Lezione 28 5/12/18 10-13

$$\lim_{x\to 0} \frac{e^{x}-e^{\operatorname{sen} x}}{\operatorname{tg} x - x}.$$

Sviluppiamo fino all' ordine 3 incluso

$$e^{x} = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+o(x^3) \quad \text{per } x\to 0$$

$$\operatorname{sen} x = x-\frac{x^3}{6}+o(x^3) \quad \text{per } x\to 0$$

$$e^{\operatorname{sen} x} = 1+\left(x-\frac{x^3}{6}+o(x^3)\right)+\frac{1}{2}\left(x-\frac{x^3}{6}+o(x^3)\right)^2$$
$$+\frac{1}{6}\left(x-\frac{x^3}{6}+o(x^3)\right)^3+o(x^3) \quad \text{per } x\to 0$$

$$\operatorname{tg} x = x+\frac{x^3}{3}+o(x^3) \quad \text{per } x\to 0$$

$$\operatorname{tg} x - x = \frac{1}{3}x^3+o(x^3) \overset{\text{per } x\to 0}{\sqrt{}} \Longrightarrow$$ «andare all' ordine 3 almeno anche al numeratore»

$$e^{\operatorname{sen} x} = 1+x+x^2\left[\frac{1}{2}\right]+x^3\left[-\frac{1}{6}+\frac{1}{6}\right]+o(x^3), \text{ per } x\to 0$$

$$\Longrightarrow \frac{e^{x}-e^{\operatorname{sen} x}}{\operatorname{tg} x - x} = \frac{x^3/6+o(x^3)}{\frac{1}{3}x^3+o(x^3)} \quad \text{per } x\to 0$$

$\Longrightarrow$ il limite cercato è $3/6 = 1/2$

1) $\dfrac{1}{\operatorname{tgh}(1/x)}$ $\quad \dfrac{7^x-1}{x}$ $\quad \dfrac{\cos\sqrt{x}-\sqrt{e}^{\sin x}}{(\log(1+2\sqrt{x}))^2}$

$\operatorname{tgh}(1/x) \longrightarrow 1$ per $x \to 0^+$

Se $a>0$ $\quad \dfrac{a^x-1}{x} \xrightarrow[x\to 0^+]{} ?$

$$\frac{a^x-1}{x} = \frac{e^{x\log a}-1}{x}$$

$$\Rightarrow \lim_{x\to 0^+} \frac{a^x-1}{x} \underset{\uparrow}{=} \lim_{x\to 0^+} \frac{d}{dx}(a^x-1)\Big|_{x=0}$$

teo 25.10.1

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\log a} = \log a \, e^{x\log a}$$

$$\Rightarrow \frac{d}{dx}a^x\Big|_{x=0} = \log a$$

$$\Rightarrow \frac{a^x-1}{x} \xrightarrow[x\to 0^+]{} \log a$$

Altro modo: Taylor

Resta $\displaystyle\lim_{x\to 0^+} \frac{\cos\sqrt{x}-e^{\frac{\sin x}{2}}}{(\log(1+2\sqrt{x}))^2}$

$$\frac{\log\left(1+2\sqrt{x}\right)}{2\sqrt{x}} \xrightarrow[x\to 0^+]{} 1$$

$$\frac{\log(1+z)}{z} \xrightarrow[z\to 0]{} 1 \qquad \frac{\frac{1}{1+z}}{1} \longrightarrow 1$$

$$\frac{\cos\sqrt{x} - e^{\frac{1}{2}\operatorname{sen} x}}{\left(2\sqrt{x}\right)^2} \quad \underbrace{\frac{\left(2\sqrt{x}\right)^2}{\left(\log\left(1+2\sqrt{x}\right)\right)^2}}$$

$$\downarrow x\to 0^+$$

$$1$$

L'unica cosa da fare restante è:

$$\lim_{x\to 0^+} \frac{\cos\sqrt{x} - e^{\frac{1}{2}\operatorname{sen} x}}{4x}$$

$$\cos y = 1 - \frac{y^2}{2} + o(y^2) \quad \text{per } y\to 0$$

$$\cos\sqrt{x} = 1 - \frac{x}{2} + o(x)$$

$$\operatorname{sen} x = x - \frac{x^3}{6} + o(x^3) \quad \text{per } x\to 0$$

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + o(t^3) \quad \text{per } t\to 0$$

$$e^{\frac{1}{2}\operatorname{sen} x} = 1 + \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^3)\right) + \frac{1}{2}\left[\frac{1}{2}\left(x - \frac{x^3}{6} + o(x^3)\right)\right]^2$$

$$+ o(x^2)$$

$$\frac{\cos\sqrt{x} - e^{\frac{1}{2}\operatorname{sen} x}}{4x} = \frac{1 - \frac{x}{2} - 1 - \frac{1}{2}x + o(x)}{4x}$$

$$= -\frac{1}{4} + o(x) \longrightarrow -\frac{1}{4}$$

Limite finale:

$$1 \cdot \log 7 \cdot \left(-\frac{1}{4}\right) = -\frac{1}{4}\log 7.$$

2) $$\sum_{n=1}^{+\infty}\left(\frac{(-1)^n}{4} + c\right)^n, \quad c > \frac{1}{4}$$

$$\Longrightarrow \frac{(-1)^n}{4} + c > 0 \quad \forall n \in \mathbb{N}$$

$\Longrightarrow$ La serie è a termini positivi.

Se $c \geq \frac{3}{4}$ $\quad a_n := \frac{(-1)^n}{4} + c$

≥ 1

se n è pari

Dunque se $c \geq \frac{3}{4}$ (a_n) non è infinitesima e dunque la serie non può convergere

Resta da discutere

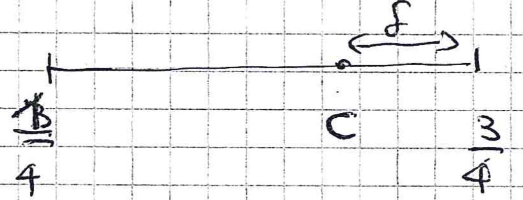
$$\frac{1}{4} < c < \frac{3}{4}$$

$$\sqrt[n]{a_n} = \frac{(-1)^n}{4} + c \leq q < 1$$

per un qualche q:

$$\frac{(-1)^n}{4} + c \leq \frac{1}{4} + c \overset{?}{\leq} q$$

$$c = \frac{3}{4} - \delta, \qquad \delta \in \left(0, \tfrac{1}{2}\right)$$



$$\frac{1}{4} + c = 1 - \delta =: \underset{\underset{1}{\wedge}}{q} = 1 - \left(\frac{3}{4} - c\right)$$

Dunque $\exists q < 1 : \forall n \in \mathbb{N} \quad \sqrt[n]{a_n} \leq q$

$\Longrightarrow$ la serie converge.

<u>Nota</u>: $\nexists \lim_{n \to +\infty} \sqrt[n]{a_n}$

4) $$\lim_{x\to 0^-} F(x) = c + \int_{-1}^{0} \frac{t}{t^2+t-2}\,dt$$

Infatti $$\phi(x) = \int_{-1}^{x} \underbrace{\frac{t}{t^2+t-2}}_{\text{si annulla in } -2,\,1}\,dt$$

$$\Longrightarrow \frac{t}{t^2+t-2} \in \mathcal{C}^0((-2,0])$$

$$\Longrightarrow \phi \in \mathcal{C}^1((-2,0])$$

$$\Longrightarrow \phi \in \mathcal{C}^0((-2,0])$$

$$\Longrightarrow \lim_{x\to 0^+} \phi(x) = \phi(0)$$

$$\lim_{x\to 0^+} F(x) = \frac{1}{2}\int_0^0 \frac{t}{t^4+1}\,dt = 0$$

(anche $x \longrightarrow \int_0^x \frac{t}{t^4+1}\,dt$ è continua in 0 $= \frac{1}{2}\operatorname{arctg}(x^2)$)

Dunque F è continua in 0 se

$$C + \int_{-1}^{0} \frac{t}{t^2+t-2}\,dt = 0$$

$$\Longrightarrow C = \int_0^{-1} \frac{t}{t^2+t-2}\,dt = -\int_{-1}^{0} \frac{t}{t^2+t-2}\,dt$$

$$F'(x) = \begin{cases} \dfrac{x}{x^2+x-2} & x \in (-2,0) \\ -\dfrac{1}{2}\,\dfrac{x}{x^4+1} & x \in [0,+\infty) \end{cases}$$

$$\lim_{x\to 0^-} F'(x) = 0 = \lim_{x\to 0^+} F'(x) \Rightarrow$$

$\exists\ F'(0) = 0$ (Conseguenza di De l'Hopital 25.10.1 applicato a $\frac{F(x)-F(x_0)}{x-x_0}$)

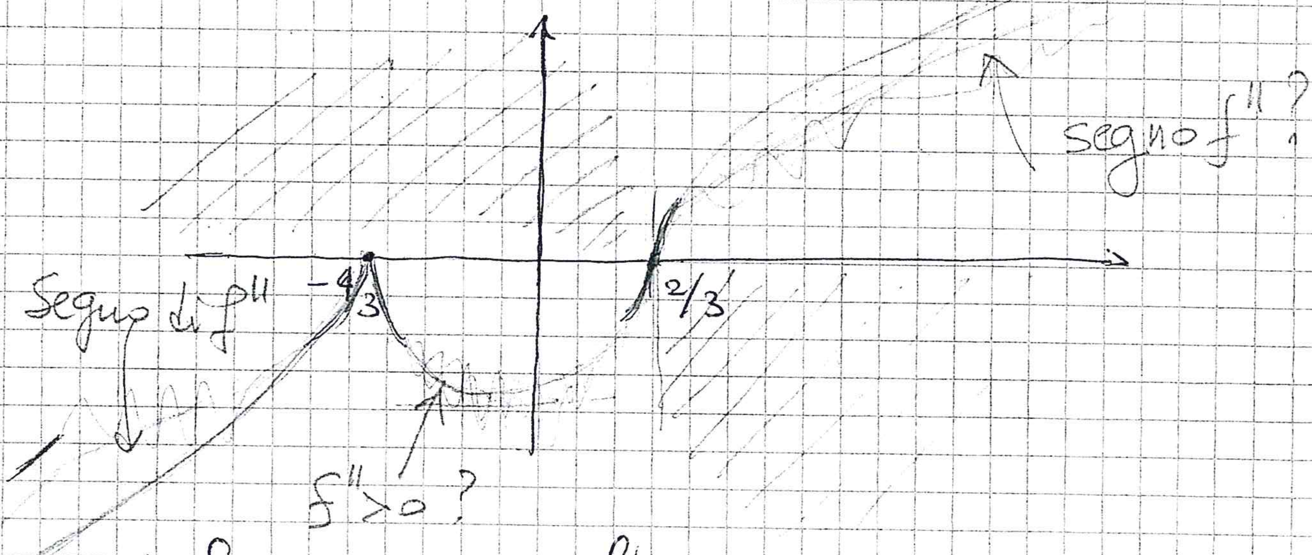
3) $f(x) = \frac{1}{3}\left((3x+4)^2(3x-2)\right)^{1/3}$

$dom(f) = \mathbb{R}$; $f \in C^0(\mathbb{R})$, $f \in C^\infty(\mathbb{R} \setminus \{-\frac{4}{3}, \frac{2}{3}\})$

$f = 0 \iff x = -\frac{4}{3}$, $x = 2/3$;

Se $x > 2/3 \Rightarrow f(x) > 0$

Se $x < 2/3 \Rightarrow f(x) < 0$



$\sup f = +\infty = \lim_{x\to+\infty} f(x) \Rightarrow$ non ci sono max assoluti

$\inf f = -\infty = \lim_{x\to-\infty} f(x) \Rightarrow$ non ci sono min assoluti

$x = -4/3$ max loc stretto

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = 1 \; ; \quad f(x) - x = \overbrace{c}^{=2/3} + o\left(\frac{1}{x}\right)$$

per $x \to +\infty$

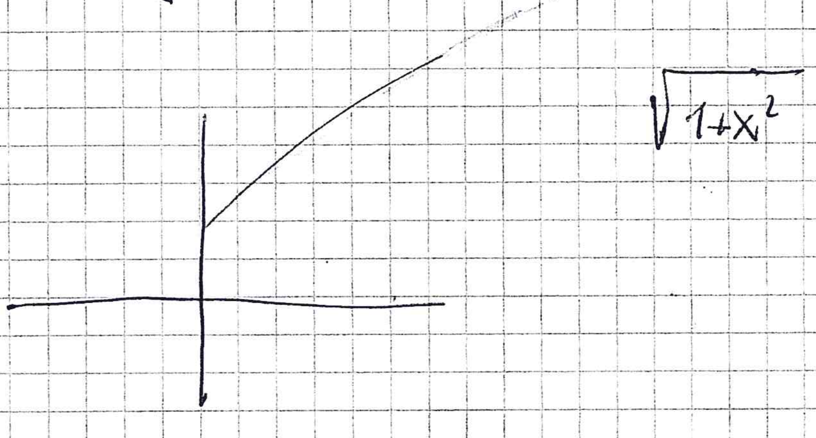
pendenza $\underbrace{\text{asintoto obliquo}}_{\;\to\; y = x + 2/3}$ a $\pm\infty$ è 1

Per $x \neq -\frac{4}{3}, \frac{2}{3}$,

$$f'(x) = \frac{1}{9} \frac{3(6x+8)(3x-2) + 3(3x+4)^2}{\left((3x+4)^2(3x-2)\right)^{2/3}}$$

$$= \frac{1}{9} \frac{\cancel{3}(3x+4)\left[2(3x-2) + (3x+4)\right]}{\left((3x+4)^2(3x-2)\right)^{2/3}}$$

$$= \frac{3(3x+4)\,x}{\left((3x+4)^2(3x-2)\right)^{2/3}}$$



$$\int \frac{t}{t^2+t-2}\,dt = \frac{1}{2}\int \frac{2t}{t^2+t-2}\,dt$$

$$= \frac{1}{2}\int \frac{2t+1}{t^2+t-2}\,dt - \frac{1}{2}\int \frac{1}{t^2+t-2}\,dt$$

$$= \frac{1}{2}\log\left(t^2+t-2\right) + c - \frac{1}{2}\int \frac{1}{t^2+t-2}\,dt$$

$$\int \frac{1}{t^2+t-2}\,dt = \int \frac{1}{\left(t+\frac{1}{2}\right)^2 - \frac{9}{4}}\,dt$$

$$t + \tfrac{1}{2} = \tau \implies \int \frac{1}{\tau^2 - \frac{9}{4}}\,dt$$

$$\int \frac{1}{x^2-a^2}\,dx \sim -\int \frac{-1}{x^2-1}\,dx = -\int \frac{1}{1-x^2}\,dx$$

$$\|$$

$$\frac{1}{a^2}\int \frac{1}{\left(\frac{x}{a}\right)^2-1}\,dx \qquad \frac{x}{a} = y$$

$$\frac{1}{a^2}\, a \int \frac{1}{y^2-1}\,dy$$

legato all' inversa della tgh

$$\int \frac{P(x)}{Q(x)}\,dx \qquad P, Q \text{ polinomi}$$

a coefficienti reali. A meno di dividere P e Q, posso supporre $gr(P) < gr(Q)$ (unico caso interessante ai fini del calcolo dell'integrale).

<u>ES</u> 28.10.1 $P(x) = x$, $Q(x) = x^2 + x - 2$

Si trovano tutte le radici di $Q(x)$:

$r_1, \dots, r_k$ radici reali, con molteplicità $m_1, \dots, m_k$.

$z_1, \dots, z_h$ radici complesse di molteplicità $\mu_1, \dots, \mu_h$

(anche $\overline{z_1}, \dots, \overline{z_h}$ sono radici, di molteplicità $\mu_1, \dots, \mu_h$)

<u>ES</u> 28.10.2 $r_1 = -2$, $r_2 = 1$

$m_1 = 1$, $m_2 = 1$

non ci sono radici complesse

TEO 28.11.1

$\exists$ E possibile scrivere allora, per opportune $a_{ij}, b_{rn}, c_{rs} \in \mathbb{R}$ unici

$$\frac{P(x)}{Q(x)} = \frac{a_{11}}{x-r_1} + \dots + \frac{a_{1m_1}}{(x-r_1)^{m_1}}$$

$$+ \dots + \frac{a_{k1}}{x-r_k} + \dots + \frac{a_{km_k}}{(x-r_k)^{m_k}}$$

$$+ \frac{b_{11}x + c_{11}}{(x-z_1)(x-\bar{z}_1)} + \dots + \frac{b_{1\mu_1}x + c_{1\mu_1}}{[(x-z_1)(x-\bar{z}_1)]^{\mu_1}}$$

$$+ \dots + \frac{b_{h1}x + c_{h1}}{(x-z_h)(x-\bar{z}_h)} + \dots + \frac{b_{h\mu_h}x + c_{h\mu_h}}{[(x-z_h)(x-\bar{z}_h)]^{\mu_h}}$$

ES 28.11.1 $r_1 = -2$, $r_2 = 1$, $m_1 = m_2 = 1$, $K = 2$

$$\frac{x}{x^2+x-2} = \frac{a_{11}}{x+2} + \frac{a_{21}}{x-1} \quad \forall x \in I$$

$$= \frac{a_{11}(x-1) + a_{21}(x+2)}{(x+2)(x-1)}$$

$$x = a_{11}(x-1) + a_{21}(x+2) \quad \forall x \in I$$

$$x(1 - a_{11} - a_{21}) + a_{11} - 2a_{21} = 0 \quad \underline{\forall x \in I}$$

$$\Longrightarrow \begin{cases} 1 - a_{11} - a_{21} = 0 \\ a_{11} - 2a_{21} = 0 \end{cases} \qquad \begin{matrix} a_{21} = 1/3 \\ a_{11} = 2/3 \end{matrix}$$

$$\Rightarrow \quad \frac{x}{x^2+x-2} = \frac{2}{3}\,\frac{1}{x+2} + \frac{1}{3}\,\frac{1}{x-1}$$

$$\Rightarrow \quad \int \frac{x}{x^2+x-2}\,dx = \frac{2}{3}\int \frac{1}{x+2}\,dx + \frac{1}{3}\int \frac{1}{x-1}\,dx$$

$$= \frac{2}{3}\log|x+2| + \frac{1}{3}\log|x-1| + C$$