Efficient computation of ℓ_1 uncertainty model from an impulse response set

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Abstract

In this paper the problem of constructing the minimum ℓ_1 uncertainty model containing a finite set of assigned models is addressed. This problem is relevant to ℓ_1 robust control design techniques, in order to minimize the size of the uncertainty set associated to the nominal plant. The problem is formulated as a (conditional) Chebyshev center problem and an algorithm for its solution is proposed. The algorithm converges in a finite number of steps showing computational efficiency for large size problems.

Key words: Uncertainty model, ℓ_1 uncertainty, robust control, impulse response, linear programs.

1 Introduction

The construction of uncertainty model sets is a key step in robust control. A typical example is given by additive unstructured uncertainty models $G + \Delta$, in which G represents the nominal model, and Δ is the uncertainty block assumed bounded in some suitable norm. The choice of the norm depends on the control objective in terms of robust performances. The ℓ_1 robust control setup is motivated by robustness requirements with respect to the peak-to-peak gain and has been widely investigated in the last twenty years (see e.g. [5,8,4] for a thorough treatment).

When an uncertainty model set is not available, a possible approach is to estimate it from data. Robust identification in the ℓ_1 setting has been addressed by a number of researchers, along different approaches (see e.g. [9] and references therein). Recent contributions have concerned ℓ_1 identification with mixed stochastic/deterministic uncertainty models [12] and the ℓ_1 error quantification problem, in which the nominal model G is given and the aim is to estimate the ℓ_1 uncertainty bound from data [10].

In this paper, it is assumed that the information available on the system is given as a collection of feasible models. Such models are representative of the behavior of the system under various operating conditions: for example, they may be generated by identification experiments performed under different conditions (e.g., different input signals), or they may originate from linearization of a nonlinear system around several operating points. In this setting, the problem of constructing the best uncertainty model set consists of estimating a nominal model G and the minimum size of the uncertainty Δ so that all the given feasible models are contained in the uncertainty model set. This uncertainty measure is generally needed to apply classical robust control techniques. Moreover, since the structure of the nominal model is generally selected considering different classes of restricted complexity approximating models (see e.g. [11,13,14]), the solution of this problem may be useful in choosing the most appropriate model structure for representing the plant through a minimal size uncertainty ball. As an example, consider the case when an orthonormal basis function expansion is used to approximate the plant transfer function and one has to choose the optimal pole location of the basis functions (see e.g. [2]).

In this paper, the above problem is addressed in the ℓ_1 setup and an algorithm is proposed for computing the best ℓ_1 uncertainty model set containing a finite set of assigned models. It is shown that the problem can be formulated as the computation of the ℓ_1 *conditional Cheby-*

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shev center of a suitable set, i.e. the center of the minimum ℓ_1 ball containing the set, subject to some predefined constraints. Conditional centers play a key role in the set-membership estimation literature (see e.g. [6,3]) and their computation typically requires the solution of min-max optimization problems. In the ℓ_1 setting, the problem can be cast as a Linear Program (LP), whose dimension may make it untractable even for small size problems. Conversely, the proposed algorithm computes the ℓ_1 conditional center by solving a finite sequence of small linear programs, and it is shown to be computationally feasible even for large size problems.

The paper is organized as follows. Section 2 contains the problem formulation and shows how the construction of the best uncertainty model set can be cast as a conditional Chebyshev center problem. In Section 3, an algorithm for the computation of the Chebyshev center is presented. Computational issues are treated in Section 4. An illustrative example is reported in Section 5 and some conclusions are given in Section 6.

2 Problem formulation

In the following, SISO discrete-time LTI systems with impulse responses bounded in the ℓ_1 norm are considered. The problem addressed in this paper is that of constructing an ℓ_1 uncertainty model set

$$\mathcal{G} = \{P : \ P = G + \Delta, \ \ G \in \mathcal{M}, \ \ \|\Delta\|_1 \leq \gamma\} \qquad (1)$$

containing a given set of models $\mathcal{H} = \{H_1, \ldots, H_n\}$, i.e. such that $\mathcal{H} \subseteq \mathcal{G}$. In (1), $\mathcal{M}$ denotes the model class to which the nominal model G must belong. In particular, the aim is that of finding the nominal model G so that the size γ of the ℓ_1 uncertainty is minimized.
In the ℓ_1 framework, it is convenient to cast the problem in terms of impulse responses. Let us denote by $g = \{g(k)\}_{k=0}^{+\infty}$ and $h_j = \{h_j(k)\}_{k=0}^{+\infty}$ the impulse responses of G and H_j, respectively. Therefore, the problem is that of computing g^* such that

$$g^* = \arg \inf_{g \in \mathcal{M}} \sup_{h \in \mathcal{H}} \|g - h\|_1 \qquad (2)$$

and the corresponding ℓ_1 uncertainty bound $\gamma^* = \sup_{h \in \mathcal{H}} \|g^* - h\|_1$.

Let $T_N : \ell_1 \to \mathbb{R}^N$ denote the truncation operator $T_N g = [g(0) \ g(1) \ \ldots \ g(N-1)]'$, and R_N be the corresponding remainder operator, such that $g = \begin{bmatrix} T_N g \\ R_N g \end{bmatrix}$.
The following nominal model class will be considered

$$\mathcal{M} = \{g : \ T_N g = M\theta, \ \theta \in \mathbb{R}^p; \ \ R_N g = 0\}. \qquad (3)$$

The matrix M can be chosen so that its columns form a p-dimensional basis of truncated impulse responses of linear filters. Typical examples are Laguerre or Kautz functions [13,14] or generalized orthonormal basis functions [11]. Clearly, the set $\mathcal{M}$ contains the class of FIR filters as a special case (when M is the identity matrix and $p = N$).

Problem (2) with the model class (3) boils down to a min-max optimization problem in a finite dimensional space. Indeed, for any $g \in \mathcal{M}$ one has $\|g - h\|_1 = \|M\theta - T_N h\|_1 + \|R_N h\|_1$. By setting $\delta_j = \|R_N h_j\|_1$, for $j = 1, \ldots, n$, one gets

$$\inf_{g \in \mathcal{M}} \sup_{h \in \mathcal{H}} \|g-h\|_1 = \inf_{\theta \in \mathbb{R}^p} \sup_{j=1,\ldots,n} \left\| \begin{pmatrix} M \\ 0 \end{pmatrix} \theta - \begin{pmatrix} T_N h_j \\ \delta_j \end{pmatrix} \right\|_1 . \qquad (4)$$

Problem (4) is an instance of the so-called *conditional Chebyshev center* problem. For a set $\mathcal{X}$, an ℓ_1 Chebyshev center is defined as the center of a minimum ℓ_1 ball containing the set $\mathcal{X}$, i.e.

$$c^* = \arg \inf_{c} \sup_{x \in \mathcal{X}} \|x - c\|_1, \qquad (5)$$

while the *Chebyshev radius* of $\mathcal{X}$ is

$$r^* = \sup_{x \in \mathcal{X}} \|x - c^*\|_1. \qquad (6)$$

Notice that the Chebyshev center may not be unique, while the Chebyshev radius is unique.
When the center is constrained to belong to a given set $\mathcal{C}$, it is denoted as conditional Chebyshev center (with respect to $\mathcal{C}$), i.e.

$$c^*_{\mathcal{C}} = \arg \inf_{c \in \mathcal{C}} \sup_{x \in \mathcal{X}} \|x - c\|_1, \qquad (7)$$

while the conditional radius of $\mathcal{X}$ is defined as

$$r^*_{\mathcal{C}} = \sup_{x \in \mathcal{X}} \|x - c^*_{\mathcal{C}}\|_1. \qquad (8)$$

If we define $x_j \in \mathbb{R}^{N+1} = \begin{bmatrix} T_N h_j \\ \delta_j \end{bmatrix}$, $j = 1, \ldots, n$, and introduce the sets $\mathcal{X} = \{x_1, \ldots, x_n\}$ and $\mathcal{C} = \{c \in \mathbb{R}^{N+1} : \ T_N c = M\theta, \ \theta \in \mathbb{R}^p; \ c_{N+1} = 0\}$, it immediately turns out that problem (4) is equivalent to the computation of the ℓ_1 conditional Chebyshev radius of $\mathcal{X}$, with respect to $\mathcal{C}$.

In the literature, the computation of conditional Chebyshev centers has been addressed in the ℓ_2 and ℓ_∞ norm ([6,3]), while no efficient procedure is available for the ℓ_1 case. In fact, the known solution to the problem is in general unfeasible from the computational point of view. In the next section, an efficient algorithm for the computation of the ℓ_1 Chebyshev center and radius will be introduced.

3 Computation of the ℓ_1 Chebyshev center

Let $\mathcal{X} = \{x_1, \ldots, x_n\} \subset \mathbb{R}^m$. A natural way to solve (5)–(6) is to write the equivalent problem

$$\begin{cases} \inf r \\ s.t.: \ \|x_j - c\|_1 \le r \quad , \quad j = 1, \ldots, n \end{cases}$$

which can be converted into the following LP problem

$$\begin{cases} \inf r \\ s.t.: \\ -q_{j,i} \le (x_{j,i} - c_i) \le q_{j,i} \ , \ j = 1, \ldots, n, \ i = 1, \ldots, m \\ \sum_{i=1}^m q_{j,i} \le r \ , \ j = 1, \ldots, n \end{cases} \tag{9}$$

where $x_{j,i}$ denotes the i-th component of vector x_j. The optimal c and r are the Chebyshev center and radius of $\mathcal{X}$, respectively. The LP (9) has $mn+m+1$ variables and $2mn+n$ constraints and turns out to be computationally unfeasible also for small size problems such as, e.g., $n = 100, \ m = 100$.

To overcome this issue, an efficient algorithm to solve (5)–(6) will be presented. Let us firstly define the key notion of improving direction.

Definition 1 *Let $c \in \mathbb{R}^m$. A vector d is said an* improving direction *for c, if there exists $\varepsilon > 0$ such that* $\max_{x \in \mathcal{X}} \|x - (c + \varepsilon\, d)\|_1 < \max_{x \in \mathcal{X}} \|x - c\|_1$.

Given any starting point $c^{(0)}$, the algorithm described in the following returns a sequence of candidate centers $c^{(k)}$ and of associated improving directions $d^{(k)}$, such that $c^{(k+1)} = c^{(k)} + d^{(k)}$. It will be shown that there exists a finite K such that $\max_{x \in \mathcal{X}} \|x - c^{(k)}\|_1 < \max_{x \in \mathcal{X}} \|x - c^{(k-1)}\|_1$, for $k = 1, \ldots, K$, and $c^{(K)} = c^*$. To simplify the notation, we will denote by c and d the candidate center and the improving direction at a generic step k, omitting the superscripts when they are not necessary.

At a given step k, let us define, for any $j = 1, \ldots, n$,

$$t_j = (x_j - c). \tag{10}$$

Let us define $\underline{d}, \overline{d} \in \mathbb{R}^m$, so that the elements of these vectors are given by

$$\underline{d}_i = \max_{j:\ t_{j,i} < 0} \{t_{j,i}\} \quad , \quad i = 1, \ldots, m \tag{11}$$

$$\overline{d}_i = \min_{j:\ t_{j,i} > 0} \{t_{j,i}\} \quad , \quad i = 1, \ldots, m \tag{12}$$

where $t_{j,i}$ denotes the i-th component of vector t_j. Given i, if there does not exist a $t_{j,i} < 0$, then it is assumed $\underline{d}_i = -\infty$. Similarly, if there does not exist a $t_{j,i} > 0$, then it is assumed $\overline{d}_i = +\infty$. Define the hyperbox

$$\mathcal{D} = \{d \in \mathbb{R}^m \colon \ \underline{d}_i \le d_i \le \overline{d}_i \ , \ i = 1, \ldots, m\}. \tag{13}$$

Let us introduce the notion of optimal Chebyshev step.

Definition 2 *A vector $\tilde{d} \in \mathbb{R}^m$ is an* optimal Chebyshev step *for c on $\mathcal{D}$ if*

$$\tilde{d} = \arg \inf_{d \in \mathcal{D}} \max_{x \in \mathcal{X}} \|x - (c + d)\|_1. \tag{14}$$

Notice that $c + \tilde{d}$ is a conditional Chebyshev center of $\mathcal{X}$ with respect to the set $c + \mathcal{D}$.

Let $I = \{i = 1, \ldots, m\}$, $I_j^+ = \{i \in I \colon t_{j,i} > 0\}$, $I_j^- = \{i \in I \colon t_{j,i} < 0\}$, and $I_j^0 = \{i \in I \colon t_{j,i} = 0\}$. The computation of an optimal Chebyshev step is addressed in the following lemma.

Lemma 1 *Let $\mathcal{X} = \{x_1, \ldots, x_n\} \subset \mathbb{R}^m$ and $c \in \mathbb{R}^m$. Let $\mathcal{D}$ be defined as in (10)-(13). Let $[\tilde{\gamma}, \tilde{d}, \tilde{q}]'$ be a solution of the LP problem*

$$\begin{cases} \min \gamma \\ s.t.: \\ \displaystyle\sum_{i \in I_j^-} d_i + \sum_{i \in I_j^+} (-d_i) + \sum_{i \in I_j^0} q_i - \gamma \le -\|t_j\|_1 \ , \ j = 1, \ldots, n \\ -q_i \le d_i \le q_i \ , \ \underline{d}_i \le d_i \le \overline{d}_i \ , \ i = 1, \ldots, m. \end{cases} \tag{15}$$

Then, $\tilde{d}$ is an optimal Chebyshev step for c on $\mathcal{D}$. Moreover, c is a Chebyshev center of $\mathcal{X}$ if and only if $\tilde{\gamma} = \max\limits_{x \in \mathcal{X}} \|x - c\|_1$.

Proof. Let us rewrite the optimization problem (14) as

$$\inf_{d \in \mathcal{D}} \max_{j=1,\ldots,n} \|t_j - d\|_1. \tag{16}$$

By (11)-(13), it follows that

$$\begin{aligned} \max_{j=1,\ldots,n} \|t_j - d\|_1 &= \max_{j=1,\ldots,n} \sum_{i=1}^m |t_{j,i} - d_i| \\ &= \max_{j=1,\ldots,n} \left[\sum_{i \in I_j^+} (t_{j,i} - d_i) + \sum_{i \in I_j^-} (d_i - t_{j,i}) + \sum_{i \in I_j^0} |d_i| \right] \\ &= \max_{j=1,\ldots,n} \left[\sum_{i \in I_j^+} t_{j,i} + \sum_{i \in I_j^-} (-t_{j,i}) + \sum_{i \in I_j^-} d_i + \sum_{i \in I_j^+} (-d_i) + \sum_{i \in I_j^0} |d_i| \right] \\ &= \max_{j=1,\ldots,n} \left[\|t_j\|_1 + \sum_{i \in I_j^-} d_i + \sum_{i \in I_j^+} (-d_i) + \sum_{i \in I_j^0} |d_i| \right]. \end{aligned}$$

Therefore, it is possible to rewrite (16) as

$$\inf_{d\in\mathcal{D}} \max_{j=1,\ldots,n} \left[\|t_j\|_1 + \sum_{i\in I_j^-} d_i + \sum_{i\in I_j^+} (-d_i) + \sum_{i\in I_j^0} |d_i| \right]$$

which is equivalent to

$$\begin{cases} \min \gamma \\ s.t.: \\ \|t_j\|_1 + \sum_{i\in I_j^-} d_i + \sum_{i\in I_j^+} (-d_i) + \sum_{i\in I_j^0} |d_i| \le \gamma \ , \ j=1,\ldots,n \\ \underline{d}_i \le d_i \le \overline{d}_i \ , \ i=1,\ldots,m. \end{cases} \tag{17}$$

It can be easily checked that (17) is equivalent to (15). Finally, being $\tilde{\gamma} = \max_{x\in\mathcal{X}} \|x-(c+\tilde{d})\|_1$, one has that $\tilde{\gamma} \le \max_{x\in\mathcal{X}} \|x-c\|_1$ (because $\mathcal{D}$ contains the null vector by definition). If $\tilde{\gamma} = \max_{x\in\mathcal{X}} \|x-c\|_1$, than there does not exist any improving direction for c on $\mathcal{D}$. Since problem (5) is convex in c, it turns out that c is the Chebyshev center of $\mathcal{X}$. Conversely, if c is a Chebyshev center then, by definition, there are no improving directions for c on $\mathcal{D}$, and hence $\tilde{\gamma} = \max_{x\in\mathcal{X}} \|x-c\|_1$. □

Remark 1 *Given any point $c \in \mathbb{R}^m$, the LP (15) in Lemma 1 can be used to test whether c is the Chebyshev center of $\mathcal{X}$. In fact, c is the center of $\mathcal{X}$ if and only if $\tilde{\gamma} = \max_{x\in\mathcal{X}} \|x-c\|_1$. Notice that this occurs whenever the optimal Chebyshev step $\tilde{d}$ obtained by solving (15) is not an improving direction for c.*

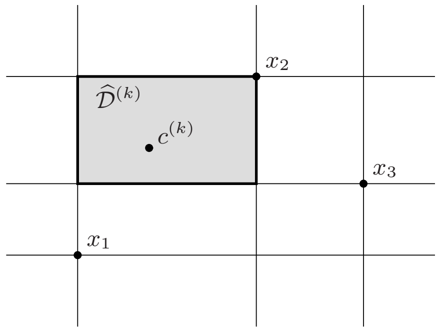
Let us consider at a given iteration k, the candidate center $c^{(k)}$ and the associated hyperbox $\mathcal{D}^{(k)}$ defined by (10)-(13). Let $\tilde{d}^{(k)}$ be an optimal Chebyshev step for $c^{(k)}$ on $\mathcal{D}^{(k)}$ and

$$\tilde{\gamma}^{(k)} = \inf_{d\in\mathcal{D}^{(k)}} \max_{x\in\mathcal{X}} \|x-(c^{(k)}+d)\|_1. \tag{18}$$

By Lemma 1, if $\tilde{\gamma}^{(k)} = \max_{x\in\mathcal{X}} \|x-c^{(k)}\|_1$, then $c^{(k)}$ is the Chebyshev center of $\mathcal{X}$ and the algorithm stops. Otherwise, if $\tilde{\gamma}^{(k)} < \max_{x\in\mathcal{X}} \|x-c^{(k)}\|_1$, let us set the k-th improving direction $d^{(k)}$ equal to an optimal Chebyshev step for $c^{(k)}$ on $\mathcal{D}^{(k)}$, i.e. $d^{(k)} = \tilde{d}^{(k)}$ and $c^{(k+1)} = c^{(k)} + \tilde{d}^{(k)}$.

Since $\tilde{d}^{(k)} \in \mathcal{D}^{(k)}$, it follows that $c^{(k+1)} \in \widehat{\mathcal{D}}^{(k)}$, where $\widehat{\mathcal{D}}^{(k)} = c^{(k)} + \mathcal{D}^{(k)}$. Thus, one has

$$\max_{j:\, x_{j,i} < c_i^{(k)}} \{x_{j,i}\} \le c_i^{(k+1)} \le \min_{j:\, x_{j,i} > c_i^{(k)}} \{x_{j,i}\} \ , \ i=1,\ldots,m.$$



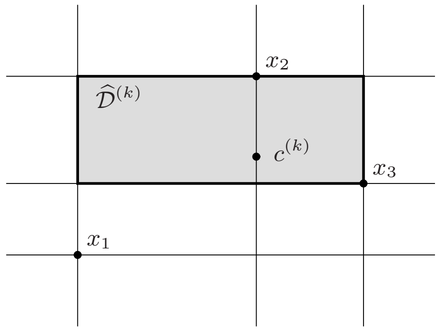


Fig. 1. Two examples of sets $\widehat{\mathcal{D}}^{(k)}$ $(m=2,\ n=3)$.

Two examples of sets $\widehat{\mathcal{D}}^{(k)}$ with $m=2$ and $n=3$ are reported in Figure 1.

The next result states that the algorithm described above converges in a finite number of steps.

Theorem 1 *Let $\mathcal{X} = \{x_1,\ldots,x_n\} \subset \mathbb{R}^m$ and $c^{(0)} \in \mathbb{R}^m$. Define the sequence*

$$c^{(k+1)} = c^{(k)} + d^{(k)} \ , \ k=0,1,\ldots$$

where $d^{(k)}$ is an optimal Chebyshev step for $c^{(k)}$ on $\mathcal{D}^{(k)}$. Then, there exists a finite K such that $c^{(K)}$ is an ℓ_1 Chebyshev center of $\mathcal{X}$ and $\tilde{\gamma}^{(K)}$ its corresponding radius.

Proof. Consider the sequence $\tilde{\gamma}^{(k)}$ defined in (18). If for some K it occurs that $\tilde{\gamma}^{(K)} = \max_{x\in\mathcal{X}} \|x-c^{(K)}\|_1$, than Lemma 1 guarantees that $c^{(K)}$ is an ℓ_1 Chebyshev center of $\mathcal{X}$ and $\tilde{\gamma}^{(K)}$ its corresponding radius. Let us assume by contradiction that the above stopping condition is never verified. Then, by construction, the sequence $\tilde{\gamma}^{(K)}$ is strictly decreasing. This requires that at each iteration k, the set $\widehat{\mathcal{D}}^{(k)}$ contains portions of the space $\mathbb{R}^m$ that were not included in any $\widehat{\mathcal{D}}^{(l)}$, $l<k$. Since the number of possible regions $\widehat{\mathcal{D}}$ is finite, it can be concluded that the stopping condition is satisfied after a finite number of steps. □

The following pseudo-programming code summarizes the proposed algorithm.

Algorithm 1

1. Choose $\mathtt{c} \in \mathbb{R}^{\mathtt{m}}$

2. $\gamma_{\mathtt{old}} = \max_{\mathtt{x}\in\mathcal{X}} \|\mathtt{x}-\mathtt{c}\|_1$

3. $\mathtt{exit} = 0$

4. while ($\mathtt{exit} == 0$*)*

5. $[\tilde{\gamma}, \tilde{\mathtt{d}}] = \mathtt{solveLP}(\mathcal{X}, \mathtt{c})$

6. if ($\tilde{\gamma} == \gamma_{\mathtt{old}}$*) then* $\mathtt{exit} = 1$

7. else $\gamma_{\mathtt{old}} = \tilde{\gamma}$*;* $\mathtt{c} = \mathtt{c} + \tilde{\mathtt{d}}$*; end if*

8. end while

9. $\mathtt{c}^* = \mathtt{c}$*;* $\mathtt{r}^* = \tilde{\gamma}$

where `solveLP` *denotes a function which solves the LP in (15) and returns* $\tilde{\gamma}$ *and* $\tilde{d}$.

Remark 2 *Note that the LP in (15) has* $2m+1$ *variables and* $4m+n$ *constraints, i.e. its computational burden is much lower than that of the LP (9) (see Section 4 for further details).*

In the following, a simple example is reported to show a step-by-step execution of Algorithm 1.

Example 1

Let $\mathcal{X} = \left\{ \begin{bmatrix}-8\\-1\end{bmatrix}, \begin{bmatrix}7\\6\end{bmatrix}, \begin{bmatrix}-1\\8\end{bmatrix}, \begin{bmatrix}-7\\5\end{bmatrix}, \begin{bmatrix}8\\-3\end{bmatrix}, \begin{bmatrix}-4\\3\end{bmatrix}, \begin{bmatrix}3\\-6\end{bmatrix} \right\}$ *and let* $c^{(0)} = [-10, -10]'$. *By applying the procedure described in Algorithm 1 it results, after 6 iterations,* $c^* = [0.5, 1]'$ *and* $r^* = 11.5$. *The candidate center at each iteration is shown in Figure 2. The grey box corresponds to* $\widehat{\mathcal{D}}^{(2)} = c^{(2)} + \mathcal{D}^{(2)}$ *and the vector* $\tilde{d}^{(2)}$ *is an optimal Chebyshev step for* $c^{(2)}$ *on* $\mathcal{D}^{(2)}$*, computed by solving (15). Then,* $c^{(3)} = c^{(2)} + \tilde{d}^{(2)}$ *and the algorithm steps to the next iteration. Numerical values for candidate center, optimal Chebyshev step and corresponding radius are reported in Table 1.*

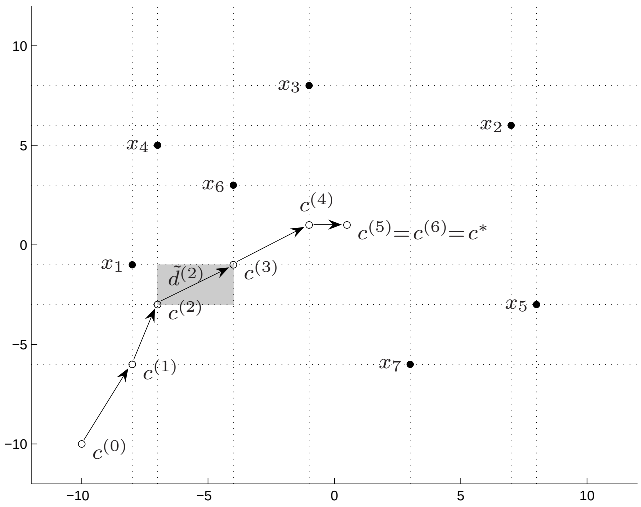


Fig. 2. Elements of $\mathcal{X}$ (points) and candidate centers $c^{(k)}$ (circles) in Example 1.

Table 1
Candidate center, optimal Chebyshev step and corresponding radius for each iteration of Algorithm 1.

k	$c^{(k)}$	$\tilde{d}^{(k)}$	$\tilde{\gamma}^{(k)}$
0	[-10, -10]'	[2, 4]'	33.0
1	[-8, -6]'	[1, 3]'	27.0
2	[-7, -3]'	[3, 2]'	23.0
3	[-4, -1]'	[3, 2]'	18.0
4	[-1, 1]'	[1.5, 0]'	13.0
5	[0.5, 1]'	[0, 0]'	11.5
6	[0.5, 1]'	-	11.5

Let us now address the ℓ_1 conditional Chebyshev center problem (7)-(8). In particular, let us consider the case in which the center is constrained to lie on a linear subspace, i.e.

$$\mathcal{C} = \{c \in \mathbb{R}^m : \; c = L\theta; \;\; \theta \in \mathbb{R}^p\} \tag{19}$$

where $L \in \mathbb{R}^{m\times p}$ is a given matrix.

It is easy to see that problem (7)-(8) can be solved by adopting the same approach outlined above for problem (5)-(6). In particular, since (19) is a linear constraint, the LP (15) in Lemma 1 can be replaced by

$$\begin{cases} \min \gamma \\ s.t.: \\ \sum_{i\in I_j^-} d_i + \sum_{i\in I_j^+}(-d_i) + \sum_{i\in I_j^0} q_i - \gamma \le -\|t_j\|_1 \;,\; j=1,\dots,n \\ d - L\,\theta = 0 \\ -q_i \le d_i \le q_i \;,\; \underline{d}_i \le d_i \le \overline{d}_i \;,\; i=1,\dots,m. \end{cases} \tag{20}$$

Then, the conditional center problem can be solved by using Algorithm 1, provided that the function `solveLP` in step 5 is changed accordingly, and the initial condition is chosen inside the set $\mathcal{C}$ (i.e., step 1 becomes: Choose $\mathtt{c} \in \mathcal{C}$).

4 Computational issues

Algorithm 1 introduced in Section 3 proceeds by successively exploring regions (hyperboxes) $\widehat{\mathcal{D}}^{(k)}$ of the space $\mathbb{R}^m$ and computing the Chebyshev center of $\mathcal{X}$ restricted to such regions. In general, although $\widehat{\mathcal{D}}^{(0)} \cup \widehat{\mathcal{D}}^{(1)} \cup \ldots \cup \widehat{\mathcal{D}}^{(K)}$ will usually cover only a small portion of the space $\mathbb{R}^m$, the number of regions to be explored can be in principle very high, thus requiring long time for the algorithm to reach the solution.

To overcome such issue, in this section a modified version of Algorithm 1 is introduced. It is based on a heuristic which is expected to reduce the computational burden when the cardinality of the set $\mathcal{X}$ is high. The modified algorithm is presented for the unconditional Chebyshev center problem; its extension to the conditional case is straightforward. To describe the new algorithm, it is firstly reported in a pseudo-programming code.

Algorithm 2

1. *Choose* $\mathtt{w}_0 \in \mathcal{X}$
2. $\mathtt{w}_1 = \arg\max_{\mathtt{x}\in\mathcal{X}} \|\mathtt{x}-\mathtt{w}_0\|_1$; $\mathtt{w}_2 = \arg\max_{\mathtt{x}\in\mathcal{X}} \|\mathtt{x}-\mathtt{w}_1\|_1$
3. $\mathcal{W} = \{\mathtt{w}_1, \mathtt{w}_2\}$; $\mathtt{c} = (\mathtt{w}_1 + \mathtt{w}_2)/2$
4. $\gamma_{\mathtt{old}} = \|\mathtt{w}_1 - \mathtt{c}\|_1$
5. $\mathtt{exit} = 0$
6. *while* $(\mathtt{exit} == 0)$
7. $\quad \widehat{\mathtt{w}} = |\mathcal{W}|$
8. $\quad \mathcal{W} = \mathcal{W} \cup \{\bar{\mathtt{x}} \in \mathcal{X}:\ \bar{\mathtt{x}} = \max_{\mathtt{x}\in\mathcal{X}} \|\mathtt{x}-\mathtt{c}\|_1\}$
9. $\quad [\tilde{\gamma}, \tilde{\mathtt{d}}] = \mathtt{solveLP}(\mathcal{W}, \mathtt{c})$
10. $\quad$ *if* $((\tilde{\gamma} == \gamma_{\mathtt{old}})$ *and* $(\widehat{\mathtt{w}} == |\mathcal{W}|))$ *then* $\mathtt{exit} = 1$
11. $\quad$ *else* $\gamma_{\mathtt{old}} = \tilde{\gamma}$; $\mathtt{c} = \mathtt{c} + \tilde{\mathtt{d}}$; *end if*
12. *end while*
13. $\mathtt{c}^* = \mathtt{c}$; $\mathtt{r}^* = \tilde{\gamma}$

Steps 1-3 initialize the algorithm with a "smart" candidate center. At each iteration, Algorithm 2 is similar to Algorithm 1, with the exception that the function `solveLP` is computed on the set $\mathcal{W}$ rather than on the whole set $\mathcal{X}$. According to step 8, the set $\mathcal{W}$ is augmented at each iteration by including the farthest points of $\mathcal{X}$ with respect to the current candidate center c. Convergence in a finite number of steps is still guaranteed, as shown by the next result.

Theorem 2 *For any set $\mathcal{X} = \{x_1, \ldots, x_n\}$, Algorithm 2 converges to an ℓ_1 Chebyshev center of $\mathcal{X}$ in a finite number of steps.*

Proof. Let us assume that, on step 10, at a given iteration, $\tilde{\gamma} = \gamma_{old}$ and $\widehat{w} = |\mathcal{W}|$, i.e., the candidate radius γ and the set $\mathcal{W}$ are the same as in the previous iteration. This means that there does not exist any improving direction for the current candidate center c with respect to the set $\mathcal{W}$, i.e. c is the Chebyshev center of $\mathcal{W}$. Since by construction $\mathcal{W} \subseteq \mathcal{X}$ and $\mathcal{W} \supseteq \{\bar{x} \in \mathcal{X}:\ \bar{x} = \arg\max_{x\in\mathcal{X}} \|x-c\|_1\}$, it follows that $\max_{x\in\mathcal{X}} \|c-x\|_1 = \max_{x\in\mathcal{W}} \|c-x\|_1 \leq r^*$, where the inequality comes from the fact that c is a center of $\mathcal{W}$ and $\mathcal{W} \subseteq \mathcal{X}$. Therefore c is also a center of $\mathcal{X}$.
It remains to prove that the algorithm stops in a finite number of steps. At any iteration, one has by construction $\mathcal{W}^{(k)} \subseteq \mathcal{W}^{(k+1)}$. If at a given iteration $\mathcal{W} = \mathcal{X}$, then Theorem 1 applies and the algorithm converges in a finite number of steps. Conversely, let us assume that there does not exist any iteration for which $\mathcal{W} = \mathcal{X}$, and let us denote by $\overline{\mathcal{W}} \subset \mathcal{X}$ the set such that $\overline{\mathcal{W}} \supseteq \mathcal{W}^{(k)}$ for any k. Then, Theorem 1 applies to the set $\overline{\mathcal{W}}$, and the algorithm converges in a finite number of steps to a Chebyshev center of $\overline{\mathcal{W}}$, which is also, according to the first part of the proof, a Chebyshev center of $\mathcal{X}$. □

Remark 3 *The main difference between Algorithm 1 and Algorithm 2 is that the latter solves LP problems on the sets $\mathcal{W}$, whose cardinality is generally much smaller than the cardinality of $\mathcal{X}$, with an obvious improvement in terms of computational time.*

In order to reduce the computational burden required by the `solveLP` function, instead of solving (15), we solve the following LP which has a simpler constraint structure:

$$\begin{cases} \min \gamma \\ s.t.: \\ \sum_{i\in I_j^-}(d_i^+ - d_i^-) + \sum_{i\in I_j^+}(d_i^- - d_i^+) \\ \qquad + \sum_{i\in I_j^0}(d_i^+ + d_i^-) - \gamma \leq -\|t_j\|_1\ ,\ j=1,\ldots,n \\ 0 \leq d_i^+ \leq \overline{d}_i\ ,\ 0 \leq d_i^- \leq -\underline{d}_i\ ,\ i=1,\ldots,m. \end{cases} \tag{21}$$

Proposition 1 *Let $[\tilde{\gamma}, \tilde{d}^+, \tilde{d}^-]$ be a solution of (21), and let $\tilde{d} \triangleq \tilde{d}^+ - \tilde{d}^-$ and $\tilde{q} \triangleq \tilde{d}^+ + \tilde{d}^-$. Then, $[\tilde{\gamma}, \tilde{d}, \tilde{q}]$ is a solution of (15).*

Proof. Let us rewrite (21) by substituting $d = d^+ - d^-$ and $q = d^+ + d^-$. One obtains

$$\begin{cases} \min \gamma \\ s.t.: \\ \displaystyle\sum_{i\in I_j^-} d_i + \sum_{i\in I_j^+}(-d_i) + \sum_{i\in I_j^0} q_i - \gamma \leq -\|t_j\|_1\ ,\ j=1,\ldots,n \\ 0 \leq \frac{q_i+d_i}{2} \leq \overline{d}_i\ ,\ i=1,\ldots,m \\ 0 \leq \frac{q_i-d_i}{2} \leq -\underline{d}_i\ ,\ i=1,\ldots,m \end{cases} \tag{22}$$

which can be easily rearranged into

$$\begin{cases} \min \gamma \\ s.t.: \\ \displaystyle\sum_{i\in I_j^-} d_i + \sum_{i\in I_j^+}(-d_i) + \sum_{i\in I_j^0} q_i - \gamma \leq -\|t_j\|_1\ ,\ j=1,\ldots,n \\ -q_i \leq d_i \leq q_i\ ,\ i=1,\ldots,m \\ -2\,\underline{d}_i + q_i \leq d_i \leq 2\,\overline{d}_i - q_i\ ,\ i=1,\ldots,m. \end{cases} \tag{23}$$

Since the constraints $-q_i \leq d_i \leq q_i$ are equivalent to $q_i \geq |d_i|$, it follows that the last constraints in (23) are more restrictive than the constraints $\underline{d}_i \leq d_i \leq \overline{d}_i$ in (15). This means that any solution $[\tilde{\gamma}, \tilde{d}, \tilde{q}]$ of (23) satisfies the constraints in (15). Now, we prove that $[\tilde{\gamma}, \tilde{d}, \tilde{q}]$ is also a solution of (15). By contradiction, let us assume that $[\hat{\gamma}, \hat{d}, \hat{q}]$ is a solution of (15) such that $\hat{\gamma} < \tilde{\gamma}$ and such that $\hat{q}_i = |\hat{d}_i|$ for $i = 1, \ldots, m$ (notice that it can be easily checked by Lemma 1 that such a solution always exists). By substituting $[\hat{\gamma}, \hat{d}, \hat{q}]$ into (23) one has that $[\hat{\gamma}, \hat{d}, \hat{q}]$ is a feasible solution thus obtaining a contradiction. $\square$

In Table 2, the number of iterations and the computation time of Algorithms 1 and 2 are reported for randomly generated sets $\mathcal{X}$ with different m and n. Moreover, the maximum cardinality of the set $\mathcal{W}$ generated by Algorithm 2 is shown, as well as the cardinality of the so-called *core-set* of $\mathcal{X}$, defined as $\mathcal{Q} = \{x \in \mathcal{X} : \|x - c^*\|_1 = r^*\}$. The computation time of (9) is also reported (denoted by "1 LP"); "OUT" means "out of memory".

Computations have been performed under Matlab by using *CPLEX* [7,1] to solve the LPs, on a Pentium 4 at 3.20 GHz with 2 GB of RAM. Time is expressed in seconds. The results demonstrate that both algorithms are computationally feasible for standard solvers and hardware, and that the heuristic adopted in Algorithm 2 is effective when the cardinality n of the set $\mathcal{X}$ grows.

It can be noticed that the cardinality of $\overline{\mathcal{W}}$ is typically slightly greater than the cardinality of $\mathcal{Q}$. This means that the total number of LPs solved by Algorithm 2 is of the same order as the cardinality of $\mathcal{Q}$, which is in general much less than n.

5 Illustrative example

In order to show a typical behavior of the proposed algorithm for constructing ℓ_1 uncertainty model sets, let us consider a set of models $\mathcal{H} = \{H_1, \ldots, H_n\}$, where $H_j = H_0 + \Delta_j$, $j = 1, \ldots, n$. In this example, let $H_0 = L_1\theta_1 + L_2\theta_2$, where L_1 and L_2 are the first two filters of the Laguerre basis with pole equal to 0.9, and $\theta = [1 \quad -0.6]'$. Then, let $n = 20$ and

$$\Delta_j(z) = \alpha \frac{1 - |\lambda_j|}{z - \lambda_j}, \quad j = 1, \ldots, 20,$$

where the poles λ_j have been randomly chosen in the interval $[-0.95, 0.95]$. By construction, $\|\Delta_j(z)\|_1 = \alpha$, $\forall j$ (in the following, we set $\alpha = 2$).

Table 2
Number of iterations, cardinality of the set $\overline{\mathcal{W}}$ and $\mathcal{Q}$, and computation time of Algorithms 1, 2 and (9).

		Algorithm 1		Algorithm 2				1 LP
n	m	# iter.	time	# iter.	$\lvert\overline{\mathcal{W}}\rvert$	$\lvert\mathcal{Q}\rvert$	time	time
10	10	8	0.02	4	5	3	0.02	0.02
10	100	6	0.04	9	10	10	0.06	0.52
10	500	6	0.17	9	10	10	0.21	OUT
10	1000	6	0.46	9	10	10	0.64	OUT
50	10	26	0.05	5	6	6	0.03	0.13
50	100	24	0.40	28	27	25	0.18	OUT
50	500	17	1.94	38	39	39	2.30	OUT
50	1000	13	4.95	46	47	47	9.59	OUT
100	10	46	0.11	13	14	4	0.03	0.58
100	100	39	1.05	34	35	33	0.29	OUT
100	500	29	8.21	60	61	61	6.41	OUT
100	1000	31	33.69	71	72	71	27.82	OUT
500	10	31	0.23	10	11	6	0.03	OUT
500	100	191	17.21	42	42	37	0.52	OUT
500	500	154	143.58	78	79	79	13.64	OUT
500	1000	126	529.34	111	112	112	112.02	OUT
1000	10	322	5.13	16	17	6	0.05	OUT
1000	100	307	67.97	47	48	47	0.90	OUT
1000	500	291	455.73	94	95	93	23.27	OUT
1000	1000	199	1173.88	139	140	138	230.93	OUT

First, let us consider the nominal model class of FIR filters of order $N = 100$. The aim is to compute the minimum ℓ_1 uncertainty model set containing the set $\mathcal{H}$, i.e. to solve the problem $g^* = \arg \inf_{g \in FIR_{100}} \sup_{h \in \mathcal{H}} \|g - h\|_1$. According to the treatment in Section 2, this boils down to solve the conditional Chebyshev center problem (4) with $M = I_{100}$ and $\theta \in \mathbb{R}^{100}$. By applying the conditional version of Algorithm 2, one obtains the impulse response g^* such that the corresponding ℓ_1 uncertainty bound is $\gamma^* = 1.68$. In Figure 3, g^* along with the 20 impulse responses h_j are depicted.

Now, let us now consider as nominal model class the truncated Laguerre expansion of order 2. Let us denote by $\mathcal{M}_a$ the Laguerre basis of order 2 with pole a. The problem to be solved is $g^* = \arg \inf_{g \in \mathcal{M}_a} \sup_{h \in \mathcal{H}} \|g - h\|_1$. Let us first choose the same Laguerre pole of H_0, i.e. $a = 0.9$. By applying the conditional version of Algorithm 2, one obtains a g^* such that $g^* = M\,\theta^*$ with $\theta^* = [1 \quad -0.6]'$ and $\gamma^* = 2$, as expected. Let us now pick $a = 0.85$. In this case, one gets $\theta^* = [1.21 \quad -0.70]'$ and $\gamma^* = 2.92$, where the increase of the ℓ_1 uncertainty bound is due to the different choice of the Laguerre pole. The impulse responses g^* obtained for $a = 0.9$ and $a = 0.85$ are reported in Figure 4, together with those of the models $H_j \in \mathcal{H}$.

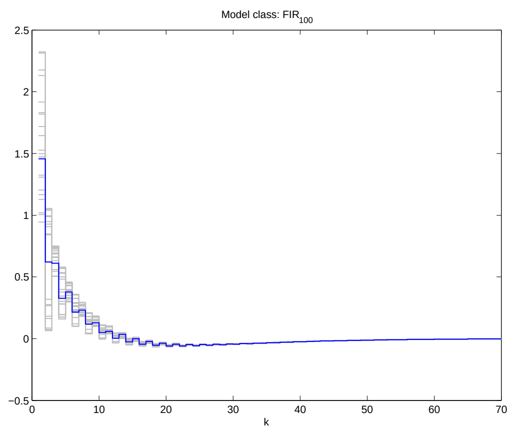


Fig. 3. Impulse response g^* for the model class FIR_{100} (dark), and impulse responses of models H_j, $j = 1, \ldots, 20$ (light).

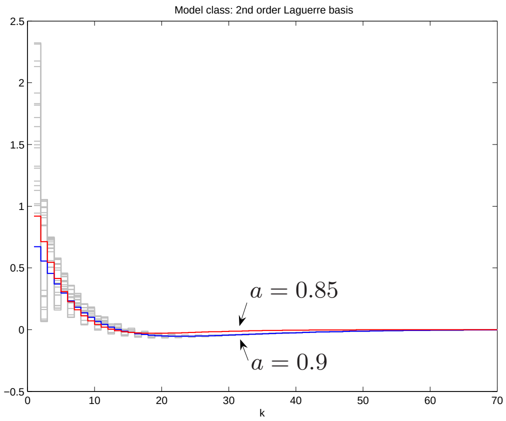


Fig. 4. Impulse response g^* for the Laguerre model class with two choices of the pole a, and impulse responses of models H_j, $j = 1, \ldots, 20$ (light).

6 Conclusions

An efficient algorithm for the computation of the ℓ_1 conditional Chebyshev center has been presented. The algorithm can be employed to construct the minimum ℓ_1 uncertainty model set containing a given set of impulse responses. From a computational point of view, it boils down to a sequence of small LP problems. The number of LPs to be solved can be very high in principle, but in practice the algorithm is computationally feasible for large size problems.

The ongoing research concerns two main directions. First, it would be interesting to analyze the computational complexity of the algorithm, for example by establishing upper bounds to the number of steps it requires to converge. On the other side, construction of model sets for different structures of uncertainty could be useful for wider application of ℓ_1 control design techniques.

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