of the game with superior pursuers given in [23] requires to compute the intersections of two Apollonius circles.

4 Numerical simulations

In this section, results of numerical simulations are reported to compare the optimal strategies with some non-optimal ones. Games have been simulated in closed loop through a discrete-time implementation with a sampling time equal to 0.001. For all the examples, the capture radius is set to $r = 1$. The initial player positions at $t = 0$ are $P_1(0) = [-5\ 0]'$, $P_2(0) = [5\ 0]'$ and $E(0) = [3\ 0.4]'$. It is easy to see that $E(0) \in \mathcal{C}$ in (7), and then the evader can be captured in finite time. In the figures illustrating the games, squares represent the agent initial positions and dots the final ones. The capture circle at the final time t_f is dashed.

Fig. 3 shows the game in which all the agents play the optimal strategy, i.e., they all point to $(0,\ y^*_{ef})$. Capture occurs at $t_f = t^*_f = 12.388$ and the evader is captured simultaneously by both pursuers on the perpendicular bisector of the segment connecting $P_1(0)$ and $P_2(0)$, as expected.

In the example shown in Fig. 4, pursuers move in an optimal way according to (13), while the evader keeps moving upward. In this case, the evader is captured by P_2 at $t_f = 2.965$, that is at a time much less than t^*_f. Notice that now the pursuer trajectories are not straight lines anymore, due to the

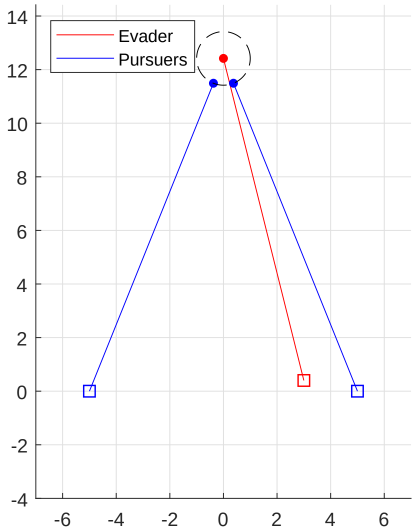

Fig. 3 Player trajectories when both the pursuers and the evader play the optimal strategy.

fact that the evader has chosen a non-optimal strategy. Games in which the players adopt these strategies (optimal for the pursuers, straight upwards for the evader) have been simulated for 10000 randomly selected initial conditions $E(0) \in \mathcal{C}$. As expected, capture always occurs at a time $t_f < t_f^*$. The average ratio t_f/t_f^* is equal to 0.454.

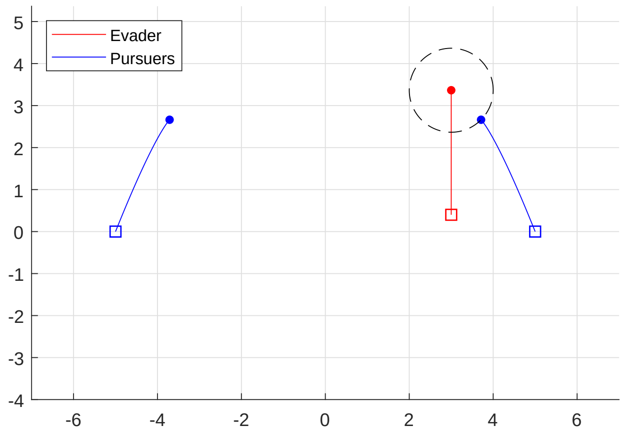

Fig. 4 Player trajectories when the pursuers play the optimal strategy and the evader moves upward.

Now, assume that the evader plays the closed-loop optimal strategy defined by (12), while the pursuers play a "pure-pursuit" game, that is they head towards the evader position at each time. The behavior of this game is reported in Fig. 5. At time $t = 0.952$ the simulation was stopped because the evader left the capture set, and then it can escape indefinitely. In view of this result, one may argue that if the pursuers play a pure-pursuit game and the evader plays at its best, capture never occurs. Such conjecture is not true, in general. In fact, capture is strictly related to the initial conditions. For instance, if $P_1(0) = [-5\ 0]'$, $P_2(0) = [5\ 0]'$ and $E(0) = [4\ 0.25]'$, capture occurs at time $t_f = 4.988$, see Fig. 6. The optimal capture time provided by (15), when also the pursuers play the optimal strategy, is $t_f^* = 4.374$. Also in this case, 10000 randomly initial conditions $E(0) \in \mathcal{C}$ have been simulated. Capture is achieved only in 0.49% of the games, in which the average ratio t_f/t_f^* is equal to 1.174. The region containing the initial conditions $E(0)$ that lead to the evader capture is depicted in Fig. 7.

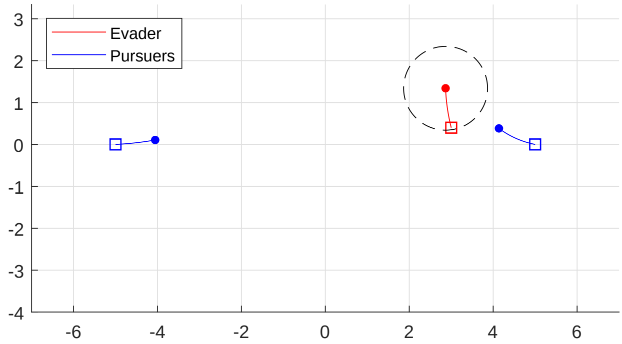

Fig. 5 Player trajectories when the evader plays the optimal strategy and the pursuers play the pure-pursuit strategy, with $E(0) = [3\ 0.4]'$.

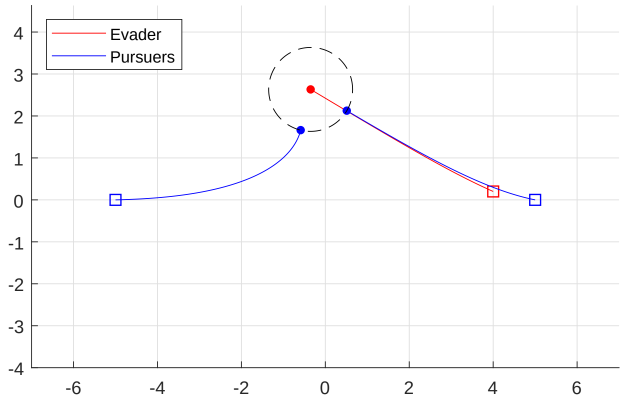

Fig. 6 Player trajectories when the evader plays the optimal strategy and the pursuers play the pure-pursuit strategy, with $E(0) = [4\ 0.25]'$.

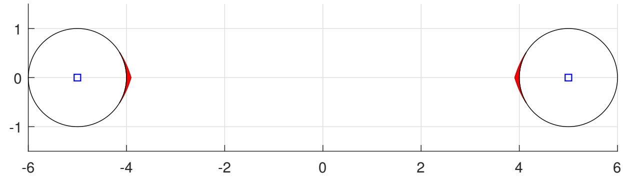

Fig. 7 Region of evader initial conditions for which capture with pure pursuit occurs (in red).

5 Conclusions

A two-pursuer one-evader game with all the agents moving at the same speed has been studied. It has been shown how the capture set depends on the capture radius and the optimal strategies of all players have been derived. The game value function has been also characterized, thus guaranteeing that the devised strategies constitute a saddle point of the game and they are optimal closed-loop control actions. The solution of this game is not only interesting in itself, but also as a tool that may be useful in multi-pursuer games with a larger number of pursuers. For example, novel strategies may be conceived that promote the evolution of the multi-pursuer game towards the most favorable two-versus-one winning condition. This topic is the subject of ongoing research.

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