A recursive technique for tracking the feasible parameter set in bounded error estimation

M. Casini, A. Garulli and A. Vicino

Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche
Università di Siena - Via Roma 56, 53100 Siena, Italy

Abstract

In this paper a new recursive algorithm is proposed for tracking parameter changes of a time-varying linear system. Since a bounded error approach is adopted for modeling both the measurement noise and the parameter change process, the problem addressed amounts to the design of a procedure for updating an estimate of the feasible parameter set. The approximating regions considered are in the form of outbounding orthotopes. The novelty of the approach lies in the use of a selection technique which keeps track only of a special subset of the constraints defining the feasible set. These inequalities represent the binding constraints of suitable linear programs of limited size. The devised algorithm is tested on several numerical examples, showing remarkable performance both in terms of computational burden, which is comparable to that of classical recursive estimation algorithms like RLS, and quality of the set estimate as compared to alternative techniques available in the literature.

Keywords: Set membership estimation; time-varying systems; recursive identification; linear programming.

1 Introduction

Adaptive tracking of model parameters is a classic research area in the field of system identification and adaptive control. By far the most popular approaches are founded on a stochastic framework assuming that the measurement as well as the process noises are stochastic processes with partially known statistics (see [1, 2, 3] and references therein). In spite of the extremely rich literature on stochastic techniques, relatively few contributions tackle the problem in the *set membership* framework, where uncertainty is assumed to be unknown but bounded (see [4, 5, 6] and references therein). In the latter framework, it is usual to study the so-called *feasible parameter set* (FPS), defined as the set of model parameters consistent with the measured data and the a priori knowledge on noise bounds. Assume for instance the case when a linear model is considered and measurement noise is bounded according to the max norm. In this setup the FPS is represented by a convex polytope in the parameter space, which can be described either through a set of linear inequalities or a set of vertices. In either case, the complexity of the FPS generally increases with the number of available measurements. If we add to this setup the hypothesis that model parameters change over time, the FPS changes accordingly and its description becomes quickly intractable in almost any realistic application. This situation arises in any setting where the goal is to capture the whole set of feasible parameters, assuming that each element of the FPS may change over time. Notice that the situation is completely different if, like in statistical approaches, a parameter point estimate is tracked together with its statistics, typically its covariance matrix.

While a large number of contributions is available in the set membership framework for identification of time-invariant systems, few works have been specifically dedicated to the problem of tracking time-varying parameters in the bounded error setting. The papers [7, 8] can be recognized as the first ones to propose modified versions of classic recursive schemes, coping with the

UBB assumption. Adaptations of the celebrated LMS and RLS algorithms have been presented in [9]. Recursive algorithms computing outer approximations of the FPS have been proposed, based on different classes of sets such as polyhedra [10], ellipsoids [11, 12, 13, 14], orthotopes [15, 16], parallelotopes [17] or zonotopes [18, 19]. A method for computing a polytopic set containing the FPS has been introduced in [20], in the more general context of LPV system identification. The bounded error approach to parameter tracking has also been widely employed within fault detection problems (see e.g., [21, 22, 23]).

In this paper, a recursive technique is proposed for bounding the FPS of a time-varying linear system through an orthotopic approximating region. It is well known that the naive approach consisting in solving at each step a number of linear programs equal to twice the number of parameters, provides the minimum bounding orthotope. However, the related computational burden increases consistently over time. Our purpose is to find an algorithm with the typical computational burden of classical recursive schemes, like e.g. RLS, providing an estimate of the bounding orthotope whose quality is comparable with that obtainable through the naive approach. This target is inspired by the fact that several recursive schemes available in the literature involving more complex approximating regions, like ellipsoids or parallelotopes, lead to estimates whose quality may be surprisingly worse than that of the tight outbounding orthotope, because of the intrinsic conservatism introduced at each time step. In our scheme, the approximating orthotope is updated by exploiting the information contained in a special subset of the constraints defining the FPS. Constraint selection at each step is performed according to the concept of *binding constraints*, which consist in the constraints which are active at the solution point of a suitable linear program.

The proposed scheme is tested on several numerical examples, each including randomly generated realizations of both the measurement noise and parameter change process. The obtained results show a significant improvement of the quality of estimates, measured according to different indicators, with respect to bounding approximations provided by recursive schemes based on ellipsoids or parallelotopes. Moreover, the conservatism of the approximating orthotope compared to that provided by the naive batch approach turns out to be negligible, in spite of the noticeable computational burden reduction. Preliminary results in this respect can be found in [24].

The paper is organized as follows. Section 2 introduces the notation and the definitions used in the paper. The problem formulation is given in Section 3. Section 4 illustrates the proposed recursive FPS approximation procedure. Three numerical examples are presented in Section 5 to evaluate the quality of the approximation provided by the proposed approach, while concluding remarks are given in Section 6.

2 Notation and definitions

Let $A \in \mathbb{R}^{m\times n}$. Then, $|A|$ denotes the matrix whose entries are the absolute values of the entries of A. The jth row of A is denoted by A_j, while if $\mathcal{I} \subset \mathbb{N}$, $A_{\mathcal{I}}$ is the matrix obtained by selecting the rows of A with indexes in $\mathcal{I}$.

An *orthotope*, or axis-aligned box, is defined as

$$\mathcal{O} = \mathbf{o}(\overline{\theta}, d) = \{\theta : \ \theta = \overline{\theta} + \mathrm{diag}(d)w, \ \|w\|_\infty \leq 1\}, \tag{1}$$

where $\overline{\theta}, d, w \in \mathbb{R}^n$, $d_i \geq 0$, $i = 1, \ldots, n$, and $\mathrm{diag}(d)$ is a diagonal matrix with diagonal equal to d. We denote by $\mathcal{F}_i = \{\theta \in \mathcal{O} : \ \theta_i = \overline{\theta}_i + d_i\}$ and $\mathcal{F}_{i+n} = \{\theta \in \mathcal{O} : \ \theta_i = \overline{\theta}_i - d_i\}$, for $i = 1, \ldots, n$, the $(n-1)$-dimensional *faces* of the orthotope $\mathcal{O}$. The weighted ℓ_∞ unit ball is given by

$$\mathcal{B}_\infty(\epsilon) = \{w : |w_i| \leq \epsilon_i, \ \ i = 1, \ldots, n\} = \mathbf{o}(0, \epsilon). \tag{2}$$

The symbol $\oplus$ denotes the Minkowski sum of sets

$$\mathcal{C}_1 \oplus \mathcal{C}_2 = \{\theta : \ \theta = \alpha + \beta, \ \alpha \in \mathcal{C}_1, \ \beta \in \mathcal{C}_2\}.$$

Trivially, $\mathbf{o}(\overline{\theta}_a, d_a) \oplus \mathbf{o}(\overline{\theta}_b, d_b) = \mathbf{o}(\overline{\theta}_a + \overline{\theta}_b, d_a + d_b)$. The minimum volume orthotope containing a set $\mathcal{C}$ is denoted by $\mathbf{o}^*(\mathcal{C})$. If $\mathcal{C}$ is a polytope, $\mathbf{o}^*(\mathcal{C})$ can be computed by solving the $2n$ LPs

$$\beta_i = \begin{array}{c} \max \quad e_i^T\theta \\ \text{s.t.} \\ \theta \in \mathcal{C} \end{array}, \qquad \beta_{i+n} = \begin{array}{c} \min \quad e_i^T\theta \\ \text{s.t.} \\ \theta \in \mathcal{C} \end{array}, \tag{3}$$

for $i = 1, \ldots, n$, where the vectors e_i denote the columns of the identity matrix. Then, $\mathbf{o}^*(\mathcal{C}) = \mathbf{o}(\overline{\theta}^*, d^*)$, where

$$\overline{\theta}_i^* = \frac{\beta_i + \beta_{i+n}}{2}, \quad d_i^* = \frac{\beta_i - \beta_{i+n}}{2}, \quad i = 1, \ldots, n.$$

Consider the linear program (LP)

$$\begin{array}{c} \max \quad c^T x \\ \text{s.t.} \\ Ax \leq b. \end{array} \tag{4}$$

Let $\mathcal{S}^*$ be the solution set of the LP. A constraint $A_i x \leq b_i$ is said to be *active* at x if there exists a feasible x for which $A_i x = b_i$. A constraint $A_i x \leq b_i$ is a *binding constraint* of the LP if there exists a solution $x^* \in \mathcal{S}^*$ of (4) such that $A_i x^* = b_i$. The region defined by the binding constraints of an LP is called *binding set* (obviously, it contains the entire feasible region of the LP). Notice that a binding set contains at least n constraints. Moreover, if the constraint set is bounded there always exists one solution $x^* \in \mathcal{S}^*$ such that at least n linearly independent constraints are active at x^*.

3 Problem formulation

Consider the time-varying linear regression model

$$y(t) = \varphi^T(t)\theta(t) + e(t), \tag{5}$$

where $\theta(t) \in \mathbb{R}^n$ is the parameter vector to be estimated, $\varphi(t)$ is a known vector containing past values of the system input and output signals, and $e(t)$ is an unknown-but-bounded noise such that

$$|e(t)| \leq \delta(t), \quad \forall t, \tag{6}$$

where $\delta(t)$ is a known nonnegative sequence. The parameter vector varies over time according to

$$\theta(t+1) = \theta(t) + w(t), \tag{7}$$

where $w(t) \in \mathbb{R}^n$ is UBB in the weighted ℓ_∞ norm, i.e.

$$|w_i(t)| \leq \epsilon_i(t), \quad \forall t, \quad i = 1, \ldots, n, \tag{8}$$

and $\epsilon_i(t)$ are known nonnegative sequences.

The evolution of the FPS $\Theta(t)$ is described according to the following recursion

$$\Theta(t+1) = \Theta^+(t+1) \bigcap \mathcal{S}(t+1) \tag{9}$$

where

$$\Theta^+(t+1) = \Theta(t) \oplus \mathcal{B}_\infty(\epsilon(t)) \tag{10}$$

and

$$\mathcal{S}(t+1) = \{\theta : |y(t+1) - \varphi^T(t+1)\theta| \leq \delta(t+1)\} \tag{11}$$

is the measurement feasibility set at time $t+1$.

The FPS $\Theta(t)$ is a generic polytope, whose number of faces tends to grow over time. For this reason, a wide variety of recursive set approximation techniques have been proposed in the literature. They are usually based on a family of sets $\mathcal{R}$ of fixed complexity, and proceed by subsequent approximations satisfying the inclusions

$$\mathcal{R}(t+1) \supseteq \mathcal{R}^{+}(t+1) \bigcap \mathcal{S}(t+1). \tag{12}$$

where

$$\mathcal{R}^{+}(t+1) \supseteq \mathcal{R}(t) \oplus \mathcal{B}_{\infty}(\epsilon(t)) \tag{13}$$

By choosing an initial approximation $\mathcal{R}(0) \supseteq \Theta(0)$, where $\Theta(0)$ is the initial FPS (usually derived from a priori information on the system parameters), one has the guarantee that $\Theta^{+}(t) \subseteq \mathcal{R}^{+}(t)$ and $\Theta(t) \subseteq \mathcal{R}(t)$, at every time instant t. Typical choices of the approximating regions $\mathcal{R}$ include orthotopes, ellipsoids, parallelotopes, zonotopes, limited complexity polyhedra. The computation of $\mathcal{R}^{+}(t+1)$ and $\mathcal{R}(t+1)$ in (12)-(13), can be based on various criteria, such as volume minimization or other indicators assessing the overall parametric uncertainty.

In this paper, the sets $\mathcal{R}$ are orthotopes, defined as in (1). It is well known that if this class of sets is exploited in the recursive scheme (12)-(13), and the volume is minimized at each step, the resulting approximation of the FPS is extremely coarse. On the other hand, the minimum volume orthotope containing the FPS, $\mathbf{o}^{*}(\Theta(t))$, can be computed by solving $2n$ LPs at each time t, whose number of constraints increases with t. In the next section, a constraint selection technique is proposed which allows one to compute a recursive orthotopic approximation of the FPS by solving a much smaller number of LPs, with a number of constraints which remains approximately constant over time.

4 Constraint selection technique

In this section, a procedure able to recursively approximate the FPS is reported. First, a proposition instrumental to the formulation of the proposed algorithm is stated. After that, the devised procedure is described step-by-step. Finally, the properties of the algorithm are discussed.

Proposition 1 *Let $\mathcal{C} = \{\theta : \; A\theta \leq b\}$ be a polytope in $\mathbb{R}^n$ and let $\mathbf{o}^{*}(\mathcal{C}) = \mathbf{o}(\overline{\theta}, d)$. Then,*

$$\mathcal{C} \oplus \mathcal{B}_{\infty}(\epsilon) \subseteq \overline{\mathcal{C}} \bigcap \overline{\mathcal{O}}$$

where

$$\overline{\mathcal{C}} = \{\theta : \; A\theta \leq b + |A|\epsilon\} \tag{14}$$

$$\overline{\mathcal{O}} = \mathbf{o}^{*}(\mathcal{C}) \oplus \mathcal{B}_{\infty}(\epsilon) = \mathbf{o}(\overline{\theta}, d+\epsilon). \tag{15}$$

Proof: Let $\theta \in \mathcal{C} \oplus \mathcal{B}_{\infty}(\epsilon)$. Then, there exist $\alpha \in \mathcal{C}$ and $\beta \in \mathcal{B}_{\infty}(\epsilon)$, such that $\theta = \alpha + \beta$. Therefore, one has

$$A\theta = A(\alpha+\beta) \leq b + A\beta \leq b + |A|\epsilon$$

which yields $\theta \in \overline{\mathcal{C}}$. Moreover, being $\alpha \in \mathcal{C}$, one has also $\alpha \in \mathbf{o}^{*}(\mathcal{C})$ and hence

$$\theta = \alpha + \beta \in \mathbf{o}^{*}(\mathcal{C}) \oplus \mathcal{B}_{\infty}(\epsilon) = \overline{\mathcal{O}}.$$

□

Let us now introduce the identification algorithm which will be referred to as Recursive Outer Bounding Orthotope with Binding Constraints (ROBO-BC). Let us consider separately the *time update* step (13) and the *measurement update* step (12).

ROBO-BC Time Update

Step T0. At a generic time t, let

- $\mathcal{C}(t) = \{\theta : \; A(t)\theta \leq b(t)\}$ be a polytope such that $\mathcal{C}(t) \supseteq \Theta(t)$;
- $\mathcal{O}(t) = \mathbf{o}(\overline{\theta}(t), d(t))$ be an orthotope such that $\mathcal{O}(t) \supseteq \Theta(t)$;

- $\mathcal{V}(t) = \left\{v^{(i)}(t),\ i = 1, \ldots, 2n\right\}$ such that $v^{(i)}(t)$ belongs to the ith face of the orthotope $\mathbf{o}^*(\mathcal{V}(t))$ and there are at least n linearly independent constraints in $\mathcal{C}(t)$ that are active at $v^{(i)}(t)$.

Define the approximating set as $\mathcal{D}(t) \triangleq \mathcal{C}(t) \bigcap \mathcal{O}(t)$.
Step T1. Define the enlarged set $\mathcal{D}^+(t+1)$ as

$$\mathcal{D}^+(t+1) = \mathcal{C}^+(t+1) \bigcap \mathcal{O}^+(t+1)$$

where

$$\begin{aligned}
\mathcal{C}^+(t+1) &= \{\theta :\ A(t)\theta \leq b(t) + |A(t)|\epsilon(t)\} \\
\mathcal{O}^+(t+1) &= \mathcal{O}(t) \oplus \mathcal{B}_\infty(\epsilon(t)) = \mathbf{o}(\overline{\theta}(t), d(t) + \epsilon(t))\ .
\end{aligned}$$

Step T2. For each $v^{(i)}(t)$, $i = 1, \ldots, 2n$, define

$$\mathcal{I}_i = \{j_h \in \mathbb{N} :\ A_{j_h}(t)v^{(i)}(t) = b_{j_h}(t),\ h = 1, \ldots, n\} \tag{16}$$

as a set of n indexes corresponding to n linearly independent constraints of $\mathcal{C}(t)$ that are active at $v^{(i)}(t)$, and set

$$\hat{v}^{(i)}(t+1) = \left[A_{\mathcal{I}_i}(t)\right]^{-1}\left[b(t) + |A(t)|\epsilon(t)\right].$$

ROBO-BC Measurement Update
Step M0. Let $\mathcal{C}^+(t+1)$, $\mathcal{O}^+(t+1)$, $\hat{v}^{(i)}(t+1)$, $i = 1, \ldots, 2n$ be given, according to procedure T0-T2, and let $\mathcal{S}(t+1)$ be the new measurement set at time $t+1$. For $i = 1, \ldots, 2n$, define the sets

$$\mathcal{C}_i^+(t+1) = \{\theta :\ A_{\mathcal{I}_i}(t)\theta \leq b(t) + |A(t)|\epsilon(t)\}$$

with $\mathcal{I}_i$ given by (16). Notice that by construction the constraints in $\mathcal{C}_i^+(t+1)$ are a subset of constraints in $\mathcal{C}^+(t+1)$ that are active at $\hat{v}^{(i)}(t+1)$.
Step M1. For each $i = 1, \ldots, 2n$, if $\hat{v}^{(i)}(t+1) \notin \mathcal{S}(t+1)$ solve the LP

$$\begin{gathered}
\max\,(\min)\ e_i^T\theta \\
\text{s.t.} \\
\theta \in \mathcal{C}^+(t+1) \bigcap \mathcal{S}(t+1)
\end{gathered} \tag{17}$$

and set[1]

- $\mathcal{C}_i(t+1)$ as the binding set of the LP (17),
- $v^{(i)}(t+1)$ as an element of the solution set of (17) such that n linearly independent binding constraints of the LP are active at $v^{(i)}(t+1)$.

Otherwise, if $\hat{v}^{(i)}(t+1) \in \mathcal{S}(t+1)$, set $v^{(i)}(t+1) = \hat{v}^{(i)}(t+1)$ and $\mathcal{C}_i(t+1) = \mathcal{C}_i^+(t+1)$.
Step M2. Set

$$\begin{aligned}
\mathcal{C}(t+1) &= \bigcap_{i=1}^{2n} \mathcal{C}_i(t+1) \\
\mathcal{V}(t+1) &= \{v^{(i)}(t+1),\ i = 1, \ldots, 2n\}
\end{aligned}$$

and compute $\widehat{O}(t+1) = \mathbf{o}^*(\mathcal{V}(t+1)) = \mathbf{o}(\hat{\overline{\theta}}(t+1), \hat{d}(t+1))$, where

$$\hat{\overline{\theta}}_i(t+1) = \frac{v_i^{(i)}(t+1) + v_i^{(i+n)}(t+1)}{2}\ ,$$

[1] In (17), the notation max(min) means that max holds for $i = 1, \ldots, n$, while min holds for $i = n+1, \ldots, 2n$.

$$\hat{d}_i(t+1) = \frac{v_i^{(i)}(t+1) - v_i^{(i+n)}(t+1)}{2} \ ,$$

for $i = 1, \ldots, n$.
Step M3. Set

$$\mathcal{O}(t+1) = \widehat{\mathcal{O}}(t+1) \bigcap \mathcal{O}^+(t+1) = \mathbf{o}(\overline{\theta}(t+1), d(t+1)),$$

where

$$\overline{\theta}(t+1) = \frac{u+l}{2} \ , \quad d(t+1) = \frac{u-l}{2} \ ,$$

with

$$\begin{aligned} u_i &= \min \left\{ \hat{\overline{\theta}}_i(t+1) + \hat{d}_i(t+1), \overline{\theta}_i^+(t+1) + d_i^+(t+1) \right\}, \\ l_i &= \max \left\{ \hat{\overline{\theta}}_i(t+1) - \hat{d}_i(t+1), \overline{\theta}_i^+(t+1) - d_i^+(t+1) \right\}. \end{aligned}$$

Then, proceed with the next time update at time $t+1$.

Notice that the time update step of the algorithm is inspired by the result of Proposition 1, which states that the exact time update (10) can be approximated by computing the intersection between two sets: the enlarged version of the polytope $\Theta(t)$, according to (14), and the enlarged bounding box $\mathbf{o}^*(\Theta(t)) \oplus \mathcal{B}_\infty(\epsilon(t))$, given by (15).

After definitions at step T0, one trivially has $\Theta(t) \subseteq \mathcal{D}(t)$. Step T1 guarantees that $\Theta^+(t+1) \subseteq \mathcal{D}^+(t+1)$. In fact, by using Proposition 1

$$\begin{aligned} \Theta^+(t+1) &\triangleq \Theta(t) \oplus \mathcal{B}_\infty(\epsilon(t)) \subseteq \mathcal{D}(t) \oplus \mathcal{B}_\infty(\epsilon(t)) = \left(\mathcal{C}(t) \bigcap \mathcal{O}(t)\right) \oplus \mathcal{B}_\infty(\epsilon(t)) \\ &= (\mathcal{C}(t) \oplus \mathcal{B}_\infty(\epsilon(t))) \bigcap (\mathcal{O}(t) \oplus \mathcal{B}_\infty(\epsilon(t))) \\ &\subseteq \mathcal{C}^+(t+1) \bigcap (\mathbf{o}^*(\mathcal{C}(t)) \oplus \mathcal{B}_\infty(\epsilon(t))) \bigcap \mathcal{O}^+(t+1) \\ &\subseteq \mathcal{C}^+(t+1) \bigcap \mathcal{O}^+(t+1) = \mathcal{D}^+(t+1). \end{aligned}$$

Notice that, after step M3, $\mathcal{C}(t+1)$, $\mathcal{O}(t+1)$ and $\mathcal{V}(t+1)$ satisfy by construction the same properties of their counterparts at time t, and hence the procedure can be iterated.

The measurement update (12) has been inspired by the strategy proposed in [15, 25]. The main idea is that the information provided by the new measurement set $\mathcal{S}(t+1)$ contributes to tighten the ith face of the approximating orthotope $\mathcal{O}^+(t+1)$ only if $\hat{v}^{(i)}(t+1) \notin \mathcal{S}(t+1)$. Whenever this occurs, an LP is solved and only the constraints in the binding set of the LP are deemed to be useful constraints and are kept in the polytope $\mathcal{C}(t+1)$. Otherwise, if $\hat{v}^{(i)}(t+1) \in \mathcal{S}(t+1)$ nothing is done and the set of constraints is left unchanged. Notice that instrumentally this allows one to propagate also a polytopic approximation $\mathcal{C}(t)$ of the FPS, besides the orthotopic one.

It should be remarked that the approach proposed in [25] has been modified to cope with the fact that the system parameters are not constant and hence the approximated FPS must be enlarged at every time update.

5 Numerical examples

In this section, the ROBO-BC procedure for the estimation of time-varying parameters is compared to other recursive approaches taken from the literature. In particular, four algorithms will be considered.

a) A recursive outer bounding *ellipsoidal* estimator (ROBE), performing the measurement update according to the algorithm proposed in [11] (minimum volume ellipsoid containing the intersection between an ellipsoid and the measurement set $\mathcal{S}(t)$), and the time update by first computing the minimum ellipsoid containing $\mathcal{B}_\infty(\epsilon)$ and then applying the algorithm in [26] which provides the optimal bounding ellipsoid for the vector sum of two ellipsoids.

b) A recursive outer bounding *parallelotopic* estimator (ROBP), corresponding to the algorithm proposed in [17].

c) A recursive outer bounding *orthotopic* estimator, corresponding to the Recursive Central Estimate (RCE) proposed in [15, 16]. This method was devised for time-invariant systems and has been adapted to time-varying parameters by using the same time update procedure proposed in Section 4. The comparison with RCE is included because it adopts a set representation similar to the one used in this paper, but a different updating strategy. In fact, the proposed technique requires the solution of a single LP including only the binding constraints for each update of the approximating orthotope, while the RCE relies on a fixed number of constraints and requires the computation of the inverse of n matrices of dimension $n \times n$.

d) The Recursive Outer Bounding Orthotope with Full Constraints (ROBO-FC) which differs from the proposed ROBO-BC algorithm only because all the active constraints in $\Theta(t)$ are kept track of in the polytope $\mathcal{C}(t)$. Notice that this technique requires to solve $2n$ LPs whenever at least one of the constraints in $\mathcal{S}(t+1)$ intersects the time updated polytope $\Theta^+(t+1)$. Performing this check requires the computation of two LPs, so at each time step the number of solved LPs may range from 2 to $2n+2$. It should be remarked that the approximation provided by the ROBO-FC algorithm is by construction contained in that of the ROBO-BC, and therefore is used as a benchmark for all the other techniques (being the exact estimation of the FPS too computationally demanding).

In order to compare the performance of the different algorithms, the uncertainty associated to each parameter is reported. For orthotopic estimators, this is given directly by the semi-length of the corresponding side of the orthotope (the elements of d in (1)). For the ROBE and ROBP algorithms, the minimum orthotope containing respectively the ellipsoidal and parallelotopic approximating set is considered.

5.1 Example 1

The first case study is taken from [27]. Consider the FIR model

$$y(t) = b_1(t)u(t) + b_2(t)u(t-1) + e(t) \tag{18}$$

where the time-varying parameters $b_1(t)$, $b_2(t)$ have been generated according to

$$\begin{aligned} b_1(t) &= 1.5 + \sin(2\pi t/3000) \\ b_2(t) &= 0.5 + \sin(2\pi t/1500) \end{aligned}$$

and $e(t)$ is unknown-but-bounded as in (6). The input signal $u(t)$ has been randomly generated within the interval $[-1, 1]$. The bounds on the parameter variations have been considered constant and set to $\epsilon(t) = [2\pi/3000 \; 2\pi/1500]'$, $\forall t$. The noise bound $\delta(t)$ is constant and equal to 1, and the noise signal $e(t)$ has been generated according to two different distributions (uniform and Gaussian). All results are averaged over 500 identification experiments with 5000 data points each.

Figure 1 shows these uncertainties for the parameters $b_1(t)$ and $b_2(t)$ for a uniformly distributed noise $e(t)$, while Figure 2 does the same for the case of Gaussian noise. It can be observed that the proposed technique provides a tighter approximation of the parametric uncertainty with respect to the ROBE, ROBP and RCE algorithms. Moreover, and more importantly, such approximation is almost indistinguishable from that given by the ROBO-FC, thus testifying that the constraint selection technique has captured the essential constraints which contribute to define the true FPS. However, the computational burden required by the proposed technique is much smaller than that necessary to keep track of all the active constraints. This is shown in Table 1, reporting the average number of active constraints and of LPs to be performed per time step.

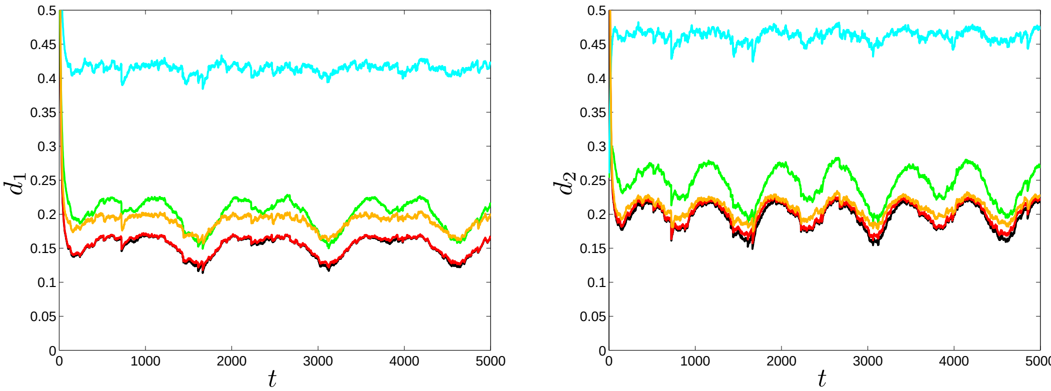

Figure 1: Example 1, uniform noise: uncertainty associated to parameters $b_1(t)$ and $b_2(t)$: ROBO-FC (black), ROBO-BC (red), ROBP (green), ROBE (cyan), RCE (yellow).

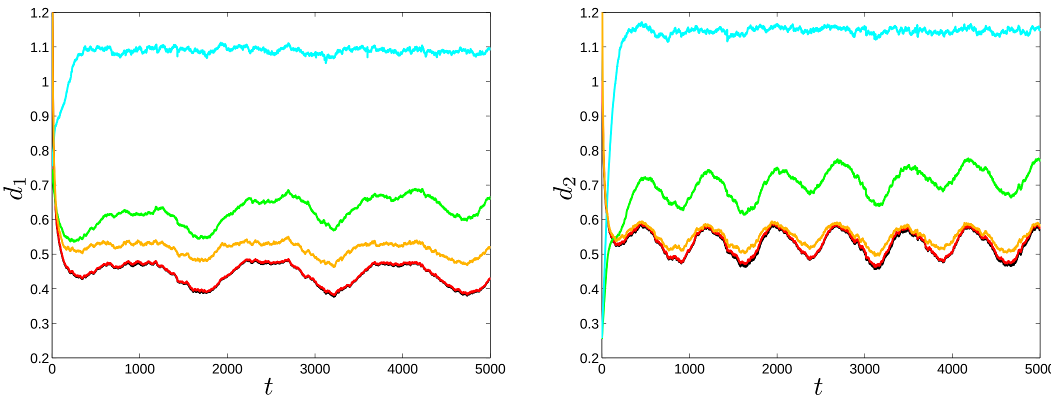

Figure 2: Example 1, Gaussian noise: uncertainty associated to parameters $b_1(t)$ and $b_2(t)$: ROBO-FC (black), ROBO-BC (red), ROBP (green), ROBE (cyan), RCE (yellow).

5.2 Example 2

Let us consider a variant of the FIR system in Example 1, where b_1 and b_2 show a linear drift, i.e.,

$$\begin{aligned} b_1(t) &= 1.5 + 0.002\,t \\ b_2(t) &= 0.5 + 0.001\,t. \end{aligned}$$

Noise e is uniformly distributed in $[-0.2, 0.2]$, while bounds on parameter variations are set to $\epsilon(t) = [0.004\ 0.002]'$, $\forall t$. Notice that, in order to simulate a real scenario, bounds $\epsilon(t)$ are overestimated w.r.t. actual values by a factor 2. The experiment length is set to 5000 samples. Figure 3 compares the uncertainty (averaged over 500 experiments) of the two unknown parameters obtained with the proposed technique and other recursive algorithms. As in the previous example, the uncertainty associated to parameters obtained by the proposed algorithm is almost indistin-

Table 1: Example 1: average number of active constraints and LPs.

	Unif. noise		Gauss. noise	
	active	LPs	active	LPs
ROBO-BC	4.98	0.25	5.80	0.18
ROBO-FC	5.87	0.80	8.92	0.77

guishable from that obtained by the ROBO-FC estimator, while the mean number of solved LPs is greatly reduced, passing from 1.83 to 0.57. Moreover, the devised algorithm clearly outperform the ROBE, ROBP and RCE algorithms.

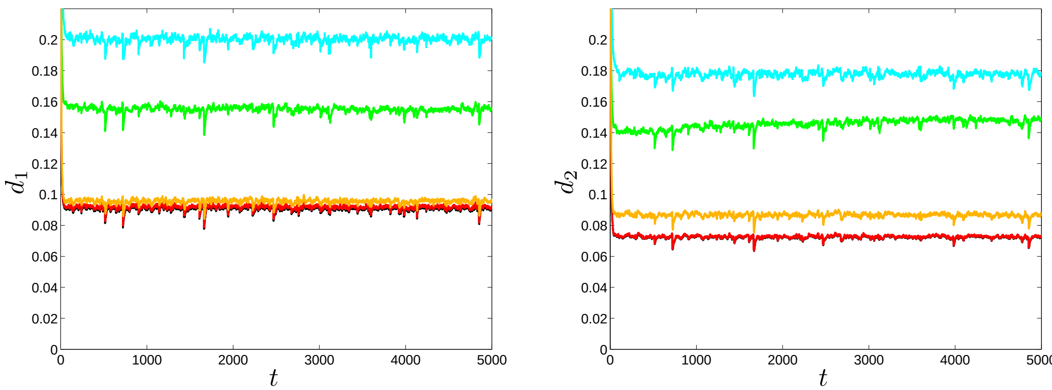

Figure 3: Example 2: uncertainty associated to parameters $b_1(t)$ and $b_2(t)$: ROBO-FC (black), ROBO-BC (red), ROBP (green), ROBE (cyan), RCE (yellow).

5.3 Example 3

Consider the time-varying ARX model

$$y(t) + a_1(t)y(t-1) + a_2(t)y(t-2) = b_1(t)u(t-1) + b_2(t)u(t-2) + e(t)$$

where the parameter vector $\theta(t) = [a_1(t)\ a_2(t)\ b_1(t)\ b_2(t)]'$ changes over time according to model (7), with process noise $w(t)$ satisfying (8), with

$$\epsilon(t) = [1/1000\ 0\ 1/100\ 0]', \quad \forall t$$

and $\theta(0) = [0.5\ \ 0.3\ \ 0.7\ \ 0.4]'$ (notice that $a_2(t)$ and $b_2(t)$ are constant parameters, while $a_1(t)$ and $b_1(t)$ are time-varying, with different variation rates). The signals $w_1(t)$ and $w_3(t)$ have been generated from a uniform random distribution, within the bounds (8). The input $u(t)$ and the measurement noise $e(t)$ have been generated according to a uniform distribution as in Example 2, with $\delta(t) = 0.05$, $\forall t$. Results are averaged over 500 different realizations of $u(t)$, $e(t)$ and $w(t)$.

Figure 4 shows the uncertainties for the auto-regressive parameters $a_1(t)$ and $a_2(t)$, while in Figure 5 the uncertainties related to the input parameters b_1 and b_2 are reported. It can be observed that the proposed technique outperforms the ROBE, ROBP and RCE algorithms, and provides a very small overestimation of the parametric uncertainty given by the ROBO-FC estimator. Indeed, the two orthotopic estimates are almost superimposed for the time-invariant parameters a_2 and b_2, while some more conservatism is observed in the approximation of the time-varying parameters a_1 and b_1. As long as the computational burden is concerned, Table 2 shows that the average number of active constraints and solved LPs per time step propagated by the ROBO-BC algorithm is significantly smaller than that employed by the ROBO-FC, for both uniform and Gaussian distributed noise. Moreover, in the proposed method, the number of active constraints of each LP remains approximately constant over time, as shown in Figure 6.

In order to analyze the behaviour of the considered algorithms w.r.t. the noise bound δ, a simulation campaign has been performed by varying δ from 10^{-3} to 10^{-1}. Twenty-one different values of δ (logarithmically spaced) have been considered; uniformly distributed noise is assumed, and results are averaged over 100 identification experiments for each choice of δ.

In Fig. 7, the averaged final uncertainties (after $N = 5000$ samples) associated to each parameter are reported for different values of the noise level δ. Notice that, differently from the other considered algorithms, the uncertainty obtained by the proposed method remains close to that

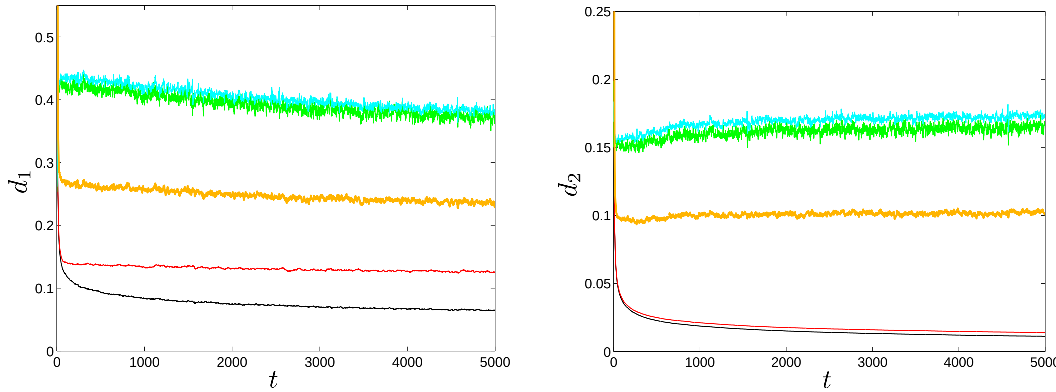

Figure 4: Example 3: uncertainty associated to parameters $a_1(t)$ and $a_2(t)$: ROBO-FC (black), ROBO-BC (red), ROBP (green), ROBE (cyan), RCE (yellow).

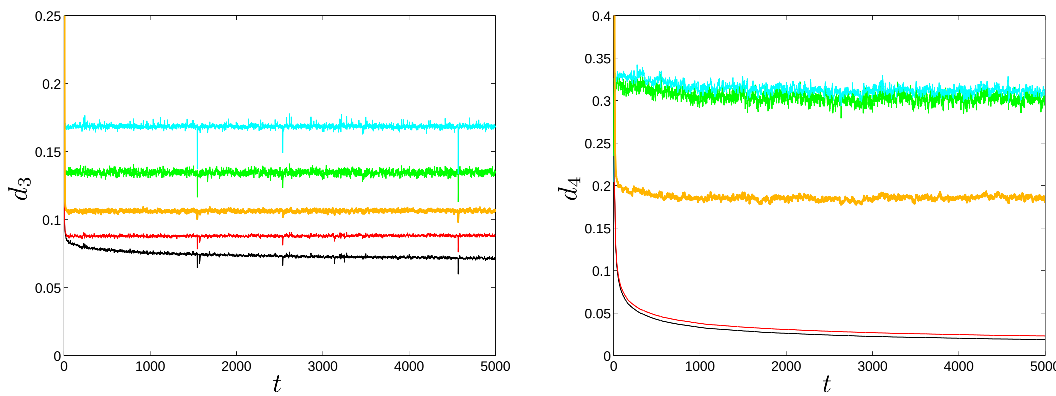

Figure 5: Example 3: uncertainty associated to parameters $b_1(t)$ and $b_2(t)$: ROBO-FC (black), ROBO-BC (red), ROBP (green), ROBE (cyan), RCE (yellow).

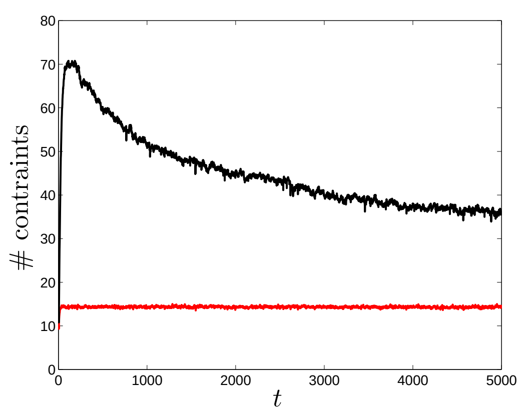

Figure 6: Example 3, Gaussian noise: average number of active constraints for each solved LP by the ROBO-FC (black) and by the ROBO-BC method (red).

Table 2: Example 3: average number of active constraints and LPs.

	Uniform noise		Gaussian noise	
	active	LPs	active	LPs
ROBO-BC	12.25	3.84	14.32	3.96
ROBO-FC	19.14	8.72	45.14	9.50

obtained by the ROBO-FC estimator for each value of δ. Regarding the computational burden, Figures 8 and 9 compare the average number of LPs and active constraints for the ROBO-BC and the ROBO-FC methods. In Figure 10, the ratio between the number of solved LPs by the ROBO-FC algorithm and by the proposed method is reported. Notice that, for different values of the noise level δ, the ROBO-FC method needs to solve a number of LPs going from 130% to 260% w.r.t. that needed by the ROBO-BC algorithm. Moreover, as reported in Figure 9, the number of active constraints of each LP is reduced in the proposed approach, leading to an additional reduction of the computational burden.

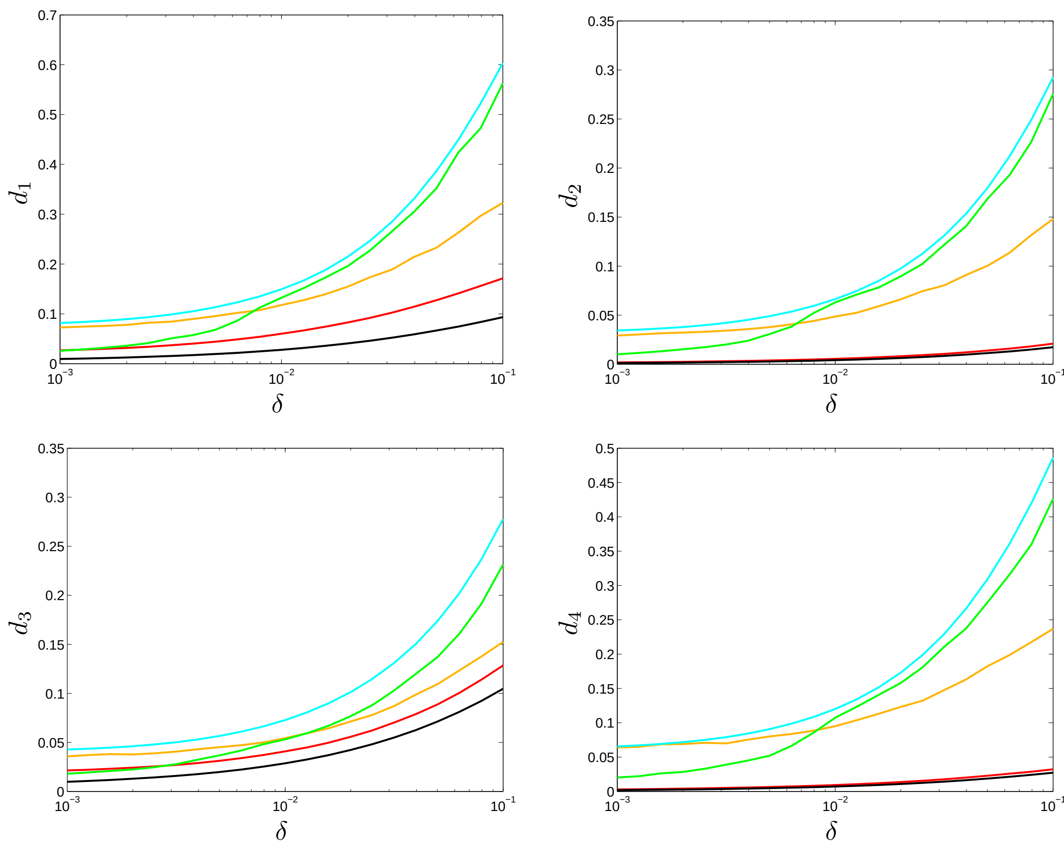

Figure 7: Example 3: Final uncertainty of each parameter: ROBO-FC (black), ROBO-BC (red), ROBP (green), ROBE (cyan), RCE (yellow).

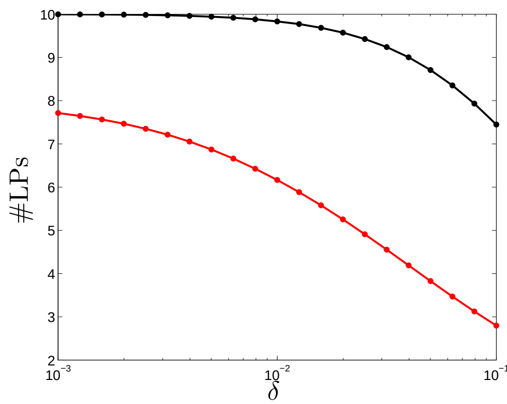

Figure 8: Example 3: Mean number of solved LPs at each iteration by the ROBO-FC (black) and by the ROBO-BC method (red).

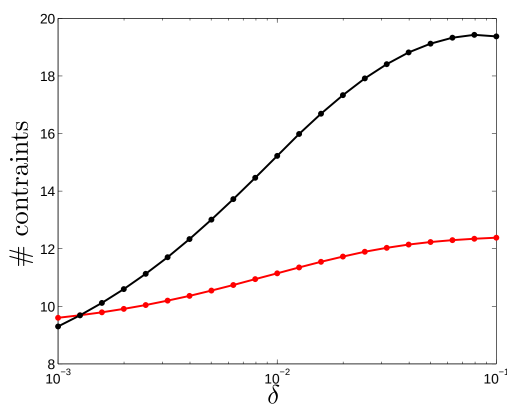

Figure 9: Example 3: Mean number of constraints for each solved LP by the ROBO-FC (black) and by the ROBO-BC method (red).

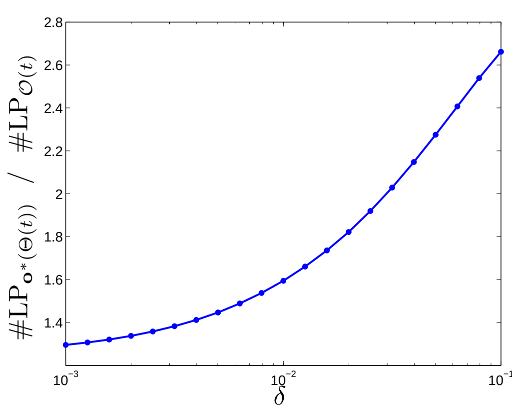

Figure 10: Example 3: Ratio between the mean number of solved LPs by the ROBO-FC and by the ROBO-BC method.

5.4 Example 4

To analyze the behaviour of the proposed algorithm for estimating systems involving a large number of parameters, a FIR model of order 10 is considered,

$$y(t) = \sum_{i=1}^{10} b_i(t)u(t-i) + e(t)\ .$$

Initial values of parameters are set to 1, i.e., $\theta(0) = [1, \ldots, 1]'$. The parameters $b_i(t)$ with odd indexes are assumed to be constant and equal to 1, while those with even indexes change according to model (7)-(8), with $\epsilon_i(t) = 1/500$, $\forall t$. Signals $w(t)$, $u(t)$ and $e(t)$ are generated as in Example 2, with $\delta(t) = 0.1$, $\forall t$. The experiment length is set to 1000 samples. Table 3 reports the uncertainty associated to each parameter, averaged both with respect to the parameters and to time. A total number of 100 experiments have been performed. Table 4 provides the average number of active constraints and solved LPs per iteration for the proposed method and for the ROBO-FC algorithm. As expected, in the latter case the active constraints increase significantly and one has to solve a number of LPs per iteration close to $2n + 2$. Conversely, such figures are much lower for the ROBO-BC technique, thus making it viable for online implementation.

Table 3: Example 4: average parametric uncertainty.

	Unif. noise	Gauss. noise
ROBE	0.4177	0.4552
ROBP	0.5557	0.5556
RCE	0.4738	0.5481
ROBO-BC	0.2115	0.2807
ROBO-FC	0.2078	0.2772

Table 4: Example 4: average number of active constraints and LPs.

	Unif. noise		Gauss. noise	
	active	LPs	active	LPs
ROBO-BC	44.89	8.53	55.31	8.99
ROBO-FC	268.79	21.99	398.94	22.00

6 Conclusions

In this paper an algorithm has been provided for tracking parameter changes of a time-varying linear system. A set membership approach to the recursive estimation problem has been adopted, where orthotopic approximating regions have been used for outbounding the feasible parameter set. The devised procedure is based on a constraint selection technique which allows one to keep a limited number of constraints bounding the feasible set at each step of the recursion. Binding constraints are used to generate limited size linear programs to be solved at each time step. Several numerical tests performed on randomly generated experiments show good performance of the recursive procedure both in terms of computational burden and quality of the obtained estimate, as compared to alternative techniques available in the literature. Ongoing work is focused on devising efficient algorithms for tracking the feasible parameter set through more structured approximating regions like ellipsoids or parallelotopes. Moreover, the extension of the approach to deal with state estimation of dynamical system is under investigation.

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