Lezione 27 28/11/2019 9-11

$$u'' + \alpha u' + \beta u = 0 \quad , \quad \boxed{\alpha, \beta \in \mathbb{R}} \qquad (27.1.1)$$

Scriviamo

$u(x) = e^{\lambda x} v(x)$ con λ da scegliere in modo astuto

$$(27.1.1) \iff v'' + (2\lambda + \alpha) v' + (\lambda^2 + \alpha\lambda + \beta) v = 0$$

Def 27.1.1 $\quad P(\lambda) = \lambda^2 + \alpha\lambda + \beta, \quad \lambda \in \mathbb{C}$

si chiama polinomio caratteristico di (27.1.1)

Dal teo fondamentale dell'algebra

P ha esattamente radici complesse λ_1, λ_2.

Poiché $\alpha, \beta \in \mathbb{R}$, se $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$

$$\Longrightarrow \lambda_2 = \overline{\lambda_1}, \quad \lambda_1 = a + ib, \ \lambda_2 = a - ib$$

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 + (-\lambda_1 - \lambda_2)\lambda + \lambda_1\lambda_2$$

$$= \lambda^2 + \alpha\lambda + \beta$$

$$\alpha = -(\lambda_1 + \lambda_2)$$

(principio di identità dei polinomi)

$$\beta = \lambda_1 \lambda_2$$

Fissiamo $\lambda = \lambda_1$: $\quad v'' + (2\lambda_1 + \alpha) v' = 0$

$$v'' + (2\lambda_1 + \alpha)\, v' = 0$$

$$v'' + (2\lambda_1 - \lambda_1 - \lambda_2)\, v' = v'' + (\lambda_1 - \lambda_2) v' = 0$$

$v' =: f$ $\qquad\qquad$ $f' + (\lambda_1 - \lambda_2) f = 0$

EDO di ordine 1 !

1° Caso: radici di P coincidenti, $\lambda_1 = \lambda_2$

$$f' = 0$$

$$\Rightarrow f(x) = \text{costante} = c$$

$$\Rightarrow v'(x) = c$$

$$\Rightarrow v(x) = cx + d$$

$$\Rightarrow u(x) = e^{\lambda_1 x}(cx + d)$$

Dunque se $\lambda_1 = \lambda_2 \in \mathbb{R}$ tutte le sol. di $u'' + \alpha u' + \beta u = 0$ sono della forma $u(x) = e^{\lambda_1 x}(cx + d)$ al variare di $c, d \in \mathbb{R}$

2° caso $\lambda_1 \neq \lambda_2$; $f' + (\lambda_1 - \lambda_2) f = 0$

$$f(x) \underset{\uparrow}{=} c_1 e^{(\lambda_2 - \lambda_1)x} = v'(x)$$

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$$v'(x) = c_1 \, e^{(\lambda_2-\lambda_1)x}$$

$$\Rightarrow \quad v(x) = \frac{c_1}{\lambda_2-\lambda_1} e^{(\lambda_2-\lambda_1)x} + d_1$$

ma $u(x) = e^{\lambda_1 x} v(x) \Longrightarrow$

$$u(x) = e^{\lambda_1 x}\left(\frac{c_1}{\lambda_2-\lambda_1} e^{(\lambda_2-\lambda_1)x} + d_1\right)$$

$$= \frac{c_1}{\lambda_2-\lambda_1} e^{\lambda_2 x} + d_1 e^{\lambda_1 x}$$

al variare di $c_1, d_1 \in \mathbb{C}$.

Equivalentemente

$$u(x) = k_2 e^{\lambda_2 x} + k_1 e^{\lambda_1 x}$$

al variare di $k_1, k_2 \in \mathbb{C}$,

$$u(x) = k_1 e^{\lambda_1 x} + k_2 e^{\lambda_2 x}.$$

Se $\lambda_1 \in \mathbb{R}$ anche $\lambda_2 \in \mathbb{R}$ e $k_1, k_2 \in \mathbb{R}$, e

$u(x) \in \mathbb{R}$. E se $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$? $\lambda_1 = a+ib$, $b \neq 0$

$\lambda_2 = a - ib$ $\qquad e^{\lambda_1 x} = e^{(a+ib)x} =$

$$e^{\lambda_1 x} = e^{(a+ib)x} = e^{ax+ibx} =$$
$$= e^{ax} e^{ibx}$$
$$= e^{ax}\left(\cos(bx) + i\,\mathrm{sen}(bx)\right)$$

$$u(x) = K_1 e^{ax}\left(\cos(bx) + i\,\mathrm{sen}(bx)\right) + K_2 e^{ax}\left(\cos(bx) - i\,\mathrm{sen}(bx)\right),$$

$$= e^{ax}\cos(bx)(K_1+K_2) + i\,e^{ax}\,\mathrm{sen}(bx)(K_1 - K_2)$$

$$= e^{ax}\left[\underbrace{(K_1+K_2)}\cos(bx) + \underbrace{i\,(K_1-K_2)}\,\mathrm{sen}(bx)\right]$$

$K_1, K_2 \in \mathbb{C}$

$$= e^{ax}\left(K_3\cos(bx) + K_4\,\mathrm{sen}(bx)\right),$$

al variare di $K_3, K_4 \in \mathbb{R}$.

Es $2u'' - 4u' + 6u = 0 \iff u'' - 2u' + 3u = 0$

$$P(\lambda) = \lambda^2 - 2\lambda + 3 = \left(\lambda - (1-\sqrt{2}i)\right)\left(\lambda - (1+\sqrt{2}i)\right)$$

$\lambda_1 = 1 - \sqrt{2}i$, $\quad \lambda_2 = \overline{\lambda_1} = 1 + \sqrt{2}i \implies$

$u(x) = e^{x}\left(K_3\cos(\sqrt{2}x) + K_4\,\mathrm{sen}(\sqrt{2}x)\right)$ $K_3, K_4 \in \mathbb{R}$

Es. 27.5.1.

$$\lim_{x\to 0^+} \left(\frac{4x}{tg(4x)}\right)^{\frac{1}{x\,sen(3x)}}$$

$$\left(\frac{4x}{tg(4x)}\right)^{\frac{1}{x\,sen(3x)}} = e^{\log\left[\left(\frac{4x}{tg(4x)}\right)^{\frac{1}{x\,sen(3x)}}\right]}$$

$$= e^{\boxed{\frac{1}{x\,sen(3x)} \log\left(\frac{4x}{tg(4x)}\right)}}$$

Devo trovare

$$(27.5.1) \quad \lim_{x\to 0^+} \frac{1}{x\,sen(3x)} \log\left(\frac{4x}{tg(4x)}\right)$$

e il risultato finale sarà (Lemma Campos.)

$$e^{(27.5.1)}$$

$$sen\, t = t - t^3/6 + Resto\,(Lag)$$

$$sen(3x) = 3x - \frac{27}{6}x^3 + Resto\,(Lag)$$

$$x\,sen(3x) = 3x^2 - \frac{27}{6}x^4 + x\,Resto\,(Lag)$$

$$\operatorname{tg} y = y + 2\, y^3/3! + \text{Resto (Lag)}$$
$$= y + \frac{y^3}{3} + \text{Resto (Lag)}$$

$$\operatorname{tg}(4x) = 4x + \frac{64}{3}x^3 + \text{Resto (Lag)}$$

$$\frac{4x}{\operatorname{tg}(4x)} = \frac{4x}{4x + \frac{64}{3}x^3 + \text{Resto (Lag)}}$$

$$= \frac{1}{1 + \frac{16}{3}x^2 + \frac{1}{4x}\text{Resto (Lag)}} = (27.6.1)$$

$$\frac{1}{1+t} = f(t) = 1 - t + t^2 - t^3 + \text{Resto (Lag)}$$

$$f'(t) = -\frac{1}{(1+t)^2}, \quad f'(0) = -1$$

$$f''(t) = \frac{2}{(1+t)^3}, \quad f''(0) = 2$$

$$f'''(t) = -\frac{6}{(1+t)^4}, \quad f'''(0) = -6$$

$$(27.6.1) = 1 - \left(\frac{16}{3}x^2 + \frac{1}{4x}\text{Resto}\right) + \left(\frac{16}{3}x^2 + \frac{1}{4x}R\right)^2 + \text{Resto}(\ldots)$$

$$= 1 - \frac{16}{3}x^2 + \ldots\ldots$$

$$\log\left(\frac{4x}{\operatorname{tg}(4x)}\right) = \log\left(1 - \frac{16}{3}x^2 + \dots\right)$$

$$\log(1+w) = w - \frac{w^2}{2} + \text{Resto}$$

$$\log\left(1 - \frac{16}{3}x^2 + \dots\right) = -\frac{16}{3}x^2 + \dots - \frac{1}{2}\left(\frac{16}{3}x^2 + \dots\right)^2 + \text{Resto}$$

$$= -\frac{16}{3}x^2 + \dots$$

$$\boxed{\frac{1}{x \operatorname{sen}(3x)} \log\left(\frac{4x}{\operatorname{tg}(4x)}\right)} = \frac{-\frac{16}{3}x^2 + \dots}{3x^2 + \dots}$$

$$\lim_{x \to 0^+} \boxed{} = -\frac{16}{9}$$

Risultato finale : $e^{-16/9}$