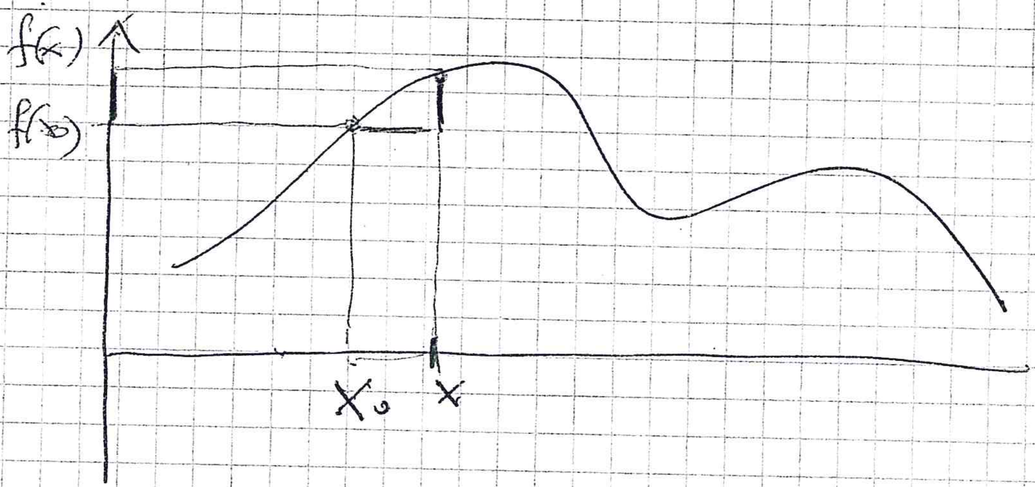
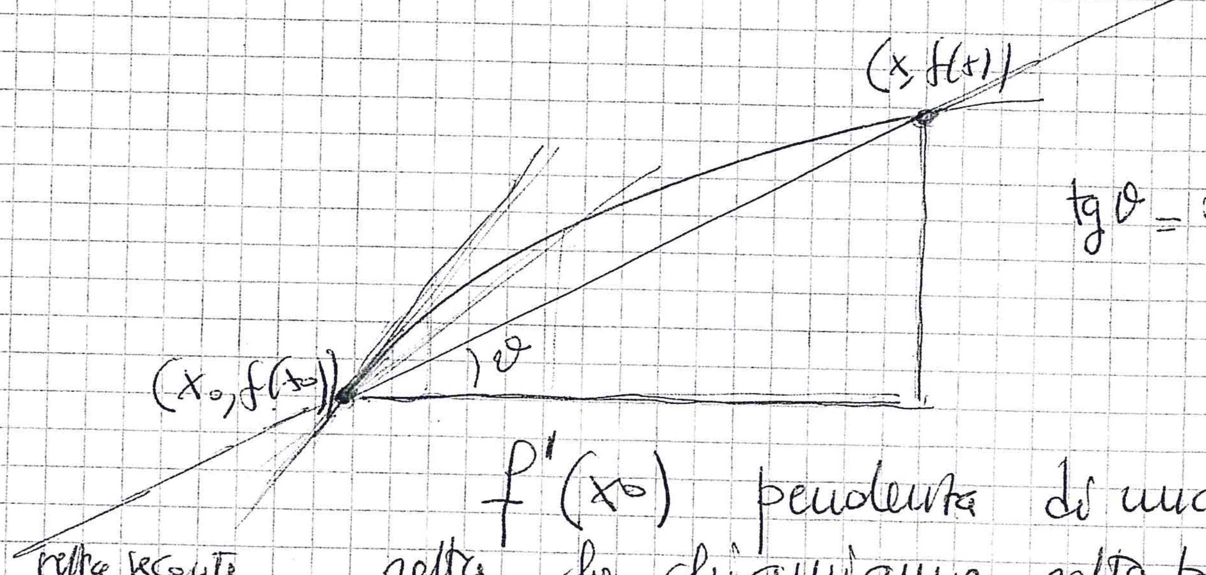
Lezione 22 21/11/18 10–13

OSS 22.1.1. La derivata in x_0 di f tiene conto solo dei valori di f in un intorno di x_0 (x_0 escluso), grazie al Lemma 18.3.1 (localizzazione)

OSS 22.1.2. Interpretazione geometrica



$$\frac{f(x)-f(x_0)}{x-x_0}, \quad x \neq x_0$$



$$tg\,\theta = \frac{f(x)-f(x_0)}{x-x_0}$$

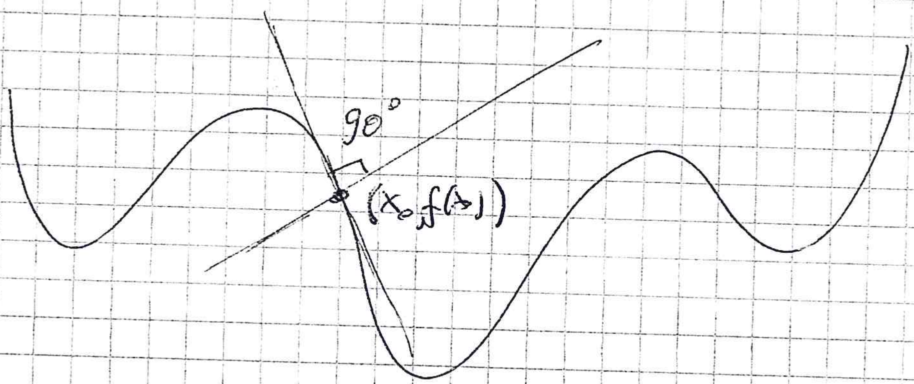
$f'(x_0)$ pendenza di una retta, che chiamiamo retta tangente al grafico di f, in $(x_0, f(x_0))$ ("limite di rette secanti")

Eq. retta tg a grafico di f in $(x_0, f(x_0))$

è definita: $y = f'(x_0)(x - x_0) + f(x_0)$

$(\text{se } \exists f'(x_0) \in \mathbb{R})$ $\quad f(x_0) = y_0, \quad y - y_0 = f'(x_0)(x - x_0)$

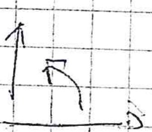
Es 22.4.1 trovare Eq. retta normale a grafico di f in $(x_0, f(x_0))$



Retta tg: $\quad y - f(x_0) - f'(x_0)(x - x_0) = 0$

$$\underbrace{(1, +f'(x_0))}_{"v"} \cdot (y - f(x_0), -x + x_0) = 0$$

$$v^{\perp} = (-f'(x_0), 1)$$



$$v \cdot v^{\perp} = 0$$

Usare $v^{\perp}$ per scrivere eq. retta normale

ES 22.3.1 se $f(x) =$ costante

$$f'(x) = 0$$

se $f(x) = ax+b$ $a \in \mathbb{R}$, $b \in \mathbb{R}$

$$f'(x) = a$$

Se $n \geq 1$ $n \in \mathbb{N}$, $f(x) = x^n$

$\forall x \in \mathbb{R}$

Dico che $f'(x) = nx^{n-1}$. Una possibile dimostrazione:

OSS 22.3.1 Se $f, g : [a,b] \longrightarrow \mathbb{R}$, $x_0 \in [a,b]$, f, g derivabili in x_0

$\Longrightarrow$ fg è derivabile in x_0

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

Infatti $\dfrac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0}$, se $x \in [a,b] \setminus \{x_0\}$, possiamo scrivere

$$= \frac{f(x)g(x) - f(x_0)g(x)}{x - x_0} + \frac{f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0}$$

$$= \underbrace{g(x)\,\frac{f(x)-f(x_0)}{x-x_0}}_{\downarrow\, x\to x_0} + \underbrace{f(x_0)\,\frac{g(x)-g(x_0)}{x-x_0}}_{\downarrow\, x\to x_0}$$

$$g(x_0)\,f'(x_0) \qquad\qquad f(x_0)\,g'(x_0)$$

t. 21.20.1

g deriv. in x_0 $\Rightarrow$ g continua in x_0

f deriv. in x_0

limite (prod) = prod (lim)

g deriv. in x_0

$$\xrightarrow[x\to x_0]{} f'(x_0)g(x_0)+f(x_0)g'(x_0)$$

Vediamo ora che $\boxed{(x^n)' = n\,x^{n-1} \quad \forall n\in\mathbb{N}}$

Se $n=1$ $\quad (x)' = 1$ $\quad$ Sì

Se la formula $\boxed{}$ è falsa, prendo il primo $\nu\in\mathbb{N}$ per cui è falsa ($\mathbb{N}$ è bene ordinato). Si ha $\nu\geq 2$

Dunque $\boxed{}$ è vera per $n=\nu-1 \geq 1$

$$(x^{\nu-1})' = (\nu-1)\,x^{\nu-2}$$

$$x^{\nu} = x\cdot x^{\nu-1} \Rightarrow (x^{\nu})' = x^{\nu-1} + x(\nu-1)x^{\nu-2}$$

$= \nu x^{\nu-1}$, assurdo.

oss. 22.3.1

22.5.1

ES V (cosa) se f, g derivabili in x_0, allora $f+g$ è derivabile in x_0,

$$(f+g)'(x_0) = f'(x_0) + g'(x_0)$$

Dunque se $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
$n \in \mathbb{N} \setminus \{0\}$

$$P'(x) = \left(a_n x^n + a_{n-1} x^{n-1} + \dots + a_0\right)'$$

ES. 22.5.1

$$= (a_n x^n)' + (a_{n-1} x^{n-1})' + \dots + (a_1 x)' + a_0'$$

$$= n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \dots + a_1$$

OSS. 22.3.1
derivata di una costante è 0

ES 22.5.2. Sia $f : [a,b] \to \mathbb{R} \setminus \{0\}$
$x_0 \in [a,b]$, f derivabile in x_0.
Allora $\frac{1}{f}$ è derivabile in x_0,

$$\left(\frac{1}{f}\right)'(x_0) = \frac{-f'(x_0)}{(f(x_0))^2} \qquad (22.5.1)$$

Infatti se $x \in [a,b]$, $x \neq x_0$

$$\frac{\frac{1}{f(x)} - \frac{1}{f(x_0)}}{x - x_0} = \frac{f(x_0) - f(x)}{f(x)f(x_0)(x-x_0)}$$

$$= \underbrace{\frac{-1}{f(x)f(x_0)}}_{\substack{\downarrow x \to x_0 \\ \text{Lemma 21.20.1} \\ -\frac{1}{(f(x_0))^2}}} \cdot \underbrace{\left(\frac{f(x) - f(x_0)}{x - x_0}\right)}_{\substack{\downarrow x \to x_0 \\ f'(x_0)}}$$

(f è deriv. in x_0)

<u>Es</u> 22.5.1 $f: [a,b] \longrightarrow \mathbb{R}$,

f derivabile in $x_0 \in [a,b]$, $g: [a,b] \to \mathbb{R} \setminus \{0\}$

g derivabile in x_0, allora $\frac{f}{g}$ è

derivabile in x_0,

(22.6.1) $$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)}{g(x_0)} - \frac{f(x_0)\, g'(x_0)}{(g(x_0))^2}$$

Se $f(x) \equiv 1$ ho la (22.5.1).

$$\left(f \cdot \frac{1}{g}\right)'(x_0) = f'(x_0)\frac{1}{g(x_0)} + f(x_0)\left(\frac{1}{g}\right)'(x_0)$$

OSS 22.3.1

ES. 22.5.2

$$= \frac{f'(x_0)}{g(x_0)} - \frac{f(x_0)\,g'(x_0)}{(g(x_0))^2}$$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)}{g(x_0)} - \frac{f(x_0)\,g'(x_0)}{(g(x_0))^2}$$

derivate di funzioni elementari:

$$\frac{d}{dx}\operatorname{sen} x = \cos x$$

$$\frac{d}{dx}\cos x = -\operatorname{sen} x$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}(c\,e^x) = c\,e^x, \quad c \in \mathbb{R}$$

$$\frac{d^2}{dx^2}\operatorname{sen} x = -\operatorname{sen} x, \quad \frac{d^2}{dx^2}\cos x = -\cos x$$

ES 22.8.1 La Def. 21.16.1 $\Longleftrightarrow$

$$\exists \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \in \mathbb{R}$$

~~su~~ $E = \{ h \in \mathbb{R} : x_0 + h \in [a,b], h \neq 0 \}$

Verifichiamo $\frac{d}{dx} \operatorname{sen} x = \cos x$

se $h \neq 0$ $\quad \frac{\operatorname{sen}(x+h) - \operatorname{sen} x}{h}$

$$= \frac{\operatorname{sen} x \cos h + \cos x \operatorname{sen} h - \operatorname{sen} x}{h}$$

$$= \frac{\operatorname{sen} x \cos h - \operatorname{sen} x}{h} + \cos x \frac{\operatorname{sen} h}{h}$$

$$= \operatorname{sen} x \underbrace{\left(\frac{\cos h - 1}{h} \right)}_{\downarrow \, 0} + \cos x \underbrace{\frac{\operatorname{sen} h}{h}}_{\downarrow \, 1}$$

$$\Rightarrow \lim_{h \to 0} \frac{\operatorname{sen}(x+h) - \operatorname{sen} x}{h} = \cos x$$

$$\frac{d}{dx} \operatorname{tg} x = 1 + (\operatorname{tg} x)^2$$

$$\frac{d}{dx} \operatorname{senh}(x) = \cosh(x)$$

$$\frac{d}{dx} \cosh(x) = \operatorname{senh}(x)$$

$$\frac{d}{dx} \operatorname{tgh}(x) = 1 - (\operatorname{tgh}(x))^2$$

$$\frac{d}{dx} \frac{x^3 + 2x + 1}{x^4 + 2x^2 + 2} = \frac{3x^2 + 2}{x^4 + 2x^2 + 2} +$$

$$+ (x^3 + 2x + 1) \frac{d}{dx} \left(\frac{1}{x^4 + 2x^2 + 2} \right)$$

$$= \frac{3x^2 + 2}{x^4 + 2x^2 + 2} + (x^3 + 2x + 1) \frac{-(4x^3 + 6x)}{(x^4 + 2x^2 + 2)^2}$$

$$= \frac{3x^2 + 2}{x^4 + 2x^2 + 2} - \frac{(x^3 + 2x + 1)(4x^3 + 6x)}{(x^4 + 2x^2 + 2)^2}$$

<u>Nota</u> $\left(\frac{f}{g}\right)' = \frac{f'}{g} - \frac{fg'}{g^2} = \frac{gf' - fg'}{g^2}$

$$\frac{d}{dx}\frac{\operatorname{sen} x}{\cos x} = \frac{d}{dx}\operatorname{tg} x$$

$$= \frac{\cos x}{\cos x} - \frac{\operatorname{sen} x \frac{d}{dx}\cos x}{(\cos x)^2} \qquad (22.10.0)$$

$$= 1 + \frac{(\operatorname{sen} x)^2}{(\cos x)^2} = 1 + (\operatorname{tg} x)^2$$

$$\frac{d}{dx}\operatorname{tgh}(x) = \frac{d}{dx}\frac{\operatorname{senh}(x)}{\cosh(x)} =$$

$$= \frac{\cosh(x)}{\cosh(x)} - \frac{\operatorname{senh}(x)\frac{d}{dx}\cosh(x)}{(\cosh(x))^2}$$

$$= 1 - (\operatorname{tgh}(x))^2.$$

Manca la derivata della funzione inversa, e la derivata della funzione composta. Questi sono due risultati non banali.

$$\text{«}\ (f\circ g)' = (f'\circ g)\cdot g'\ \text{»} \qquad (22.10.1)$$

$$\text{«}\ \frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)\ \text{»}$$

$$\text{«}\ f(g(x))' = f'(g(x))\, g'(x)\ \text{»}$$

$$\frac{d}{dx} e^{tg\,x}$$

$$\frac{d}{dx} \operatorname{sen}(\cos x)$$

$$\frac{d}{dx} \operatorname{tgh}(x^4 + \cos x)$$

$$e^{tg\,x} = f \circ g(x)$$

$$g: x \in \operatorname{dom}(tg) = \mathbb{R} \setminus \{\pi/2 + k\pi : k \in \mathbb{Z}\} \longrightarrow tg(x)$$

$$f: y \in \mathbb{R} \longmapsto e^y$$

$$\frac{d}{dx} e^{tg\,x} = f'(g(x))\, g'(x)$$

$$f'(y) = e^y$$

$$g'(x) \underset{\uparrow}{=} 1 + (tg(x))^2$$

(22.10.0)

$$\frac{d}{dx} e^{tg\,x} = e^{tg\,x} \left(1 + (tg\,x)^2\right)$$

$$\frac{d}{dx}\, \operatorname{sen}(\cos x)$$

$$g : x \in \mathbb{R} \longmapsto \cos x, \quad f : y \overset{\in \mathbb{R}}{\longmapsto} \operatorname{sen} y$$

$$\frac{d}{dx}\, \operatorname{sen}(\cos x) = -\cos(\cos x)\, \operatorname{sen} x$$

$$\frac{d}{dx}\, \operatorname{tgh}\left(x^4 + \cos x\right) = \left[1 - \left(\operatorname{tgh}(x^4 + \cos x)\right)^2\right] \cdot$$

$$\cdot \left(4x^3 - \operatorname{sen} x\right)$$

$$\frac{d}{dx}\, \underbrace{f \circ g}_{F} \circ h = (f \circ g)' \circ h \; h'$$

$$= (f' \circ g \cdot g') \circ h \; h'$$

$$f(g(h(x)))' = f'(g(h(x)))\, g'(h(x))\, h'(x)$$

$$\frac{d}{dx}\, e^{\operatorname{sen}(\cos x)} = -e^{\operatorname{sen}(\cos x)} \cos(\cos x)\, \operatorname{sen} x$$

$$h : x \in \mathbb{R} \longmapsto \cos x$$

$$g : y \in \mathbb{R} \longmapsto \operatorname{sen} y \qquad f(g(h(x))) = e^{\operatorname{sen}(\cos x)}$$

$$f : z \in \mathbb{R} \longmapsto e^z$$

$$(\underbrace{fgh}_{F})' = F'h + Fh' = f'gh + fg'h + fgh'$$

$$(fg)'' = (f'g + fg')' = f''g + f'g' + f'g' + fg''$$

$$= f''g + 2f'g' + fg''$$

$$(fg)^{(n)} \overset{?}{=} f^{(n)}g + ?f^{(n-1)}g' + ?f^{(n-2)}g'' + \dots$$

$$+ ?f'g^{(n-1)} + fg^{(n)}$$

$n = 3$

$$(fg)''' = ((fg)'')' = (f''g + 2f'g' + fg'')'$$

$$= f'''g + f''g' + 2f''g' + 2f'g'' +$$

$$+ f'g'' + fg'''$$

$$= f'''g + 3f''g' + 3f'g'' + fg'''$$

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

« $f^{-1} \circ f = id \Longrightarrow (f^{-1} \circ f)' = id'$

$$f^{-1}(f(x)) = x \qquad f^{-1}(f(x))' = 1$$

$$\Rightarrow f^{-1\,\prime}(f(x))\, f'(x) = 1 \Longrightarrow$$

$$f^{-1\,\prime}(f(x)) = \frac{1}{f'(x)}$$ »

« Se $f(x) = y$ $\quad x = f^{-1}(y)$

(22.14.1) $$f^{-1\,\prime}(y) = \frac{1}{f'(f^{-1}(y))}$$ »

$$\frac{d}{dy}\log y =$$

$$f : x \in \mathbb{R} \longmapsto e^{x} \in (0, +\infty)$$

$$f^{-1} : y \in (0, +\infty) \longrightarrow \log y$$

$$\frac{d}{dy}\log y \underset{(22.14.1)}{=} \frac{1}{e^{\log y}} = \frac{1}{y}$$

$$\frac{d}{dy}\operatorname{arcsen} y \underset{(22.14.1)}{=} \frac{1}{\cos(\operatorname{arcsen} y)}$$

$$= \frac{1}{\sqrt{1 - (\operatorname{sen}(\operatorname{arcsen} y))^2}} = \frac{1}{\sqrt{1-y^2}}$$

$$(\operatorname{sen}\vartheta)^2 + (\cos\vartheta)^2 = 1 \qquad (\cos\vartheta)^2 = 1 - (\operatorname{sen}\vartheta)^2$$

dove $\cos\vartheta \geq 0$ ho $\cos\vartheta = \sqrt{1 - (\operatorname{sen}\vartheta)^2}$

se $\cos(\quad) > 0$