On Input Design in ℓ_∞ Conditional Set Membership Identification ⋆

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Abstract

This paper deals with input design in conditional set membership identification. The problem is how to choose the input signal in order to minimize the global worst-case identification error. A characterization of the ℓ_∞ identification error is provided, showing that the optimal input is the one that minimizes the ℓ_∞ radius of a suitable set. Moreover, sufficient conditions under which the impulse input is optimal are provided.

Key words: Set membership identification; input design; reduced-complexity models.

1 Introduction

Conditional set membership identification [8,3,6,2] is a line of research that falls into the broader area of robust identification. The bottom line of this research is that the system to be identified belongs to a set which lies in a complex infinite-dimensional space and only approximate information (data, a priori knowledge, etc.) is available on it. The model class lives in a finitely parameterized linear space (see e.g. [16,17,14]) and its parameters need to be estimated according to a criterion that accounts for the worst-case modelling error in some suitable norm, with respect to the set of all feasible systems.
This paper addresses the problem of optimal input design in the context of conditional set membership identification. While the literature on robust identification has grown considerably in recent years, relatively few papers have considered the problem of input selection. Most contributions on input design in a worst-case setting concern ℓ_1 identification. Classes of binary sequences were shown to minimize the ℓ_1 worst-case error in [10,13]. In [7], it was proven that the unitary impulse is optimal under certain conditions. The evaluation of the minimum sample complexity in a worst-case identification setting has been also widely studied [12,5,4]. A general information-based approach to worst-case system identification has been introduced by Zames and coworkers [18]. In particular, they have shown that the identification error can be split in two parts, namely the inherent error and the representation error, and that only the first one is affected by the choice of the input signal [9]. Optimal input design in a different deterministic setting, not relying on the set membership paradigm, has been addressed in [15].
In this paper, the input design problem is formulated in the conditional set membership identification setting. The considered model class consists in the linear combination of truncated impulse response sequences. Under mild a priori assumptions on the system generating the data, it turns out that the optimal input design problem can be cast as a double min-max optimization in a finite dimensional space. The first contribution of this paper is to show that the ℓ_∞ identification error is given by the larger of two quantities: the representation error, which does not depend on the input signal, and the ℓ_∞ radius of a suitable set, depending on a the priori information and on the input signal. A consequence of this characterization is that in general the impulse is not the optimal input signal, in the sense of minimizing the worst-case ℓ_∞ identification error. Motivated by this, sufficient conditions under which an impulsive input is optimal are provided.
The paper is organized as follows. Section 2 introduces the main features of conditional set membership identification. In Section 3 the optimal input design problem is formulated and some useful bounds on the worst-case identification error are derived. In Section 4 a character-

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ization of the ℓ_∞ identification error is reported, while in Section 5 the impulse input is analyzed and optimality results are given. Section 6 provides some concluding remarks.

2 Conditional set membership identification

In set membership identification, the uncertainty associated to an identified model is usually measured according to the worst-case error with respect to a set of admissible systems. This set, called *feasible system set*, accounts for two main information sources: i) a priori knowledge on the plant; ii) input/output measurements. The former may be represented by a set to which the true system is assumed a priori to belong; the latter is typically a finite record of noisy data, and assumptions are made on the nature of the noise.

In this paper, LTI discrete-time SISO systems are considered. The impulse response of a system is denoted by $h = \{h_i\}_{i=0}^{\infty}$, and belongs to a linear normed space $\mathcal{H}$, equipped with the norm $\|\cdot\|_{\mathcal{H}}$. A priori knowledge on such system is expressed as $h \in \mathcal{S}$, where $\mathcal{S}$ is a set contained in $\mathcal{H}$. Let $\bar{h} \in \mathcal{S}$ be the true system generating the data. Data consists of a set of N input/output pairs $z = \{(u_k, y_k),\ k = 0, \dots, N-1\}$, related by

$$y_k = \sum_{i=0}^{k} \bar{h}_i u_{k-i} + e_k \tag{1}$$

where e_k is the disturbance affecting the measurement y_k. Equation (1) can be written in compact form as

$$y = T(u)\bar{h}^N + e \tag{2}$$

where $y = [y_0 \dots y_{N-1}]'$, $u = [u_0, \dots, u_{N-1}]'$, $\bar{h}^N = [\bar{h}_0 \dots \bar{h}_{N-1}]'$, $e = [e_0 \dots e_{N-1}]'$ and $T(u)$ is the lower triangular Toeplitz matrix of inputs, i.e.

$$T(u) = \begin{bmatrix} u_0 & 0 & \dots\dots & 0 \\ u_1 & u_0 & 0 \ \dots & \vdots \\ u_2 & u_1 & u_0 \ \ddots & \vdots \\ \vdots & \ddots & \ddots \ \ddots & 0 \\ u_{N-1} & \dots & u_2 \ u_1 & u_0 \end{bmatrix}. \tag{3}$$

It is assumed that e is unknown-but-bounded, i.e.

$$\|e\|_Y \leq \varepsilon \tag{4}$$

where $\|\cdot\|_Y$ denotes a suitable norm in $\mathbb{R}^N$ and ε is a known positive scalar. According to (2) and (4), the feasible system set is defined as

$$\mathcal{F} = \{h \in \mathcal{S} :\ \|y - T(u)h^N\|_Y \leq \varepsilon\}. \tag{5}$$

Following the terminology of the Information-Based Complexity (IBC) theory (see [11] for a thorough treatment), an *identification algorithm* ϕ is a mapping from the measurement space to the model space. When the latter coincides with the space of systems $\mathcal{H}$, the algorithm ϕ is such that: $\phi : \mathbb{R}^N \to \mathcal{H}$. The *worst-case identification error* associated to the identification algorithm turns out to be

$$E[\phi] = \sup_{h\in\mathcal{F}} \|h - \phi(y)\|_{\mathcal{H}}.$$

In robust identification, it is customary to identify reduced-complexity models that are suitable for robust control design techniques. A typical choice is to select a linearly parameterized model class such as

$$\mathcal{M} = \{g \in \mathcal{H} :\ g = M\theta,\ \theta \in \mathbb{R}^n\}$$

where M is a linear operator, $M : \mathbb{R}^n \to \mathcal{H}$ and θ is the n-dimensional parameter vector to be identified, $n < N$. A *conditional identification algorithm* is defined as a mapping $\phi : \mathbb{R}^N \to \mathcal{M}$. The model class $\mathcal{M}$ can be chosen as a collection of impulse responses of linear filters, such as Laguerre or Kautz functions [16,17], or Generalized Orthonormal Basis Functions [14].

The identification of a model within the class $\mathcal{M}$ has been addressed in the literature as *conditional* set membership identification [8,3,6,2,1]. In particular, the optimal model is given by the so-called *conditional central algorithm*, which minimizes the worst-case error among the elements of $\mathcal{M}$, i.e.

$$\phi_{cc}(y) \ = \ \arg \inf_{g\in\mathcal{M}} \sup_{h\in\mathcal{F}} \|h - g\|_{\mathcal{H}}. \tag{6}$$

A procedure for computing $\phi_{cc}(y)$ has been given in [2] for the case $\|\cdot\|_{\mathcal{H}} = \|\cdot\|_Y = \ell_2$, while suboptimal identification algorithms have been considered in [1].

3 Optimal input design

3.1 Problem formulation

Let T_N denote the truncation operator in $\mathcal{H}$, such that $T_N h = h^N = \{h_i\}_{i=0}^{N-1}$, and R_N be the remainder operator $R_N h = \{h_i\}_{i=N}^{\infty}$. In the following, it is assumed that the model class $\mathcal{M}$ is fixed, and inputs are bounded in ℓ_∞ norm (this is a common constraint essentially due to input saturations):

$$\|u\|_\infty \leq \delta. \tag{7}$$

Moreover, w.l.o.g. we assume $u_0 \neq 0$.

Let us define the following quantities:

- $\overline{\mathbb{U}} = \{U : U = T(u), u \in \mathbb{R}^N, \|u\|_\infty \leq \delta\}$ is the set containing all feasible input matrices;
- $\mathcal{B}_Y = \{e \in \mathbb{R}^N : \|e\|_Y \leq 1\}$ is the unit ball in the Y-norm.

Let u be fixed and $U = T(u)$. By definition of feasible set (5), it follows that

$$\begin{aligned} \mathcal{F} &= \{h \in \mathcal{S} : \|Uh^N - y\|_Y \leq \varepsilon\} \\ &= \{h \in \mathcal{S} : h^N = U^{-1}y + \varepsilon U^{-1}\alpha,\ \alpha \in \mathcal{B}_Y\}. \end{aligned} \tag{8}$$

Let us define the set

$$\mathcal{V}(c) = \{h : h^N = c + \varepsilon U^{-1}\mathcal{B}_Y\} \tag{9}$$

where $c = U^{-1}y = U^{-1}(U\bar{h}^N + e) = \bar{h}^N + U^{-1}e$. Notice that c is the symmetry center of $T_N\mathcal{V}(c)$.

Taking into account the prior information $\mathcal{S}$ and the bound (4) (concerning h and e respectively), one can introduce the set of admissible centers of $T_N\mathcal{V}(c)$ as $\mathcal{C} = T_N\mathcal{S} + \varepsilon U^{-1}\mathcal{B}_Y \subset \mathbb{R}^N$. Then, the feasible set (8) turns out to be $\mathcal{F} = \mathcal{V}(c) \cap \mathcal{S}$ with $c \in \mathcal{C}$.

The problem addressed in this paper is that of finding the optimal input $\overline{U}^*$ such that the worst-case error (with respect to both a priori information and noise bound) associated to the optimal model, i.e. the model provided by the conditional central algorithm (6), is minimized. In other words, one has to compute

$$\overline{U}^*(\overline{\mathbb{U}}, \varepsilon, \mathcal{S}, \mathcal{M}) = \arg \inf_{U \in \overline{\mathbb{U}}} E^*(U, \varepsilon, \mathcal{S}, \mathcal{M}) \tag{10}$$

where

$$\begin{aligned} E^*(U, \varepsilon, \mathcal{S}, \mathcal{M}) &= \sup_{\bar{h}^N \in T_N\mathcal{S}, e \in \varepsilon\mathcal{B}_Y} \inf_{g \in \mathcal{M}} \sup_{h \in \mathcal{V}(\bar{h}^N + U^{-1}e) \cap \mathcal{S}} \|h - g\|_{\mathcal{H}} \\ &= \sup_{c \in \mathcal{C}} \inf_{g \in \mathcal{M}} \sup_{h \in \mathcal{V}(c) \cap \mathcal{S}} \|h - g\|_{\mathcal{H}}. \end{aligned} \tag{11}$$

Note that E^* is the error of the conditional central algorithm for the worst possible system in $\mathcal{S}$ and noise realization in $\varepsilon\mathcal{B}_Y$.

3.2 Error bounds

In the Information-Based Complexity framework [11], E^* is usually indicated as the global error. Notice that this is the same worst-case error considered in [9]. In that paper, however, E^* is not minimized directly, but it is first separated into the *inherent error*

$$\delta(u) = \sup_{h \in \mathcal{S} : Uh^N \in \varepsilon\mathcal{B}_Y} \|h\|_{\mathcal{H}}$$

and the *representation error*

$$d = \sup_{h \in \mathcal{S}} \inf_{g \in \mathcal{M}} \|h - g\|_{\mathcal{H}}. \tag{12}$$

In [9] it was shown that

$$\max\{\delta(u), d\} \leq E^* \leq 3\max\{\delta(u), d\}$$

and the input selection was performed by minimizing $\delta(u)$. This is a suboptimal input design strategy, because minimizing $\delta(u)$ does not guarantee that E^* is minimized. In this paper, direct minimization of E^* with respect to u is considered.

The next lemma shows that the solution of (10) is achieved by an input signal such that $\|u\|_\infty = \delta$ (i.e. the upper bound in (7) is achieved).

Lemma 1 *Let $\overline{U} = T(\bar{u})$ be an optimal solution of (10) such that $\|\bar{u}\|_\infty < \delta$. Then $U = T(u)$, with $u = \frac{\bar{u}}{\|\bar{u}\|_\infty}\delta$ is an optimal solution of (10).*

Proof. Let $\alpha = \|\bar{u}\|_\infty < \delta$. Since $U = \frac{\delta}{\alpha}\overline{U}$, one has

$$c + \varepsilon U^{-1}\mathcal{B}_Y = c + \varepsilon\frac{\alpha}{\delta}\overline{U}^{-1}\mathcal{B}_Y \subset c + \varepsilon\overline{U}^{-1}\mathcal{B}_Y.$$

Therefore, $E^*(U, \varepsilon, \mathcal{S}, \mathcal{M}) \leq E^*(\overline{U}, \varepsilon, \mathcal{S}, \mathcal{M})$ and hence also U is a solution of (10). □

Without loss of generality, it is possible to assume $\varepsilon = 1$; in fact this is equivalent to scaling all signals by a factor ε and set, according to Lemma 1

$$\|u\|_\infty = \eta = \frac{\delta}{\varepsilon}. \tag{13}$$

The set of admissible input matrices becomes

$$\mathbb{U} = \{U : U = T(u),\ u \in \mathbb{R}^N,\ \|u\|_\infty = \eta\}.$$

Now, equations (10) and (11) can be rewritten as

$$U^*(\mathbb{U}, \mathcal{S}, \mathcal{M}) = \arg \inf_{U \in \mathbb{U}} E^*(U, \mathcal{S}, \mathcal{M}) \tag{14}$$

and

$$E^*(U, \mathcal{S}, \mathcal{M}) = \sup_{c \in \mathcal{C}} \inf_{g \in \mathcal{M}} \sup_{h \in \mathcal{V}(c) \cap \mathcal{S}} \|h - g\|_{\mathcal{H}} \tag{15}$$

where $\mathcal{V}(c) = \{h : h^N = c + U^{-1}\mathcal{B}_Y\}$ and $\mathcal{C} = T_N\mathcal{S} + U^{-1}\mathcal{B}_Y$.

Remark 1 *From the above scaling, it follows that if U^* is a solution of (14)-(15), the optimal input solving (10)-(11) is given by $u^* = U^*[\varepsilon, 0, \ldots, 0]'$.*

Let $\mathcal{S}$ and $\mathcal{M}$ be fixed. The next proposition gives upper and lower bounds to the worst-case identification error E^* in (15).

Proposition 1 *Define*

$$\overline{E} = \inf_{g \in \mathcal{M}} \sup_{h \in \mathcal{S}} \|h - g\|_{\mathcal{H}} \tag{16}$$

and

$$\underline{E} = \sup_{h \in \mathcal{S}} \inf_{g \in \mathcal{M}} \|h - g\|_{\mathcal{H}}. \tag{17}$$

Then,

$$\underline{E} \leq E^*(U, \mathcal{S}, \mathcal{M}) \leq \overline{E} \tag{18}$$

Proof. If $|u_0| \to \infty$ the set $T_N\mathcal{V}(c)$ collapses into a singleton (its center c). Then, this is the condition for which the feasible set is minimized (it cannot be empty if a priori assumptions are correct and data are consistent). Moreover, the set of feasible centers $\mathcal{C}$ tends to be equal to $T_N\mathcal{S}$. Therefore, one has

$$\begin{aligned} \lim_{|u_0| \to \infty} E^*(U, \mathcal{S}, \mathcal{M}) &= \sup_{c \in T_N\mathcal{S}} \inf_{g \in \mathcal{M}} \sup_{h \in \mathcal{S} : h^N = c} \|h - g\|_{\mathcal{H}} \\ &\geq \sup_{h \in \mathcal{S}} \inf_{g \in \mathcal{M}} \|g - h\|_{\mathcal{H}} = \underline{E}. \end{aligned}$$

If $\|u\| \to 0$ (or more generally $\mathcal{V}(c) \cap \mathcal{S} \equiv \mathcal{S}$), then the feasible set is maximized and one has

$$\lim_{\|u\| \to 0} E^*(U, \mathcal{S}, \mathcal{M}) = \sup_{c \in \mathcal{C}} \inf_{g \in \mathcal{M}} \sup_{h \in \mathcal{S}} \|h - g\|_{\mathcal{H}}$$

$$= \inf_{g\in\mathcal{M}} \sup_{h\in\mathcal{S}} \|h-g\|_{\mathcal{H}} = \overline{E}.$$

$\square$

Remark 2 *Note that $\underline{E}$ coincides with the* representation error *(12) (introduced in [9]), while $\overline{E}$ is the conditional radius of the set $\mathcal{S}$ with respect to the subspace $\mathcal{M}$ (see [2]).*

Problem (14)-(15) is a double min-max optimization in an infinite dimensional space $\mathcal{H}$. However, it is useful to notice that in the conditional set membership identification setting of Section 2, several problems of interest can be restricted to the finite dimensional space $\mathbb{R}^N$ (see e.g. [2], for the ℓ_2 case).

In the following, we will address the optimal input design problem in the ℓ_∞ norm, and we will consider a truncated model class $\mathcal{M}^N = T^N\mathcal{M}$, i.e. we will assume that the selected model class is a linear combination of truncated impulse sequences of length N. Under mild a priori assumptions on the system $\bar{h}$, it is easy to see that one can as well consider a truncated a priori set $\mathcal{S}^N = T^N\mathcal{S}$. Indeed, let $\mathcal{S} = \{s_i\}_{i=0}^{\infty} \subset \mathcal{H}$, with $\|\cdot\|_{\mathcal{H}} = \ell_\infty$, $|s_i| \le \gamma_i$, $\forall i$. Let us write $\mathcal{S} \triangleq \begin{bmatrix} \mathcal{S}^N \\ \mathcal{S}^+ \end{bmatrix} = \begin{bmatrix} T_N\mathcal{S} \\ R_N\mathcal{S} \end{bmatrix}$ and $\mathcal{M} \triangleq \begin{bmatrix} \mathcal{M}^N \\ \mathcal{M}^+ \end{bmatrix} = \begin{bmatrix} T_N\mathcal{M} \\ 0 \end{bmatrix}$. Then,

$$\begin{aligned} E^*(U,\mathcal{S},\mathcal{M}^N) &= \sup_{c\in\mathcal{C}} \inf_{g=\begin{bmatrix} g^N\in\mathcal{M}^N \\ g^+\in 0\end{bmatrix}} \sup_{h=\begin{bmatrix} h^N\in T_N(\mathcal{V}(c)\cap\mathcal{S}) \\ h^+\in\mathcal{S}^+\end{bmatrix}} \|h-g\|_\infty \\ &= \max\{E_a, E_b\} \end{aligned}$$

where

$$\begin{aligned} E_a &= \sup_{c\in\mathcal{C}} \inf_{g^N\in\mathcal{M}^N} \sup_{h^N\in T_N(\mathcal{V}(c)\cap\mathcal{S})} \|h^N-g^N\|_\infty \\ &= E^*(U,\mathcal{S}^N,\mathcal{M}^N) \end{aligned}$$

$$E_b = \inf_{g^+\in 0} \sup_{h^+\in\mathcal{S}^+} \|h^+-g^+\|_\infty = \sup_{h^+\in\mathcal{S}^+} \|h^+\|_\infty = \max_{i\ge N}\gamma_i.$$

So, if $E_a \ge E_b$ one has $E^*(U,\mathcal{S},\mathcal{M}^N) = E^*(U,\mathcal{S}^N,\mathcal{M}^N)$. Note that $E_a \ge E_b$ is not a strong assumption. For example, it holds whenever $\min_{j<N}\gamma_j \ge \max_{i\ge N}\gamma_i$, which is a typical a priori assumption on the impulse response (e.g., when decaying bounds on the impulse response samples are assumed). In fact, it can be checked that $E_a \ge \sup_{h^N\in\mathcal{S}^N} \inf_{g^N\in\mathcal{M}^N} \|h^N-g^N\|_\infty \ge \min_{j<N}\gamma_j$, while $E_b = \max_{i\ge N}\gamma_i$.

In the following we will assume that $E_a \ge E_b$; hence, we will set $\mathcal{H} = \mathbb{R}^N$ and consider $\mathcal{S}$, $\mathcal{F}$ and $\mathcal{M}$ as subsets of $\mathbb{R}^N$. The main consequence is that the double min-max optimization problem (14)-(15) is now defined in a finite dimensional space. This will allow us to provide a complete characterization of the worst-case ℓ_∞ identification error in the next section.

4 Characterization of the ℓ_∞ identification error

Let $\mathcal{H} = \mathbb{R}^N$, $\|\cdot\|_{\mathcal{H}} = \ell_\infty$ and $\mathcal{M}$ be a linear subspace of $\mathbb{R}^N$. Moreover, let us make the following assumption.

Assumption 1 *The a priori set $\mathcal{S}$ is an orthotope centered at the origin, i.e.*

$$\mathcal{S} = \{h = [h_0, ..., h_{N-1}]' : |h_0| \le \gamma_0, ..., |h_{N-1}| \le \gamma_{N-1}\}. \quad (19)$$

An example of a priori information commonly addressed in the literature is that of FIRs of order N with exponential decay response as

$$\mathcal{S} = \{h: |h_i| \le M\rho^i,\ M>0,\ |\rho|<1,\ i=0,\dots,N-1\},$$

which clearly satisfies (19).

If $\mathcal{F}$ is a set, we denote by $\mathrm{BOX}(\mathcal{F})$ the minimum volume box containing $\mathcal{F}$. The ℓ_∞ *radius* of $\mathcal{F}$ is defined as

$$\mathrm{rad}_\infty(\mathcal{F}) = \inf_{g\in\mathcal{H}} \sup_{h\in\mathcal{F}} \|h-g\|_\infty.$$

Moreover, let us define the ℓ_∞ *conditional radius* of a set $\mathcal{F}$ with respect to a linear manifold $\mathcal{M}$ as

$$\mathrm{rad}_\infty^{\mathcal{M}}(\mathcal{F}) = \inf_{g\in\mathcal{M}} \sup_{h\in\mathcal{F}} \|h-g\|_\infty.$$

The following result allows one to compute the ℓ_∞ conditional radius of a generic set.

Theorem 1 *Let $\mathcal{F}$ be a set and let $\mathcal{P} = \mathrm{BOX}(\mathcal{F})$. Let $\mu = \mathrm{rad}_\infty(\mathcal{F})$ and let $\mathcal{M}$ be a linear manifold. Let us define:*

$$E^+ \triangleq \sup_{h\in\mathcal{P}} \inf_{g\in\mathcal{M}} \|h-g\|_\infty. \quad (20)$$

Then

$$\mathrm{rad}_\infty^{\mathcal{M}}(\mathcal{F}) = \max\{E^+, \mu\}.$$

Proof. See Appendix A. $\square$

The next theorem provides the characterization of the identification error (15).

Theorem 2 *Let $\rho \triangleq \mathrm{rad}_\infty(\mathcal{V}(0)\cap\mathcal{S})$. Then,*

$$E^*(U,\mathcal{S},\mathcal{M}) = \max\{\underline{E}, \rho\}. \quad (21)$$

In order to prove Theorem 2, we need the following lemma.

Lemma 2 *Let $\mathcal{P} = \{h\in\mathbb{R}^s : |h_i| \le \gamma_i,\ i=1,\dots,s\}$ and $\mathcal{W}$ be a convex and balanced set. Consider the set $\mathcal{W}(c) = c + \mathcal{W}$. Then,*

$$\beta_i(\mathrm{BOX}(\mathcal{W}(c)\cap\mathcal{P})) \le \beta_i(\mathrm{BOX}(\mathcal{W}\cap\mathcal{P})),\ i=1,\dots,s.$$

where $\beta_i(B)$ denotes the length of the i-th semi-axis of the box B.

Proof. Let i, $1 \le i \le s$, be fixed and select $w, z \in \mathcal{W}$, such that $w+c, z+c \in \mathcal{W}(c) \cap \mathcal{P}$ and

$$\begin{aligned}\beta_i(\mathrm{BOX}(\mathcal{W}(c)\cap\mathcal{P})) &= \frac{1}{2}|(w_i+c_i)-(z_i+c_i)| \\ &= \frac{1}{2}|w_i - z_i|. \end{aligned} \tag{22}$$

Since $w+c, z+c \in \mathcal{P}$ one has

$$|w_j - z_j| = |(w_j+c_j)-(z_j+c_j)| \le 2\,\gamma_j \;, \forall j = 1,\ldots,s. \tag{23}$$

Let $p = \frac{1}{2}(w-z)$ and $q = \frac{1}{2}(z-w) = -p$. Since $\mathcal{W}$ is convex and symmetric, it follows that since $w, z \in \mathcal{W}$, also $p, q \in \mathcal{W}$. Moreover, due to (23)

$$|p_j| = |q_j| = \frac{1}{2}|w_j - z_j| \le \gamma_j \;\;, \forall j = 1,\ldots,s$$

which means that $p, q \in \mathcal{P}$. Therefore, $p, q \in \mathcal{W}\cap\mathcal{P}$ and, using (22)

$$\begin{aligned}\beta_i(\mathrm{BOX}(\mathcal{W}\cap\mathcal{P})) &\ge \frac{1}{2}|p_i - q_i| \\ &= \frac{1}{2}\left|\frac{1}{2}(w_i-z_i) - \frac{1}{2}(z_i-w_i)\right| \\ &= \frac{1}{2}|w_i - z_i| = \beta_i(\mathrm{BOX}(\mathcal{W}(c)\cap\mathcal{P})).\end{aligned}$$

Since the same reasoning can be repeated for all i, $1 \le i \le s$, the lemma is proved. □

Now, Theorem 2 can be proved.

Proof of Theorem 2: Let

$$r(c) \triangleq \mathrm{rad}_\infty(\mathcal{V}(c)\cap\mathcal{S})$$

and

$$\underline{E}(c) = \sup_{h\in\mathrm{BOX}(\mathcal{V}(c)\cap\mathcal{S})} \inf_{g\in\mathcal{M}} \|h-g\|_\infty.$$

By Theorem 1 one has

$$\begin{aligned}E^*(U,\mathcal{S},\mathcal{M}) &= \sup_{c\in\mathcal{C}} \inf_{g\in\mathcal{M}} \sup_{h\in\mathcal{V}(c)\cap\mathcal{S}} \|h-g\|_\infty \\ &= \sup_{c\in\mathcal{C}} \max\{\underline{E}(c), r(c)\} \\ &= \max\left\{\sup_{c\in\mathcal{C}} \underline{E}(c), \sup_{c\in\mathcal{C}} r(c)\right\}.\end{aligned}$$

Since $\mathcal{C} \supseteq \mathcal{S}$,

$$\sup_{c\in\mathcal{C}} \sup_{h\in\mathrm{BOX}(\mathcal{V}(c)\cap\mathcal{S})} \inf_{g\in\mathcal{M}} \|h-g\|_\infty = \sup_{h\in\mathcal{S}} \inf_{g\in\mathcal{M}} \|h-g\|_\infty = \underline{E}.$$

Then, it remains to show that $\sup_{c\in\mathcal{C}} r(c) = r(0) = \rho$. The proof follows immediately from Lemma 2 by noting that $r(c) = \max_i \beta_i(\mathrm{BOX}(\mathcal{V}(c)\cap\mathcal{S}))$ and $r(0) = \max_i \beta_i(\mathrm{BOX}(\mathcal{V}(0)\cap\mathcal{S})) = \mathrm{rad}_\infty(\mathcal{V}(0)\cap\mathcal{S}) = \rho$. □

Remark 3 *If $\underline{E} \ge \rho$ then all inputs are equivalent, in the sense that they provide the same worst-case identification error E^*. So, an input signal is optimal if it minimizes ρ, or it allows ρ to be less than $\underline{E}$.*

Remark 4 *The results in this section hold for any feasible set (5), independently on the norm $\|\cdot\|_Y$ used to bound the error in (4).*

5 Properties of impulse input

In this section we will evaluate the error (21) assuming that the input is an impulse, i.e. $u = [\eta, 0, \ldots, 0]'$.

Let us denote by $\mathcal{B}_Y(c, r)$ the ball in Y-norm centered in c with radius r. The following theorem holds.

Theorem 3 *Let $\mathcal{S}$ be as in Assumption 1 and $\gamma = \max_i\{\gamma_i\}$. Let $u = [\eta, 0, \ldots, 0]'$ and $U = T(u)$. Then*

$$E^*(U,\mathcal{S},\mathcal{M}) = \begin{cases} \underline{E} & \text{if } \eta \ge \frac{1}{\underline{E}} \\ \min\left\{\frac{1}{\eta}, \gamma\right\} & \text{if } \eta < \frac{1}{\underline{E}} \end{cases}$$

Proof. Since u is an impulse, then $\mathcal{F} = \mathcal{B}_Y(c,r)\cap\mathcal{S}$ with $r = \frac{1}{\eta}$. By Theorem 2, one has:

$$E^*(U,\mathcal{S},\mathcal{M}) = \max\{\underline{E}, \rho\},$$

where

$$\rho = \mathrm{rad}_\infty(\mathcal{B}_Y(0,r)\cap\mathcal{S}) = \min\{r, \gamma\}. \tag{24}$$

Moreover, we have that

$$\underline{E} = \sup_{h\in\mathcal{S}} \inf_{g\in\mathcal{M}} \|g-h\|_\infty \le \sup_{h\in\mathcal{S}} \|h\|_\infty = \gamma. \tag{25}$$

The theorem follows from (24), (25) and Theorem 2. □

Remark 5 *By (18), if $\eta \ge 1/\underline{E}$ then the impulse input is optimal. Hence, by Theorem 3 a sufficient condition for impulse optimality is that the impulse value is larger than the inverse of the representation error. If it is not possible to feed the system with an impulse of size at least $1/\underline{E}$, then the optimal input sequence solving (14)-(15) may not be an impulse.*

It is worth observing that Theorem 3 does not depend on the norm $\|\cdot\|_Y$ used to bound the noise in (4). In the following, it is shown that the choice of this norm has an influence on the optimality of the impulse input.

5.1 ℓ_2-bounded noise

For an energy-bounded noise, i.e. $\|\cdot\|_Y = \ell_2$ in (4) the feasible set is the intersection between the a priori set $\mathcal{S}$ and the ellipsoid

$$\mathcal{V}(c) = \{h : \; (h-c)'(U'U)(h-c) \le \varepsilon^2\}$$

with $c = U^{-1}y$.

The next example shows that the impulse is then not always the optimal input.

Example. Let $N = 2$, $\delta = 1$, $\varepsilon = 1$ and $\|\cdot\|_{\mathcal{H}} = \ell_\infty$. Consider problem (14)-(15) where

$$\mathcal{S} = \{h \in \mathbb{R}^2 : \; |h_0| \le 1, \, |h_1| \le 0.2\}$$

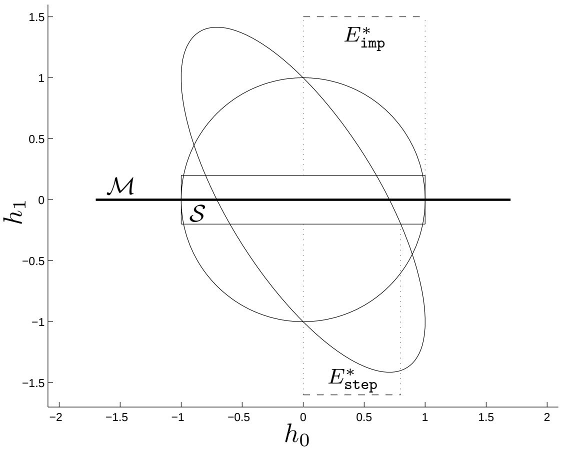

Fig. 1. Example in which the step input is better than the impulse input.

is the set containing the a priori information on the impulse response and

$$\mathcal{M} = \{g \in \mathbb{R}^2 : g = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta, \theta \in \mathbb{R}\}$$

is the model class of FIR filters of order 1. Notice that in this case one has $\overline{E} = 1$ and $\underline{E} = 0.2$ in (16) and (17). Let $u_{\texttt{imp}} = [1 \;\; 0]'$ be the impulse input and $u_{\texttt{step}} = [1 \;\; 1]'$ the step input. In Fig. 1 the worst-case errors $E^*_{\texttt{imp}}$ and $E^*_{\texttt{step}}$ are depicted for impulse and step input respectively. It can be seen that the impulse input is not the best one. Indeed, for the impulse one has $U = I_2$, $\mathcal{V}(0) = \{h : \; h'h \leq 1\}$ and $\rho_{\texttt{imp}} = \mathrm{rad}_\infty(\mathcal{V}(0) \cap \mathcal{S}) = 1$. Hence, Theorem 2 states that $E^*_{\texttt{imp}} = \max\{\underline{E}, \rho_{\texttt{imp}}\} = 1$. Conversely, for the step input

$$U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

$\mathcal{V}(0) = \{h : \; 2h_1^2 + 2h_1h_2 + h_2 \leq 1\}$ and $\rho_{\texttt{step}} = \mathrm{rad}(\mathcal{V}(0) \cap \mathcal{S}) = 0.8$. Therefore, $E^*_{\texttt{step}} = \max\{\underline{E}, \rho_{\texttt{step}}\} = 0.8 < E^*_{\texttt{imp}}$. It can be shown that the infimum with respect to U in (14) is achieved by the following input signals:

$$u = [1 \; 1]' \; , \; u = [-1 \; -1]' \; , \; u = [1 \; -1]' \; , \; u = [-1 \; 1]'.$$

Hence, the step is an optimal input in this case.

5.2 ℓ_∞-bounded noise

Let us now consider the case of ℓ_∞-bounded noise, i.e. $\|\cdot\|_Y = \ell_\infty$ in (4). In this case, the feasible set is given by $\mathcal{S} \cap \mathcal{V}(c)$ where

$$\mathcal{V}(c) = \{h : \|U(h - c)\|_\infty \leq \varepsilon\}$$

with $c = U^{-1}y$.

Let us denote by $\mathrm{PAR}(q, P)$ the parallelotope

$$\mathrm{PAR}(q, P) = \{h : \; \|Ph - q\|_\infty \leq 1\}.$$

Then, according to (13), the feasible set becomes

$$\mathcal{F} = \{h \in \mathcal{S} : \|Uh - y\|_\infty \leq 1\} = \mathrm{PAR}(y, U) \cap \mathcal{S}.$$

The next proposition shows that, if the error is ℓ_∞-bounded, then the impulse input is always optimal.

Proposition 2 *Let $\|\cdot\|_Y = \ell_\infty$ and let $u = [\eta, 0, \dots, 0]'$. Then such input minimizes $E^*(U, \mathcal{S}, \mathcal{M})$.*

Proof. By Theorem 2, in order to minimize $E^*(U, \mathcal{S}, \mathcal{M})$ it is necessary to minimize

$$\rho = \mathrm{rad}_\infty(\mathrm{PAR}(0, U) \cap \mathcal{S}).$$

Let $U_{\texttt{imp}} = \eta I$ be the impulse input matrix. By Theorem 3 it follows that

$$\begin{aligned} \rho_{\texttt{imp}} &= \mathrm{rad}_\infty(\mathrm{PAR}(0, U_{\texttt{imp}}) \cap \mathcal{S}) \\ &= \mathrm{rad}_\infty\left(\frac{1}{\eta}\mathcal{B}_\infty \cap \mathcal{S}\right) = \min\left\{\frac{1}{\eta}, \gamma\right\}. \end{aligned}$$

Now, let $U = T(u)$ be the Toeplitz matrix (3) corresponding to a generic input u (such that $\|u\|_\infty = \eta$). We will show that

$$\rho(u) = \mathrm{rad}_\infty(\mathrm{PAR}(0, U) \cap \mathcal{S}) \geq \rho_{\texttt{imp}}.$$

One has

$$\|Uh\|_\infty \leq 1 \iff \begin{cases} |u_0 h_0| \leq 1 \\ |u_0 h_1 + u_1 h_0| \leq 1 \\ |u_0 h_2 + u_1 h_1 + u_2 h_0| \leq 1 \\ \dots \end{cases} \qquad (26)$$

Let $e_0, e_1, \dots$ be the canonical base vectors; let us define $w_0^+ = \frac{1}{\eta} e_0$, $w_0^- = -\frac{1}{\eta} e_0$, $w_1^+ = \frac{1}{\eta} e_1$, …, $w_{N-1}^- = -\frac{1}{\eta} e_{N-1}$. It follows that the vectors $h = w_i^+$ and $h = w_i^-$ satisfy (26) for all $i = 0, \dots, N-1$.

Since $\mathrm{PAR}(0, U) = \{h : \|Uh\|_\infty \leq 1\}$, it follows that $w_i^+, w_i^- \in \mathrm{PAR}(0, U)$, $\forall i = 0, \dots, N-1, \forall u : \|u\|_\infty = \eta$. Let $\mathcal{W} = \{w_i^+, w_i^-, \; i = 0, \dots, N-1\}$. It can be seen that

$$\begin{aligned} \mathrm{BOX}(\mathcal{W}) \cap \mathcal{S} &= \left\{h : \; |h_i| \leq \min\left\{\frac{1}{\eta}, \gamma_i\right\}\right\} \\ &= \mathrm{BOX}(\mathrm{PAR}(0, U_{\texttt{imp}}) \cap \mathcal{S}). \end{aligned} \qquad (27)$$

Since $\mathcal{W} \subseteq \mathrm{PAR}(0, U)$ one has

$$\max_{v \in \mathrm{PAR}(0,U)} |v_i| \geq \frac{1}{\eta}\;, \quad i = 0, \dots, N-1. \qquad (28)$$

Moreover,

$$\max_{v\in\mathcal{S}} |v_i| = \gamma_i. \quad (29)$$

Since both $\mathrm{PAR}(0,U)$ and $\mathcal{S}$ are symmetric w.r.t. the origin, (28) and (29) yield

$$\max_{v\in \mathrm{PAR}(0,U)\cap\mathcal{S}} |v_i| \geq \min\left\{\frac{1}{\eta}, \gamma_i\right\} \ , \quad i = 0, \ldots, N-1$$

and hence, due to (27)

$$\mathrm{BOX}(\mathrm{PAR}(0,U)\cap\mathcal{S}) \supseteq \mathrm{BOX}(\mathrm{PAR}(0,U_{\mathtt{imp}})\cap\mathcal{S})$$

and therefore $\rho(u) \geq \rho_{\mathtt{imp}}$ for any input u. Then, the result follows from Theorem 2. □

6 Conclusions

Input design in conditional set membership identification aims at choosing the input signal that minimizes the worst-case identification error. In general, this requires the solution of a high dimensional minimax optimization problem which turns out to be intractable in most practical cases. Therefore, it is important to provide characterization results on specific classes of input signals. In the ℓ_∞ identification error measure, with energy-bounded noise, the impulse input is optimal if the impulse size is larger than the inverse of the representation error, which is a measure of the distance between the set of a priori admissible systems and the chosen model class. Conversely, when the noise is amplitude bounded, the impulse input is always optimal. Similar characterization results for different identification error norms, like ℓ_1 or ℓ_2, are the subject of the ongoing research. The results on the ℓ_∞ case presented in this paper can be seen as a useful starting point towards this aim.

A Proof of Theorem 1

Note that $E \triangleq \mathrm{rad}_\infty^{\mathcal{M}}(\mathcal{F}) = \inf_{g\in\mathcal{M}} \sup_{h\in\mathcal{P}} \|h-g\|_\infty$.
Let us now prove that if $E^+ \geq \mu$ then $E = E^+$.
Let $v^+ \in \mathcal{P}$ be a point where the sup in (20) is achieved and let G^+ be the related solution set of $\mathcal{M}$, i.e.: $G^+ \triangleq \{g\in\mathcal{M} : \|v^+ - g\|_\infty = E^+\}$. Since $\mathcal{P}$ is a box, one can choose v^+ as a vertex of $\mathcal{P}$. W.l.o.g. we may assume the box centered at the origin and $v^+ = [\beta_0, \beta_1, \ldots, \beta_{N-1}]$, $\beta_i \geq 0$, $\forall i = 0, \ldots, N-1$.
Let $v^- = -v^+$ and $g^+ \triangleq \arg\inf_{g\in G^+} \|v^- - g\|_\infty$; it is sufficient to prove that $\|v^- - g^+\|_\infty \leq E^+$.
To obtain a contradiction, let us suppose $\|v^- - g^+\|_\infty = E^- > E^+$ and let $I = \{i : \ |v_i^- - g_i^+| = E^-\}$.
Since $g^+ \in G^+$, one has $|g_i^+ - \beta_i| \leq E^+$, $\forall i$. Moreover, when $i\in I$, $|g_i^+ + \beta_i| = E^- > E^+$. Hence $g_i^+ > 0$, $\forall i \in I$.
Define $\bar{v} = \begin{cases} \beta_i & i \notin I \\ -\beta_i & i\in I \end{cases}$, and let $\bar{g} = \arg\inf_{g\in\mathcal{M}} \|g - \bar{v}\|_\infty$. By definition of E^+, one has $\|\bar{v} - \bar{g}\|_\infty \leq E^+$. It follows that

$$|\bar{g}_i - v_i^+| = |\bar{g}_i - \bar{v}_i| \leq E^+ \quad , \quad \forall i \notin I. \quad (\mathrm{A}.1)$$

Let us define $\widetilde{g} \triangleq \alpha\bar{g} + (1-\alpha)g^+$, $0 < \alpha < 1$. Due to (A.1), since $|g_i^+ - v_i^+| \leq E^+$, $\forall i$, one has

$$|\widetilde{g}_i - v_i^+| \leq E^+ \quad , \quad \forall i \notin I. \quad (\mathrm{A}.2)$$

Conversely, when $i \in I$, one has $|g_i^+ - \bar{v}_i| = E^-$ and then

$$\begin{aligned} |\widetilde{g}_i - \bar{v}_i| &\leq \alpha|\bar{g}_i - \bar{v}_i| + (1-\alpha)E^- \\ &\leq \alpha E^+ + (1-\alpha)E^- < E^- \quad , \quad \forall i\in I. \end{aligned} \quad (\mathrm{A}.3)$$

Since $g_i^+ > 0, \forall i \in I$, there always exists a small positive α such that also $\widetilde{g}_i > 0$, $\forall i \in I$. Then, using (A.3), one has

$$0 < \widetilde{g}_i < g_i^+ \quad , \quad \forall i \in I. \quad (\mathrm{A}.4)$$

Since $|g_i^+ - \beta_i| \leq E^+$ and $\beta_i \leq \mu \leq E^+, \forall i$, (A.4) implies that

$$|\widetilde{g}_i - v_i^+| = |\widetilde{g}_i - \beta_i| \leq E^+ \quad , \quad \forall i \in I. \quad (\mathrm{A}.5)$$

Therefore, from (A.2) and (A.5) it follows that $\|\widetilde{g} - v^+\|_\infty \leq E^+$ and hence $\widetilde{g} \in G^+$.
However, from (A.3) one has

$$|\widetilde{g}_i - v_i^-| = |\widetilde{g}_i - \bar{v}_i| < E^- \quad , \quad \forall i \in I. \quad (\mathrm{A}.6)$$

At the same time, if $i \notin I$, $|g_i^+ - v_i^-| < E^-$, and by a continuity argument it is always possible to find a small positive α such that

$$|\widetilde{g}_i - v_i^-| < E^+ \quad , \quad \forall i \notin I. \quad (\mathrm{A}.7)$$

Therefore, from (A.6) and (A.7) one has $\|\widetilde{g} - v^-\|_\infty < E^-$. But since $\widetilde{g} \in G^+$, this contradicts the definitions of g^+ and E^-.
To complete the proof, it remains to show that if $E^+ < \mu$, then $E = \mu$.
Let us denote the box by $\mathcal{P}(c)$ to emphasize its center and

$$E(\mathcal{P}(c), \mathcal{M}) = \inf_{g\in\mathcal{M}} \sup_{h\in\mathcal{P}(c)} \|h-g\|_\infty.$$

To obtain a contradiction, let us suppose $E(\mathcal{P}(c),\mathcal{M}) > \mu$.
Let $\mathbb{P}(c)$ be a family of boxes centered in c such that

$$\mathbb{P}(c) = \{\mathcal{P}_i(c) \text{ with semi-axes } \delta_j : \beta_j \leq \delta_j \leq \mu, j = 0, \ldots, N{-}1\}.$$

It follows that $\forall \mathcal{P}_i(c) \in \mathbb{P}(c)$ one has $\mathcal{P}(c) \subseteq \mathcal{P}_i(c)$ and so

$$E(\mathcal{P}_i(c), \mathcal{M}) \geq E(\mathcal{P}(c), \mathcal{M}) > \mu \ , \ \forall \mathcal{P}_i(c) \in \mathbb{P}(c). \quad (\mathrm{A}.8)$$

Let $\overline{\mathcal{P}}(c) \in \mathbb{P}(c)$ be the box with all semi-axes equal to μ; one has

$$\sup_{h\in\overline{\mathcal{P}}(c)} \inf_{g\in\mathcal{M}} \|h-g\|_\infty \geq \mu > E^+ = \sup_{h\in\mathcal{P}(c)} \inf_{g\in\mathcal{M}} \|h-g\|_\infty$$

and then, by a continuity argument, there exists at least one $\mathcal{P}_k(c) \in \mathbb{P}(c)$ such that

$$\sup_{h\in\mathcal{P}_k(c)} \inf_{g\in\mathcal{M}} \|h-g\|_\infty = \mu. \tag{A.9}$$

Due to (A.9), the first part of this theorem guarantees that $E(\mathcal{P}_k(c),\mathcal{M}) = \mu$, which is in contradiction with (A.8). Thus $E(\mathcal{P}(c),\mathcal{M}) \le \mu$. Since the error can not be less than μ, due to $\mu = \mathrm{rad}_\infty(\mathcal{P}(c)) \le \mathrm{rad}_\infty^{\mathcal{M}}(\mathcal{P}(c))$, it follows that $E(\mathcal{P}(c),\mathcal{M}) = \mu$. □

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