

# A FAVARD TYPE PROBLEM FOR 3-D CONVEX BODIES

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ABSTRACT. A Favard's theorem states that among all plane sets of given area and perimeter, the symmetric lens has maximum circumradius. This paper deals with the higher dimensional problem of finding the convex body in  $\mathbb{R}^3$  of given volume and mean width with the largest possible diameter. It is shown that the solution is the convex hull of a surface of revolution with constant Gauss curvature and a segment lying in the axis of revolution.

## 1. INTRODUCTION

A well-known result in convex plane geometry states that the *symmetric lens* is the unique solution of the following problem:

*Find the compact convex set in  $\mathbb{R}^2$  of given perimeter and area with the largest possible circumradius.*

We recall that a symmetric lens is the intersection of two discs with the same radius and that the circumradius of a set  $K$  is the radius of the smallest disc containing  $K$ .

This extremal property of the symmetric lens was proved by Favard in [5] and occupies a relevant place in the framework of geometric inequalities of isoperimetric type. Among these inequalities, the Bonnesen inequalities [2]

$$(1) \quad \frac{L^2}{4\pi} - A \geq \pi \left( \frac{L}{2\pi} - r \right)^2$$

$$(2) \quad \frac{L^2}{4\pi} - A \geq \pi \left( R - \frac{L}{2\pi} \right)^2$$

give lower bounds of the *isoperimetric deficit* of a set, the left-hand side, in terms of the inradius and circumradius, respectively. Here,  $L$  and  $A$  denote the perimeter and the area of the set,  $R$  and  $r$  its circumradius and inradius, the latter being the radius of the largest disc contained in the set.

In spite of the similarity, (1) and (2) differ about a main fact. Inequality (1) is sharp, since for every value of the isoperimetric deficit the convex hull of two suitable discs with the same radius gives equality. On the other hand, inequality (2) is not sharp: If the isoperimetric deficit is strictly positive, then (2) is strict also. Favard's theorem just solves the problem opened by (2). For more details on Bonnesen's inequalities and their variants, we refer to Bonnesen and Fenchel [3], Osserman [6], Schneider [9].

A natural question, that we shall deal with in the present paper, is whether it is possible to find results in higher dimensions analogous to Favard's theorem. It is worth to say at once

that the literature contains only a partial result of that type, due to Zalgaller [10]. To describe it, let  $K$  be a convex body in  $\mathbb{R}^3$ , i.e., a three-dimensional compact convex set, and  $V$  and  $S$  denote its volume and surface area, respectively. The classical isoperimetric inequality (see, for instance, [4], p.145) states that

$$S^3 \geq 36\pi V^2.$$

**Theorem 1.1** (Zalgaller [10]). *Let  $S_0$  and  $V_0$  be two numbers such that  $S_0^3 \geq 36\pi V_0^2$ . Among all three-dimensional convex bodies such  $S \leq S_0$ ,  $V \geq V_0$ , the unique body having maximum diameter is a mean curvature spindle-shaped body of surface area  $S_0$  and volume  $V_0$ .*

A mean curvature spindle-shaped body is a centrally symmetric convex body of revolution which can be seen as the convex hull of a surface of revolution with constant mean curvature and a segment lying in the axis of revolution.

Note that Zalgaller's result refers to the diameter of the set instead of the circumradius. Therefore, it answers to a three-dimensional Favard-type question in the class of bodies of revolution. The same paper [10] contains the conjecture that the solution does not change in the full class of convex bodies.

While it is natural to replace the area of a plane set with the volume of a three-dimensional one, there are two possible substitutes of the perimeter, namely, the surface area and the mean width. Indeed, the perimeter of a planar convex body is an average of all its widths. The width along a direction  $v$  is the distance between the support lines (hyperplanes, in  $\mathbb{R}^d$ ) orthogonal to  $v$ . These two possible choices are also suggested by the Steiner formula for convex parallel sets (see, for instance, [9], p.197). The  $d$ -dimensional measure of the body  $K + tB$ , where  $K$  is a convex body in  $\mathbb{R}^d$ ,  $B$  is the unit  $d$ -ball, and  $+$  stands for the Minkowski sum (vector addition), is a polynomial of degree  $d$  in  $t$ . For  $d = 2$ , the perimeter is the coefficient of the linear term. For  $d = 3$ , a multiple of the mean width and the surface area are the coefficients of  $t$  and  $t^2$ , respectively.

An analytical expression of the mean width can be given in terms of the support function  $h_K$  of a convex body  $K$ , whose definition is:

$$h_K(z) = \max_{x \in K} \langle z, x \rangle, \quad \text{for every } z \in \mathbb{S}^2,$$

where  $\langle \cdot, \cdot \rangle$  denotes the scalar product and  $\mathbb{S}^2$  the unit sphere of  $\mathbb{R}^3$ .

Thus, the mean width  $W$  of  $K$  can be expressed as

$$(3) \quad W = \frac{1}{2\pi} \int_{\mathbb{S}^2} h_K(z) dz.$$

The mean width  $W$  and the volume  $V$  of a three-dimensional convex body satisfy the following isoperimetric inequality

$$\pi W^3 \geq 6V,$$

known as Urysohn inequality (see, for instance, [4], p.145).

This paper is devoted to the following problem.

**Problem 1.2.** *Let  $W_0$  and  $V_0$  be two numbers such that  $\pi W_0^3 \geq 6V_0$ . Among all three-dimensional convex bodies such  $W \leq W_0$ ,  $V \geq V_0$ , find the bodies of maximum diameter.*

In Section 2 we shall prove that the solution exists and is unique, trivially up to isometries. The description of the solution is given in Sections 3, 4, 5 and it is summarized in the following theorem.

**Theorem 1.3.** *The unique solution of Problem 1.2 is a Gaussian curvature spindle-shaped body of mean width  $W_0$  and volume  $V_0$ .*

A Gaussian curvature spindle-shaped body is a body analogous to the one introduced by Zalgaller, where to be constant now is the Gaussian curvature.

Actually, Theorem 1.3 is inspired by Theorem 1.1. Nevertheless, our proof makes use of different tools, such as the Brunn-Minkowski inequality, the Gauss map, the area measure of a convex body, all coming from the Brunn-Minkowski theory.

The constraints on surface area and mean width give rise to conditions on the mean curvature and the Gauss curvature of the extremal bodies, respectively.

This fact can be explained by looking at the following expressions for  $S$  and  $W$ , when  $K$  is smooth:

$$(4) \quad S = \int_{\partial K} \left( \frac{1}{R_1} \frac{1}{R_2} \right) dp.$$

$$(5) \quad W = \frac{1}{4\pi} \int_{\partial K} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) dp,$$

where  $R_1, R_2$  are the principal radii of curvature of  $\partial K$  at  $p$  and the integration is performed with respect to the  $(n - 1)$ -dimensional Hausdorff measure (see [3], p. 72).

(Risultati di Zalgaller per l'inradius)

## 2. EXISTENCE AND UNIQUENESS OF THE SOLUTION

For convenience, let us denote by  $\mathcal{K}(V_0, W_0)$  the class of all three-dimensional convex bodies  $K$  such that  $V(K) \geq V_0$  and  $W(K) \leq W_0$ , and by  $d(K)$  the diameter of  $K$ . The class  $\mathcal{K}(V_0, W_0)$  is endowed with the Hausdorff metric and the functionals  $V(K)$ ,  $W(K)$ ,  $d(K)$  are continuous in such a metric.

Let  $K \in \mathcal{K}(V_0, W_0)$ . Since  $K$  contains a segment whose length is  $d(K)$ , the monotonicity of the mean width implies that

$$W(K) \geq \frac{1}{2} d(K).$$

Hence,  $\mathcal{K}(V_0, W_0)$  is, up to translations, compact. Therefore, there exists

$$D = \max_{K \in \mathcal{K}(V_0, W_0)} d(K).$$

Let us show that, if  $K$  is a solution of Problem 1.2, then

$$(6) \quad V(K) = V_0 \quad \text{and} \quad W(K) = W_0.$$

Assume that  $V(K) > V_0$  and let  $\tilde{K} \in \mathcal{K}(V_0, W_0)$  be a proper subset of  $K$  such that  $d(\tilde{K}) = D$  and  $W(\tilde{K}) < W(K)$ . For suitable  $\lambda > 1$ , we have  $\lambda\tilde{K} \in \mathcal{K}(V_0, W_0)$  and  $d(\lambda\tilde{K}) = \lambda d(\tilde{K}) = \lambda D$ ,

which contradicts the definition of  $D$ . Similarly, if  $W(K) < W_0$ , then, for suitable  $\lambda > 1$ ,  $\lambda K$  gives a contradiction.

We prove now that, *the solution of Problem 1.2 is unique in  $\mathcal{K}(V_0, W_0)$ , up to isometries.*

Assume, by contradiction, that  $K_1$  and  $K_2$  are two distinct solutions. Via a possible rigid movement,  $K_1 \cap K_2$  has still diameter  $D$ . Consider the Minkowski sum  $\tilde{K} = \frac{K_1 + K_2}{2}$ . By the Brunn-Minkowski inequality, see [9], p.309,

$$(7) \quad V(\tilde{K}) > \left( \frac{1}{2}V(K_1)^{\frac{1}{3}} + \frac{1}{2}V(K_2)^{\frac{1}{3}} \right)^3 = V_0,$$

where the strict inequality holds since we are assuming that  $K_1$  and  $K_2$  are not homothetic. By the linearity of the mean width with respect to Minkowski addition,  $W(\tilde{K}) = W_0$  and hence  $\tilde{K} \in \mathcal{K}(V_0, W_0)$ . Since  $d(\tilde{K}) = D$ , the strict inequality (7) contradicts (6).

The same argument we just used can be repeated for showing that *if  $K$  is a solution of Problem 1.2, then  $K$  is a centrally symmetric body of revolution about a diameter.*

### 3. REGULARITY OF THE SOLUTION

In order to describe the solution  $K$ , it is sufficient to characterize its section with a plane through the axis of revolution. Fix an orthonormal system  $(O; x, y)$  in the plane and assume that the diameter of  $K$  lies on the  $x$ -axis symmetrically with respect to  $O$ . The boundary of  $K$  is then obtained by rotating the meridian curve  $\gamma$ , contained in the half-plane  $y \geq 0$ , around the  $x$ -axis. Let us denote by  $C$  the section of  $K$  with the  $xy$ -plane and by  $h_C$  its support function. For simplicity, we write  $h_C(\theta)$  instead of  $h_C(\cos \theta, \sin \theta)$ .

The curve  $\gamma$  can be parametrized by

$$\begin{aligned} x(\theta) &= h_C(\theta) \cos \theta - h'_C(\theta) \sin \theta \\ y(\theta) &= h_C(\theta) \sin \theta + h'_C(\theta) \cos \theta. \end{aligned}$$

Hence

$$V(K) = \pi \int_{-D/2}^{D/2} y^2 dx = \pi \int_0^\pi [h_C(\theta) \sin \theta + h'_C(\theta) \cos \theta]^2 [h_C(\theta) + h''_C(\theta)] \sin \theta d\theta.$$

Integrating by parts gives

$$V(K) = \pi \int_0^\pi \left[ \frac{2}{3} h_C^3(\theta) \sin \theta - \frac{1}{3} h_C'^3(\theta) \cos \theta - h_C(\theta) h_C''(\theta) \sin \theta \right] d\theta.$$

Note that the last equality requires no regularity assumptions on  $\partial C$ .

Furthermore,

$$W(K) = \int_0^\pi h_C(\theta) \sin(\theta) d\theta.$$

Let us show that *every interior point of  $\gamma$  is regular, i.e., the normal cone has dimension one.*

Let us suppose by contradiction that there exists a point  $p = (p_1, p_2)$  from the interior of  $\gamma$  whose normal cone contains the arc  $[\theta_1, \theta_2]$ . Let us cut the curve  $\gamma$  with a line orthogonal to

the axis of symmetry of the arc at a distance  $\varepsilon$  from  $p$  and let  $q_1$  and  $q_2$  be the intersection points. Replace the arc of  $\gamma$  with the chord  $q_1q_2$ , and denote by  $\tilde{\gamma}$  the curve so obtained. Let  $\tilde{K}$  be the convex body enclosed by the surface of revolution of  $\tilde{\gamma}$  about the  $x$ -axis and  $\tilde{C}$  its section with the  $xy$ -plane. Clearly, for every  $\varepsilon > 0$ ,  $V(\tilde{K}) < V(K)$  and  $W(\tilde{K}) < W(K)$ . Precisely,

$$\begin{aligned} W(K) - W(\tilde{K}) &= \int_0^\pi (h_C(\theta) - h_{\tilde{C}}(\theta)) \sin \theta \, d\theta \\ &= \int_{\theta_1}^{(\theta_1+\theta_2)/2} \left[ p_1 \cos \theta + p_2 \sin \theta - \left( p_1 + \frac{\varepsilon \sin \theta_1}{\sin \frac{\theta_2 - \theta_1}{2}} \right) \cos \theta - \left( p_2 - \frac{\varepsilon \cos \theta_1}{\sin \frac{\theta_2 - \theta_1}{2}} \right) \sin \theta \right] \sin \theta \, d\theta \\ &+ \int_{(\theta_1+\theta_2)/2}^{\theta_2} \left[ p_1 \cos \theta + p_2 \sin \theta - \left( p_1 - \frac{\varepsilon \sin \theta_2}{\sin \frac{\theta_2 - \theta_1}{2}} \right) \cos \theta - \left( p_2 + \frac{\varepsilon \cos \theta_2}{\sin \frac{\theta_2 - \theta_1}{2}} \right) \sin \theta \right] \sin \theta \, d\theta \\ &= \int_{\theta_1}^{(\theta_1+\theta_2)/2} \frac{\varepsilon \sin(\theta - \theta_1)}{\sin \frac{\theta_2 - \theta_1}{2}} \sin \theta \, d\theta + \int_{(\theta_1+\theta_2)/2}^{\theta_2} \frac{\varepsilon \sin(\theta_2 - \theta)}{\sin \frac{\theta_2 - \theta_1}{2}} \sin \theta \, d\theta \\ &= \frac{\varepsilon}{2 \sin \frac{\theta_2 - \theta_1}{2}} (\cos \theta_1 - \cos \theta_2) \frac{\theta_2 - \theta_1}{2}. \end{aligned}$$

If we denote by  $\tilde{y}$  the barycenter of the triangle  $q_1pq_2$  and by  $A(\cdot)$  the area, then

$$V(K) - V(\tilde{K}) = 2\pi\tilde{y}A(q_1pq_2) = \frac{2\pi\varepsilon^2}{\tan \frac{\theta_2 - \theta_1}{2}} \left( p_2 - \frac{\varepsilon(\cos \theta_1 - \cos \theta_2)}{3 \sin \frac{\theta_2 - \theta_1}{2}} \right)$$

Let us consider the Minkowski sum  $\hat{K}$  of  $\tilde{K}$  and  $\lambda\bar{x}$ , where  $\bar{x}$  is a unit segment parallel to the  $x$ -axis. Since  $W(\lambda\bar{x}) = \lambda/2$ , if we take

$$\lambda = \frac{\varepsilon}{\sin \frac{\theta_2 - \theta_1}{2}} (\cos \theta_1 - \cos \theta_2) \frac{\theta_2 - \theta_1}{2},$$

then  $W(\hat{K}) = W(K) = W_0$ . Moreover, denoting by  $\tilde{K}|x^\perp$  the orthogonal projection of  $\tilde{K}$  onto a plane orthogonal to the  $x$ -axis,

$$\begin{aligned} V(\hat{K}) &= V(\tilde{K}) + \lambda A(\tilde{K}|x^\perp) \\ &= V(K) - \frac{2\pi\varepsilon^2}{\tan \frac{\theta_2 - \theta_1}{2}} \left( p_2 - \frac{\varepsilon(\cos \theta_1 - \cos \theta_2)}{3 \sin \frac{\theta_2 - \theta_1}{2}} \right) + \frac{\varepsilon}{\sin \frac{\theta_2 - \theta_1}{2}} (\cos \theta_1 - \cos \theta_2) \frac{\theta_2 - \theta_1}{2} A(\tilde{K}|x^\perp). \end{aligned}$$

Hence, for  $\varepsilon$  sufficiently small,  $V(\hat{K}) > V(K) = V_0$ , which is a contradiction.

#### 4. GAUSSIAN CURVATURE OF THE SOLUTION

Let us show now that *the solution is the convex hulls of a surface of revolution with positive constant Gaussian curvature (excepted the points on the axis) and a segment lying on the axis of revolution.*

Note that the above property does not exhaust the description of the solution. Indeed, two parameters need still to be determined, namely the value of the Gaussian curvature and the length of the segment. Section 5 deals with this aspect.

We are going to prove the statement by means of local variations of the solution. To do this, it is convenient to introduce the Gauss map, that we adapt here to bodies of revolution about the  $x$ -axis. Let  $H$  be such a body. For every point  $p$  from  $\partial H$ , let  $f(p)$  be the corresponding point on the meridian curve  $\gamma_H$ . We denote by  $g_H(p)$  the set of outward unit normals to  $\partial H$  at  $f(p)$ , and by  $G_H(p)$  the set of outward unit normals to  $\partial H$  at  $p$ . Note that  $g_H$  maps  $\partial H$  onto  $S^1$ , more precisely onto  $[0, \pi]$ , while  $G_H$  maps  $\partial H$  onto  $S^2$ . For simplicity, if  $\omega$  is a subset of  $[0, \pi]$ , then we define  $s(\omega) = G_H(g_H^{-1}(\omega))$ .

Let  $\omega = [\theta_1, \theta_2]$  with  $0 < \theta_1 \leq \theta_2 < \pi$  and  $\varepsilon$  a positive number. For every  $p$  from the subset  $g_K^{-1}(\omega)$  of  $\partial K$ , let us consider the segment of length  $\varepsilon$  issuing from  $p$  along the outer normal to  $\partial K$ . Define  $K(\omega, \varepsilon)$  as the convex hull of  $K$  and all these segments.

It is easy to check that

$$h_{K(\omega, \varepsilon)}(z) \leq h_K(z) + \varepsilon$$

and that in  $s(\omega)$  equality occurs.

Moreover, let  $K(\omega, -\varepsilon)$  be the largest convex body of revolution such that

$$\begin{aligned} h_{K(\omega, -\varepsilon)}(z) &\leq h_K(z) \quad , \text{ for every } z \in S^2, \\ h_{K(\omega, -\varepsilon)}(z) &\leq h_K(z) - \varepsilon \quad , \text{ for every } z \in s(\omega). \end{aligned}$$

Thus,

$$(8) \quad W(K(\omega, \varepsilon)) - W(K) = \frac{1}{2\pi} \int_{S^2} [h_{K(\omega, \varepsilon)}(z) - h_K(z)] dz \leq \frac{\varepsilon}{2\pi} A(G_{K(\omega, \varepsilon)}(\partial K(\omega, \varepsilon) \setminus \partial K)).$$

Analogously,

$$(9) \quad W(K) - W(K(\omega, -\varepsilon)) \geq \varepsilon A(s(\omega)).$$

From the definitions of  $K(\omega, \varepsilon)$  and  $K(\omega, -\varepsilon)$  easily follows that

$$(10) \quad V(K(\omega, \varepsilon)) - V(K) \geq \varepsilon A(g_K^{-1}(\omega))$$

and

$$(11) \quad V(K) - V(K(\omega, -\varepsilon)) \leq \varepsilon A(\partial K(\omega, -\varepsilon) \setminus \partial K).$$

Let  $\tilde{K} = [K(\omega_1, \varepsilon_1)](\omega_2, -\varepsilon_2)$ , where the intervals  $\omega_1, \omega_2$  are disjoint and have positive distance from the  $x$ -axis. It is easy to check that, if  $\varepsilon_1, \varepsilon_2$  are sufficiently small, then  $d(\tilde{K}) = d(K)$ .

From (8), (9), (10) and (11) we deduce that  $\tilde{K}$  belongs to  $\mathcal{K}(V_0, W_0)$  if  $\varepsilon_1$  and  $\varepsilon_2$  satisfy

$$(12) \quad \varepsilon_1 A(G_{K(\omega_1, \varepsilon_1)}(\partial K(\omega_1, \varepsilon_1) \setminus \partial K)) \leq \varepsilon_2 A(s(\omega_2)),$$

$$(13) \quad \varepsilon_1 A(g_K^{-1}(\omega_1)) \geq \varepsilon_2 A(\partial K(\omega_2, -\varepsilon_2) \setminus \partial K).$$

If (12) or (13) is strict, then  $\tilde{K}$  does not verify (6), which contradicts the fact that  $\tilde{K}$  maximizes the diameter in  $\mathcal{K}(V_0, W_0)$ . This happens if

$$(14) \quad \frac{A(\partial K(\omega_2, -\varepsilon_2) \setminus \partial K)}{A(s(\omega_2))} < \frac{A(g_K^{-1}(\omega_1))}{A(G_{K(\omega_1, \varepsilon_1)}(\partial K(\omega_1, \varepsilon_1) \setminus \partial K))}.$$

Letting  $\varepsilon_1$  and  $\varepsilon_2$  tend to zero, inequality (14) is surely fulfilled if

$$(15) \quad \frac{A(g_K^{-1}(\omega_2))}{A(s(\omega_2))} < \frac{A(g_K^{-1}(\omega_1))}{A(s(\omega_1))}.$$

Therefore, if  $K$  is the solution, then

$$(16) \quad \frac{A(g_K^{-1}(\omega))}{A(s(\omega))} = \text{constant},$$

for every closed subinterval  $\omega$  of  $[0, \pi]$  with  $g_K^{-1}(\omega)$  at positive distance from the  $x$ -axis. Since  $g_K^{-1}(\omega) = G_K^{-1}(s(\omega))$ , identity (16) states that the *area measure*  $\sigma_K$  of  $K$  is proportional to the Hausdorff measure on  $S^2$  (see [9], Th. 4.2.5), possibly excepted two conical parts with vertex on the axis of revolution. It is well known that, for a smooth convex body  $K$ , the area measure of a Borel subset  $\beta$  of  $S^2$  is the integral over  $G_K^{-1}(\beta)$  of the Gaussian curvature of  $\partial K$ . Since a body is uniquely determined, up to translation, by its area measure, the proof is concluded.

## 5. CHARACTERIZATION OF THE SOLUTION

The final step consists in understanding whether the meridian curve of the solution has two segments at the endpoints or not. Of course, the answer might depend on the values  $V_0$  and  $W_0$ . Actually, we show that the solution always contains two conical parts, excepted when  $\pi W_0^3 = 6V_0$ . In such a case the class  $\mathcal{K}(V_0, W_0)$  contains only the ball with those parameters.

Let  $K$  be a body from  $\mathcal{K}(V_0, W_0)$  candidate to be a solution, and  $y = f(x)$ ,  $-d/2 \leq x \leq d/2$ , a representation of its meridian curve  $\gamma_K$ , where  $d = d(K)$ . Denote by  $\alpha$  the angle such that  $\tan \alpha = f'(-d/2)$ ,  $0 < \alpha \leq \pi/2$  and by  $\ell$  the length of each segment on the graph of  $f(x)$ . If  $k$  denote the Gaussian curvature of the central part of  $K$ , then the function  $f(x)$  satisfies in  $[\ell \cos \alpha - d/2, d/2 - \ell \cos \alpha]$  the following differential equation

$$kf(x) (1 + f'(x)^2)^2 + f''(x) = 0.$$

A first integration yields

$$(17) \quad f'(x) = -\text{sign}(x) \sqrt{\frac{1}{kf(x)^2 + 1 - (1 + k\ell^2) \sin^2 \alpha}} - 1,$$

where we used the condition

$$f'(\ell \cos \alpha - d/2) = \tan \alpha.$$

From  $f'(0) = 0$ , we deduce

$$(18) \quad kf(0)^2 = (1 + k\ell^2) \sin^2 \alpha.$$

Integrating (17) gives

$$(19) \quad \int_{\ell \sin \alpha}^{f(x)} \frac{dt}{\sqrt{\frac{1}{kt^2+1-(1+k\ell^2)\sin^2\alpha} - 1}} = -|x| + \frac{d}{2} - \ell \cos \alpha.$$

Note that, to different choices of  $V_0$  and  $W_0$  keeping fixed  $\frac{W_0^3}{V_0}$  correspond homothetic solutions. Since we are interested in the shape of the solution, we may assume  $f(0) = 1$  as a normalization. Setting  $\rho = \ell \sin \alpha$ , from (18) and (19), the diameter is given by

$$(20) \quad d = 2\rho \frac{\cos \alpha}{\sin \alpha} + \frac{2}{\sin \alpha} \int_{\arcsin \rho}^{\pi/2} \Gamma(\alpha, \rho, t) dt$$

where

$$(21) \quad \Gamma(\alpha, \rho, t) = \sqrt{1 - \rho^2 - \sin^2 \alpha \cos^2 t}.$$

The volume  $V$  of  $K$  is the sum of the volume of the two cones and the one of the central part. Therefore, we obtain

$$(22) \quad V = \frac{2}{3}\pi\rho^3 \frac{\cos \alpha}{\sin \alpha} + \frac{2\pi}{\sin \alpha} \int_{\arcsin \rho}^{\pi/2} \Gamma(\alpha, \rho, t) \sin^2 t dt.$$

Finally, let us show that the mean width  $W$  of  $K$  can be expressed, in terms of  $\alpha$ ,  $\rho$ ,  $V$  and  $d$ , as follows

$$(23) \quad W = \frac{3V \sin^2 \alpha + \pi d(\cos^2 \alpha - \rho^2)}{2\pi(1 - \rho^2)}.$$

Indeed, by definition (3),

$$W = \frac{1}{2\pi} \int_{\Sigma_1} h_K(z) dz + \frac{1}{2\pi} \int_{\Sigma_2} h_K(z) dz,$$

where  $\Sigma_1 = s((\frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha))$  and  $\Sigma_2 = S^2 \setminus \Sigma_1$ .

It is convenient to use here the following expression for the volume of  $K$  (see [3], p. 64):

$$V = \frac{1}{3} \int_{S^2} h_K(z) d\sigma_K(z).$$

Consequently,

$$V = \frac{1}{3} \int_{\Sigma_1} \frac{h_K(z)}{k} dz + \frac{1}{3} \int_{\Sigma_2} h_K(z) d\sigma_K(z).$$

Hence,

$$2\pi W - 3kV = \int_{\Sigma_2} h_K(z) dz - k \int_{\Sigma_2} h_K(z) d\sigma_K(z).$$

Note that in  $\Sigma_2$  the area measure is concentrated on the boundary and the support function can be easily evaluated. Thus, (23) follows.

Now, looking at the expressions (20), (22) and (23), we note that  $K$  has minimal mean width among all bodies with the same volume and diameter. Indeed, if  $L$  is a body such that

$$V(L) = V(K), \quad d(L) = d(K), \quad W(L) < W(K),$$

then  $L$  belongs to  $\mathcal{K}(V_0, W_0)$ , solves Problem 1.2, but contradicts (6).

Thus, let us consider the function  $W$  in (23) under the constraint

$$(24) \quad \frac{V^{1/3}}{d} = \mu,$$

being  $V$  and  $d$  defined by (22) and (20). By the isodiametric inequality (see [4], p. 145),  $\mu \leq (\frac{\pi}{6})^{1/3}$ , where equality holds only for balls.

In order to show the existence of the conical parts, for every fixed  $\mu < (\frac{\pi}{6})^{1/3}$ , it is sufficient to prove that the minimum of  $W$  is not attained when  $\rho = 0$ .

To this end, let us remark that  $\frac{\partial}{\partial \alpha} \frac{V^{1/3}}{d}$  vanishes, for  $\rho = 0$ , only for  $\alpha = \frac{\pi}{2}$ . Hence, for every point  $(0, \alpha)$ , with  $\alpha < \frac{\pi}{2}$ , the level set (24) issuing from it defines in a positive interval of  $\rho = 0$  a function  $\alpha = \alpha(\rho)$ . Consequently,  $W$  can be thought of as a function of  $\rho$ , that, for simplicity, we denote by  $W(\rho)$ . We claim that  $W'(0) = W''(0) = 0$  and  $W'''(0) < 0$ .

Indeed, calculations lead to the following asymptotic expansion:

$$(25) \quad W'(\rho) = \frac{3\Gamma_2 \sin^2 \alpha (6\Gamma_2 - 3\Gamma_4 - 2\Gamma_1) - \Gamma_1 \Gamma_4 \cos^2 \alpha}{\sin \alpha \cos \alpha (3\Gamma_2 \Gamma_3 - \Gamma_1 \Gamma_4)} \rho^2 + O(\rho^3),$$

where

$$\begin{aligned} \Gamma_1 &= \int_0^{\pi/2} \Gamma(\alpha, 0, t) dt, & \Gamma_2 &= \int_0^{\pi/2} \Gamma(\alpha, 0, t) \sin^2 t dt, \\ \Gamma_3 &= \int_0^{\pi/2} \frac{dt}{\Gamma(\alpha, 0, t)}, & \Gamma_4 &= \int_0^{\pi/2} \frac{\sin^2 t}{\Gamma(\alpha, 0, t)} dt. \end{aligned}$$

First,  $3\Gamma_2\Gamma_3 - \Gamma_1\Gamma_4 > 0$ . Indeed, by Hölder inequality,

$$1 = \int_0^{\pi/2} \sin t dt \leq (\Gamma_2 \Gamma_3)^{1/2},$$

while  $\Gamma_1 \leq \frac{\pi}{2}$  and  $\Gamma_4 \leq 1$ .

Let us check that the numerator in (25) is negative. By Hölder inequality,  $\Gamma_1 \Gamma_4 \geq 1$  and

$$\frac{\pi}{4} = \int_0^{\pi/2} \sin^2 t dt \leq (\Gamma_2 \Gamma_4)^{1/2},$$

$$\frac{1}{2} \left( \frac{\alpha}{\sin \alpha} + \cos \alpha \right) = \int_0^{\pi/2} \Gamma(\alpha, 0, t) \sin t dt \leq (\Gamma_1 \Gamma_2)^{1/2},$$

$$\Gamma_2^2 \leq \left( \int_0^{\pi/2} \sin^2 t dt \right) \left( \int_0^{\pi/2} \Gamma^2(\alpha, 0, t) \sin^2 t dt \right) = \frac{\pi^2}{16} \left( 1 - \frac{1}{4} \sin^2 \alpha \right).$$

Therefore, the numerator is bounded from above by

$$\left(\frac{3}{16}\pi^2 + 1\right) \sin^2 \alpha - \frac{9}{32}\pi^2 \sin^4 \alpha - 1,$$

which is negative for every  $\alpha$ .

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