

# Asymptotic behaviours of a class of threshold models for collective action in social networks

Andrea Garulli, Antonio Giannitrapani

Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche  
Università di Siena  
Via Roma 56, 53100 Siena  
Italy.

## ARTICLE HISTORY

Compiled June 9, 2017

## ABSTRACT

A class of dynamic threshold models is proposed, for describing the upset of collective actions in social networks. The agents of the network have to decide whether to undertake a certain action or not. They make their decision by comparing the activity level of their neighbours with a time-varying threshold, evolving according to a time-invariant opinion dynamic model. Key features of the model are a parameter representing the degree of self-confidence of the agents, and the mechanism adopted by the agents to evaluate the activity level of their neighbours. The case in which a radical agent, initially eager to undertake the action, interacts with a group of ordinary agents, is considered. The main contribution of the paper is the complete characterisation of the asymptotic behaviours of the network, for three different graph topologies. The asymptotic activity patterns are determined as a function of the self-confidence parameter and of the initial threshold of the ordinary agents. Numerical validation on a real ego network shows that the theoretical results obtained for simple graph structures provide useful insights on the network behaviour in more complex settings.

## KEYWORDS

threshold models; opinion dynamics; collective action; consensus; social networks

## 1. Introduction

A key problem in social sciences is that of understanding the complex relationships between the attitude of individuals and their collective behaviour. The main challenge in this context is posed by modelling and analysing the way in which the evolution of the individuals' opinions affects their will of undertaking or not a certain action. In fact, it is widely recognised that this underlying mechanism is at the basis of crucial phenomena, such as the spread of behaviours and the arising of collective actions within social networks (Centola, 2010; Montanari and Saberi, 2010; Siegel, 2009).

Opinion dynamics is a well established research topic in the social science research field, which is receiving increasing attention in the dynamic systems literature (see, e.g., Friedkin, 2015). Starting from the celebrated DeGroot model (DeGroot, 1974) and the numerous variations on it, a large body of contributions has been devoted to the widely

investigated consensus problem, which has an impact also on other key problems in several different fields (see Moreau, 2005; Olfati-Saber et al., 2007; Ren and Beard, 2005; Xiao and Wang, 2006 and references therein). In recent years, researchers have concentrated their attention on models which present a richer variety of possible dynamic patterns, in order to describe the multiplicity of phenomena observed in social networks. Notable research lines along this path are the works on the Hegselmann-Krause model (Blondel et al., 2009; Etesami and Basar, 2015; Jing et al., 2014; Yang et al., 2014), the studies on the spread of misinformation (Acemoglu et al., 2010; Fardad et al., 2012) and the analysis of networks with stubborn agents (Acemoglu et al., 2013; Ghaderi and Srikant, 2014; Ravazzi et al., 2015).

Using opinion dynamics models to predict behaviours of groups of individuals is a key problem in social psychology (Friedkin, 2010). The problem is that to link the evolution of the individuals' opinion to their inclination about undertaking a certain action. In this respect, the simplest approach assumes the existence of a threshold, so that the individual becomes "active" whenever its opinion exceeds the threshold. This leads to the formulation of a so-called *threshold model*.

Threshold models have been first introduced in (Granovetter, 1978); since then, they have been employed to explain the collective behaviour of a community of individuals in many different contexts: typical examples are the spread of technological innovations among large portions of the population; the attitude of masses towards new trends in popular culture; political phenomena such as riots, strikes, and so on. In this context, threshold models are well suited to predict the occurrence of cascade effects, i.e. the possibility that a behaviour adopted by a small number of influential agents will propagate to a large part of the network (Adam et al., 2012). In (Acemoglu et al., 2011b), threshold models are adopted to analyse how innovations spread into a network starting from a set of promoters. Such a model has been later generalised in (Rosa and Giua, 2013) to account for the possibility for a member of the network to abandon a previously adopted innovation. Moreover the effect of the presence of a group of agents which maintain the innovation for a finite time despite the behaviour of their neighbours is analysed.

In (Hassanpour, 2014), a dynamic threshold model has been adopted to describe the mechanisms underlying the formation of a collective action taking place during political unrest or social revolutions. In particular, the aim is to determine whether a radical agent is able to eventually persuade all the individuals of a network to engage in the demonstration: this occurs when the activity level of the individual's neighbours exceeds a certain threshold, which in turn evolves dynamically according to a classical DeGroot opinion model. The author uses this approach to analyse the effect of media interruption during the 2011 Egyptian revolution. Properties of this model have been studied in (Liu et al., 2012).

In this paper, starting from the simple model proposed in (Hassanpour, 2014; Liu et al., 2012), a more general class of dynamic threshold models is proposed and analysed. The main novelty is the introduction of a parameter which represents the relative confidence level that an agent has on her own opinion, with respect to that of her neighbours. This provides a new degree of freedom, which allows one to characterise the behaviour of conservative networks, with respect to groups of individuals more inclined to change their attitude. Another feature of the proposed model is the use of two different mechanisms for deciding whether an agent becomes active or not. A non progressive model is adopted, meaning that each agent can change its actions multiple times, by comparing her current opinion with an indicator of the average activity level of her neighbours. In the proposed model, such an indicator can be either

the fraction of active neighbours (as in Liu et al., 2012), or a weighted average of the number of active neighbours which takes into account the self-confidence of each agent. The considered model can be also seen as an extension of the framework adopted in global games (Dahleh et al., 2013; Morris and Shin, 2003; Touri and Shamma, 2014), in which the agents make a decision upon undertaking an action according to a similar threshold-based mechanism, but once they have made their decision they do not change anymore their activity status.

The main contribution of the paper is a detailed theoretical analysis of the asymptotic behaviour of the network for different graph topologies. In particular, the complete analytic characterisation of the asymptotic activity pattern of the network is determined for three topologies: the complete graph, the star graph and the ring graph. A wide variety of collective limiting behaviours is observed. Moreover, the influence of the self-confidence parameter and of the chosen decision mechanism on the emerging behaviours is highlighted. This is a relevant application of the considered models, as it allows one to explain why certain behaviours become predominant while others disappear or remain latent.

Although the considered structures are admittedly simple, the obtained results are important because they provide a basis for the study of more complex networks, allowing one to predict the arising of interesting phenomena even in settings for which a theoretical analysis would be cumbersome or even impossible. To support this statement, a simulation study on a real ego network is presented in the final part of the paper, showing that the main features of the asymptotic behaviour can be inferred from the theoretical analysis on a network with simpler topology. A preliminary version of this work has been presented in (Garulli et al., 2015).

The paper is organised as follows. In Section 2, the considered class of threshold models is introduced, together with the two different decision schemes. The analytic results on the network asymptotic behaviours are presented in Sections 3, 4 and 5, for the cases of complete, star and ring graph topologies, respectively. Results from numerical simulations carried out for a real ego network are reported in Section 6. Concluding remarks and future developments are provided in Section 7.

## 2. Problem formulation

A network of  $n$  agents is described by an undirected graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V}$  denotes the vertex set and  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  is the edge set. Two agents  $i$  and  $j$  are *neighbours* if  $(i, j) \in \mathcal{E}$ . Let  $\mathcal{N}_i$  denote the set of neighbours of agent  $i$  and  $n_i$  be its cardinality. In this work, an agent is always considered a neighbour of itself, i.e.  $(i, i) \in \mathcal{E}$  for all  $i$ , and the network topology is assumed to be time-invariant.

In order to model the agents' behaviour, two variables are associated to agent  $i$ : the *threshold*  $\theta_i(t) \in [0, 1]$  and the *action*  $a_i(t) \in \{0, 1\}$ . The variable  $a_i$  discriminates whether the  $i$ th agent is undertaking a certain action at time  $t$  ( $a_i(t) = 1$ ) or not ( $a_i(t) = 0$ ). The threshold  $\theta_i(t)$  is used to model the attitude of the  $i$ th agent towards the possibility of becoming active. Depending on the context, it may represent the agent's opinion on a certain topic, or its intention to participate in some collective movement.

The agent behaviour is described by the time evolution of the threshold and the action variables. At each time step, an agent updates its threshold to a weighted

average of its neighbours' threshold

$$\theta_i(t+1) = \sum_{j \in \mathcal{N}_i} f_{ij} \theta_j(t), \quad i = 1, \dots, n, \quad (1)$$

where the weights are such that  $0 < f_{ij} < 1$  and  $\sum_h f_{ih} = 1, \forall i, j$ . Notice that (1) is the classic De Groot model, which has been widely employed in the literature on consensus and opinion dynamics (DeGroot, 1974).

Besides updating the threshold, each agent computes the activity level of its neighbours as

$$p_i(t+1) = \sum_{j \in \mathcal{N}_i} g_{ij} a_j(t), \quad i = 1, \dots, n, \quad (2)$$

where  $0 < g_{ij} < 1$  and  $\sum_h g_{ih} = 1, \forall i, j$ . The action value of each agent is obtained by comparing the activity level  $p_i(t)$  with the threshold  $\theta_i(t)$ , according to

$$a_i(t) = \begin{cases} 1 & \text{if } p_i(t) \geq \theta_i(t) \\ 0 & \text{else} \end{cases}, \quad i = 1, \dots, n. \quad (3)$$

By setting  $f_{ij} = g_{ij} = 0$  whenever  $j \notin \mathcal{N}_i$ , equation (1) can be rewritten in matrix form as  $\theta(t+1) = F\theta(t)$  and (2) becomes

$$p(t+1) = Ga(t), \quad (4)$$

where  $\theta = [\theta_1, \dots, \theta_n]'$ ,  $a = [a_1, \dots, a_n]'$ ,  $p = [p_1, \dots, p_n]'$ , and  $F$  and  $G$  are matrices whose  $ij$ -th entries are  $f_{ij}$  and  $g_{ij}$ , respectively. By introducing the function

$$\phi(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{else,} \end{cases}$$

and exploiting (3),(4), one gets

$$\theta(t+1) = F\theta(t), \quad (5)$$

$$a(t+1) = \phi(Ga(t) - F\theta(t)), \quad (6)$$

where the function  $\phi(\cdot)$  is to be intended componentwise.

In this work, the entries of matrix  $F$  are chosen as

$$f_{ij} = \begin{cases} \frac{\beta}{\beta+n_i-1} & \text{if } i = j, \\ \frac{1}{\beta+n_i-1} & \text{if } j \in \mathcal{N}_i, j \neq i, \\ 0 & \text{else,} \end{cases} \quad (7)$$

where  $\beta > 0$  is the relative weight each agent assigns to its current threshold value compared to that of its neighbours. In other words,  $\beta$  can be interpreted as the *relative confidence* that each agent has on its own opinion, with respect to that of the other members of the network. Two different ways of computing the neighbours' activity level are considered:

- a) *Weighted Activity Level (WAL)*: in this setting  $G = F$ , i.e. the same relative weight is adopted both for computing the activity level of the neighbours and for weighting the neighbours' threshold;
- b) *Uniform Activity Level (UAL)*: in this scenario

$$g_{ij} = \begin{cases} \frac{1}{n_i} & \text{if } j \in \mathcal{N}_i, \\ 0 & \text{else,} \end{cases}$$

so that  $p_i(t+1)$  in (2) represents the fraction of neighbours of agent  $i$  that are active at time  $t$ .

Notice that in the UAL scenario each agent decides whether to become active or not by just “counting” the number of active neighbours. Conversely, in the WAL scenario an agent weights in a different way the fact that its neighbours are active with respect to its own activity status. This is consistent with the idea that a self-confident individual, weighting its own opinion  $\beta$  times that of its neighbours, will also consider in a different way its own behaviour with respect to that of its neighbours.

**Remark 2.1.** If  $\beta = 1$ , one has  $f_{ij} = g_{ij} = \frac{1}{n_i}$ ,  $\forall i, \forall j \in \mathcal{N}_i$ , and hence there is no difference between the two considered scenarios. The threshold update rule (5) consists in computing the average of the neighbours' thresholds, and the activity level (4) is equal to the fraction of active neighbours. Notice that in this special case, the setting considered in (Liu et al., 2012) is recovered.

The objective of this work is to study the asymptotic behaviour of the nonlinear system (5),(6), when the network initially contains one *radical* agent (hereafter labelled with index 1) and  $n - 1$  *ordinary* agents. The radical agent is keen on undertaking an action and would like to convince the other agents to do the same: to this aim, at time  $t = 0$  its threshold is equal to zero and its activity variable is equal to 1. The ordinary agents are initially inactive and their threshold is equal to  $\tau$ , with  $0 < \tau < 1$ . This corresponds to the initial condition

$$\theta(0) = [0, \tau, \dots, \tau]', \quad a(0) = [1, 0, \dots, 0]'. \quad (8)$$

Loosely speaking,  $\tau$  represents the initial reluctance of the ordinary agents towards the action put forth by the radical one. The problem addressed in the paper is to determine the asymptotic value of the action vector

$$a_\infty = \lim_{t \rightarrow +\infty} a(t)$$

under the initial condition (8), as a function of the initial reluctance of ordinary agents  $\tau$  and of the relative confidence parameter  $\beta$ . It is easy to check that  $a_e = 0$  and  $a_e = \mathbb{1}$  (where  $\mathbb{1}$  denotes a vector whose entries are all equal to 1) are always equilibria for system (6), irrespective of the weighting matrices  $F$  and  $G$ <sup>1</sup>. However, several other equilibria may arise, which do depend on the topology of the interconnection network and on the values of  $\beta$  and  $\tau$ . In the next sections, three different topologies will be analysed in detail.

---

<sup>1</sup>With a slight abuse of notation, we assume that when  $p = 0$ , then  $a = 0$ , even if  $\theta = 0$ . We do not modify the definition of  $\phi$  in this sense, to keep notation simple.

### 3. Complete graph

Let the graph  $\mathcal{G}$  be complete, i.e.,  $(i, j) \in \mathcal{E}$ , for all  $i, j$ . The threshold evolution is characterised by the following results.

**Lemma 3.1.** *Consider the dynamic model (5), with  $F$  chosen as in (7). If the interconnection graph is complete and  $\theta(0) = [0, \tau, \dots, \tau]'$ ,  $0 < \tau < 1$ , then*

$$\theta_1(t) = \frac{n-1}{n} \tau (1 - \lambda_c^t), \quad (9)$$

$$\theta_i(t) = \frac{n-1}{n} \tau \left( 1 + \frac{1}{n-1} \lambda_c^t \right), \quad i = 2, \dots, n, \quad (10)$$

where  $\lambda_c = \frac{\beta-1}{\beta+n-1}$ .

**Proof.** When the graph is complete, from (7) one gets

$$\begin{aligned} F &= \frac{\beta}{\beta+n-1} I_n + \frac{1}{\beta+n-1} (\mathbb{1}\mathbb{1}' - I_n) \\ &= \frac{\beta-1}{\beta+n-1} I_n + \frac{n}{\beta+n-1} \frac{\mathbb{1}\mathbb{1}'}{n} \\ &= \frac{\mathbb{1}\mathbb{1}'}{n} + \frac{\beta-1}{\beta+n-1} \left( I_n - \frac{\mathbb{1}\mathbb{1}'}{n} \right). \end{aligned}$$

Then, it is easy to check that, for every  $t \geq 0$ , one has

$$F^t = \frac{\mathbb{1}\mathbb{1}'}{n} + \left( \frac{\beta-1}{\beta+n-1} \right)^t \left( I_n - \frac{\mathbb{1}\mathbb{1}'}{n} \right).$$

For the initial condition  $\theta(0) = [0, \tau, \dots, \tau]'$  one gets

$$\begin{aligned} \theta(t) &= F^t \theta(0) \\ &= \frac{n-1}{n} \tau \mathbb{1} + \left( \frac{\beta-1}{\beta+n-1} \right)^t \left( \theta(0) - \frac{n-1}{n} \tau \mathbb{1} \right) \end{aligned}$$

from which (9)-(10) immediately follow.  $\square$

**Corollary 3.1.** *Consider the dynamic model (5), with the weights chosen as in (7). If the interconnection graph is complete and  $\theta(0) = [0, \tau, \dots, \tau]'$ ,  $0 < \tau < 1$ , then*

$$\lim_{t \rightarrow \infty} \theta(t) = \frac{n-1}{n} \tau \mathbb{1}. \quad (11)$$

Moreover, if  $\beta > 1$ , then

$$\theta_1(t+1) > \theta_1(t), \quad (12)$$

$$\theta_i(t+1) < \theta_i(t), \quad i = 2, \dots, n, \quad (13)$$

for all  $t \geq 0$ .

### 3.1. *Weighted activity level*

Let us consider the WAL setting first, i.e.,

$$G = F = \frac{\beta}{\beta + n - 1} I_n + \frac{1}{\beta + n - 1} (\mathbb{1}\mathbb{1}' - I_n). \quad (14)$$

Define the functions of  $\beta$ :

$$\gamma_1(\beta) = \frac{n}{n-1} \frac{1}{\beta + n - 1}, \quad (15)$$

$$\gamma_2(\beta) = \frac{n}{n-1} \frac{\beta}{\beta + n - 1}, \quad (16)$$

$$\gamma_3(\beta) = \frac{1}{\beta + n - 2}. \quad (17)$$

Such functions are shown in Figures 1-2 for  $n = 5$  and  $n = 20$ , respectively. The following result holds.

**Theorem 3.1.** *System (5),(6), with  $G = F$  given by (14) and initial condition (8), exhibits the following asymptotic behaviours.*

- i) For  $\beta > 1$ ,
  - if  $\tau < \gamma_1(\beta)$ , then  $a_\infty = \mathbb{1}$ ;
  - if  $\tau > \gamma_2(\beta)$ , then  $a_\infty = 0$ ;
  - if  $\gamma_1(\beta) \leq \tau \leq \gamma_2(\beta)$ , then  $a(t) = a(0)$ ,  $\forall t \geq 1$ .
- ii) For  $\beta \leq 1$ , if  $\tau \leq \gamma_3(\beta)$ , then  $a_\infty = \mathbb{1}$ ; otherwise  $a_\infty = 0$ .

**Proof.** According to Lemma 3.1, one has  $\theta_2(t) = \theta_3(t) = \dots = \theta_n(t)$ ,  $\forall t \geq 0$ . Since from (8) it also holds  $a_2(0) = a_3(0) = \dots = a_n(0)$ , this clearly implies  $a_2(t) = a_3(t) = \dots = a_n(t)$ ,  $\forall t \geq 0$ . Hence, in the sequel we will refer only to  $\theta_2(t)$  and  $a_2(t)$ .

i) Given the initial condition (8), from Lemma 3.1 one has

$$\theta_1(1) = \frac{n-1}{\beta + n - 1} \tau, \quad (18)$$

$$\theta_2(1) = \frac{\beta + n - 2}{\beta + n - 1} \tau. \quad (19)$$

Being  $p(1) = Ga(0)$ , one gets

$$p_1(1) = \frac{\beta}{\beta + n - 1},$$

$$p_2(1) = \frac{1}{\beta + n - 1},$$

and hence  $a(1) = \mathbb{1}$  if and only if the following conditions are satisfied

$$\beta \geq (n-1)\tau, \quad (20)$$

$$1 \geq (\beta + n - 2)\tau. \quad (21)$$

Similarly, when both conditions (20)-(21) are violated, one has  $a(1) = 0$ . When  $\frac{1}{\beta+n-2} < \tau \leq \frac{\beta}{n-1}$ , one has  $a(1) = a(0)$ . If  $\beta > 1$ , due to Corollary 3.1 one has that  $\theta_1(t)$  is increasing, while  $\theta_2(t)$  is decreasing. Therefore, one will have  $a(t) = a(0)$  and  $p(t) = p(0)$  until either  $p_1(t) < \theta_1(t)$  or  $p_2(t) \geq \theta_2(t)$ . From (11), both conditions will never occur if the following inequalities hold

$$\frac{\beta}{\beta+n-1} \geq \frac{n-1}{n} \tau, \quad (22)$$

$$\frac{1}{\beta+n-1} \leq \frac{n-1}{n} \tau, \quad (23)$$

which correspond to  $\gamma_1(\beta) \leq \tau \leq \gamma_2(\beta)$ . If  $\tau > \gamma_2(\beta)$ , (22) does not hold: therefore,  $p_1(t) < \theta_1(t)$  for some  $t$  and hence  $a_\infty = 0$ . Similarly, if  $\tau < \gamma_1(\beta)$ , (23) does not hold, thus leading to  $a_\infty = 1$ .

ii) From (20)-(21) we have that  $a(1) = 1$  if and only if  $\tau \leq \min \left\{ \frac{\beta}{n-1}, \frac{1}{\beta+n-2} \right\}$ , while  $a(1) = 0$  if and only if  $\tau > \max \left\{ \frac{\beta}{n-1}, \frac{1}{\beta+n-2} \right\}$ . Since  $\beta \leq 1$  implies  $\frac{\beta}{n-1} \leq \frac{1}{\beta+n-2}$  it remains to discuss the case in which

$$\frac{\beta}{n-1} < \tau \leq \frac{1}{\beta+n-2}, \quad (24)$$

which, according to the above discussion, leads to  $a(1) = [0, 1, \dots, 1]'$ . Hence  $p(2) = Ga(1)$  is such that

$$\begin{aligned} p_1(2) &= \frac{n-1}{\beta+n-1}, \\ p_2(2) &= \frac{\beta+n-2}{\beta+n-1}. \end{aligned}$$

Being from Lemma 3.1

$$\begin{aligned} \theta_1(2) &= \frac{n-1}{n} \tau (1 - \lambda_c^2), \\ \theta_2(2) &= \frac{n-1}{n} \tau \left( 1 + \frac{1}{n-1} \lambda_c^2 \right), \end{aligned}$$

through long but straightforward calculations it is possible to verify that, under the assumptions (24), one has

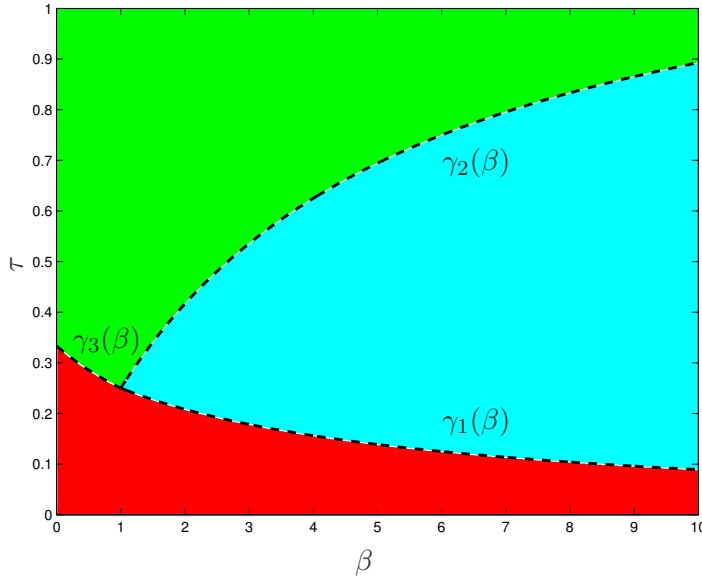
$$p_1(2) \geq \theta_1(2), \quad p_2(2) \geq \theta_2(2).$$

Therefore,  $a(2) = 1$  and one can conclude that  $a_\infty = 1$  for every  $\tau \leq \frac{1}{\beta+n-2} = \gamma_3(\beta)$ .  $\square$

A byproduct of the proof of Theorem 3.1 is the characterisation of the cases for which the asymptotic behaviour is reached in one step.

**Corollary 3.2.** *System (5),(6), with  $G = F$  given by (14) and initial condition (8). satisfies*





**Figure 1.** Complete graph, WAL setting,  $n = 5$ .

- $a(t) = 1$  for all  $t \geq 1$ , if and only if  $\tau \leq \min \left\{ \frac{\beta}{n-1}, \gamma_3(\beta) \right\}$ ;
- $a(t) = 0$  for all  $t \geq 1$ , if and only if  $\tau > \max \left\{ \frac{\beta}{n-1}, \gamma_3(\beta) \right\}$ .

Theorem 3.1 gives the complete characterisation of the asymptotic behaviour of system (5),(6), with initial condition (8). Notice that there are three possible asymptotic activity profiles: i) all the agents become active; ii) all the agents become inactive; iii) the situation remains always the same as in the initial condition (i.e., agent 1 is active and all the others are inactive). From the proof of Theorem 3.1 it is apparent that the asymptotic value  $a_\infty$  is always reached in a finite number of steps. However, it is not possible to give an a priori upper bound to such a number, which can be arbitrarily high. For example, if  $n = 5$ ,  $\tau = 0.99$  and  $\beta = 15$ , one has that  $a(t) = 0$  only for  $t \geq 19$ . Similarly, if  $n = 5$ ,  $\tau = 0.01$  and  $\beta = 118$ , one has that  $a(t) = 1$  only for  $t \geq 56$ .

Figures 1 and 2 show the asymptotic behaviours achieved for different values of  $\beta$  (relative confidence parameter) and  $\tau$  (initial threshold of the ordinary agents), in the cases of 5 and 20 agents, respectively. The different regions correspond to:  $a_\infty = 1$  (red);  $a_\infty = 0$  (green);  $a(t) = a(0)$ ,  $\forall t$  (light blue). The dashed curves represent the functions  $\gamma_i(\beta)$  defined in (15)-(17). Notice that these curves intersect at  $\beta = 1$ . It can be observed that for  $n = 20$  the curves  $\gamma_1(\tau)$  and  $\gamma_3(\tau)$  are almost indistinguishable. As expected, the area in which all the agents end up to be inactive grows with  $n$ , while the region in which all the agents become active tends to shrink, as well as that in which the initial condition  $a(0)$  is maintained indefinitely.

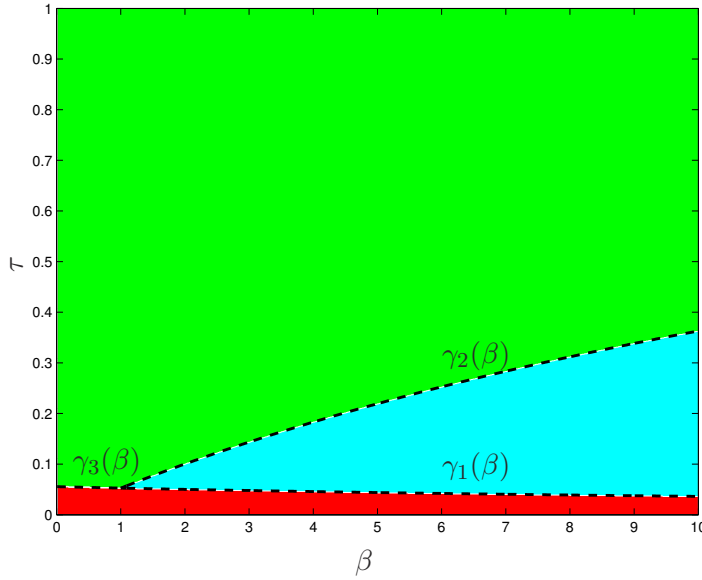


Figure 2. Complete graph, WAL setting,  $n = 20$ .

### 3.2. Uniform activity level

Now, let us consider setting UAL. When the interconnection graph is complete, this means that

$$G = \frac{\mathbb{1}\mathbb{1}'}{n}, \quad (25)$$

and  $F$  is given by (14). Let us define the function

$$\eta(\beta) = \frac{\beta + n - 1}{n(\beta + n - 2)}.$$

Then, the following result holds.

**Theorem 3.2.** *System (5),(6), with  $F$  defined as in (14),  $G$  given by (25) and initial condition (8), has the following asymptotic behaviours:*

- i) For  $\beta > 1$ ,
  - if  $\tau < \frac{1}{n-1}$ , then  $a_\infty = \mathbb{1}$ ;
  - if  $\tau > \frac{1}{n-1}$ , then  $a_\infty = 0$ ;
  - if  $\tau = \frac{1}{n-1}$ , then  $a(t) = a(0)$ ,  $\forall t \geq 1$ .
- ii) For  $\beta \leq 1$ , if  $\tau \leq \eta(\beta)$ , then  $a_\infty = \mathbb{1}$ ; otherwise  $a_\infty = 0$ .

**Proof.** i) By following the same reasoning as in the proof of Theorem 3.1, one gets (18)-(19) and, being  $G$  given by (25),  $p_1(1) = p_2(1) = \frac{1}{n}$ . Hence,  $a(1) = \mathbb{1}$  if and only

if the following conditions are satisfied

$$\begin{aligned}\frac{1}{n} &\geq \frac{n-1}{\beta+n-1} \tau, \\ \frac{1}{n} &\geq \frac{\beta+n-2}{\beta+n-1} \tau,\end{aligned}$$

while  $a(1) = 0$  if and only if both conditions are violated. When  $\frac{\beta+n-1}{n(\beta+n-2)} < \tau \leq \frac{\beta+n-1}{n(n-1)}$ , one has  $a(1) = a(0)$ . Notice that this can occur only if  $\beta > 1$ , which means that (12)-(13) in Corollary 3.1 hold. Hence,  $a(t) = a(0)$  until either  $p_1(t) < \theta_1(t)$  or  $p_2(t) \geq \theta_2(t)$ , are verified. Since  $p(t+1) = Ga(t) = \frac{1}{n}\mathbb{1}$ , these conditions correspond respectively to

$$\frac{1}{n} < \frac{n-1}{n} \tau (1 - \lambda_c^t), \quad (26)$$

$$\frac{1}{n} \geq \frac{n-1}{n} \tau \left(1 + \frac{1}{n-1} \lambda_c^t\right), \quad (27)$$

where  $\lambda_c = \frac{\beta-1}{\beta+n-1}$  as in Lemma 3.1. Being  $0 < \lambda_c < 1$ , (26)-(27) lead respectively to

$$\begin{aligned}\lambda_c^t &< 1 - \frac{1}{(n-1)\tau}, \\ \lambda_c^t &\leq \frac{1}{\tau} - n + 1.\end{aligned}$$

Hence, one eventually gets  $a(t) = 0$ , for some  $t$ , whenever

$$1 - \frac{1}{(n-1)\tau} > \frac{1}{\tau} - n + 1,$$

which corresponds to  $\tau > \frac{1}{n-1}$ . Conversely, if  $\tau < \frac{1}{n-1}$ , (27) will occur before (26), thus leading to  $a(t) = \mathbb{1}$ . Finally, for  $\tau = \frac{1}{n-1}$ , (26)-(27) are never satisfied and therefore  $a(t) = a(0)$  indefinitely.

ii) Let  $\beta \leq 1$ . Similarly to the proof of item ii) in Theorem 3.1, it is possible to show that if

$$\frac{\beta+n-1}{n(n-1)} < \tau \leq \frac{\beta+n-1}{n(\beta+n-2)}$$

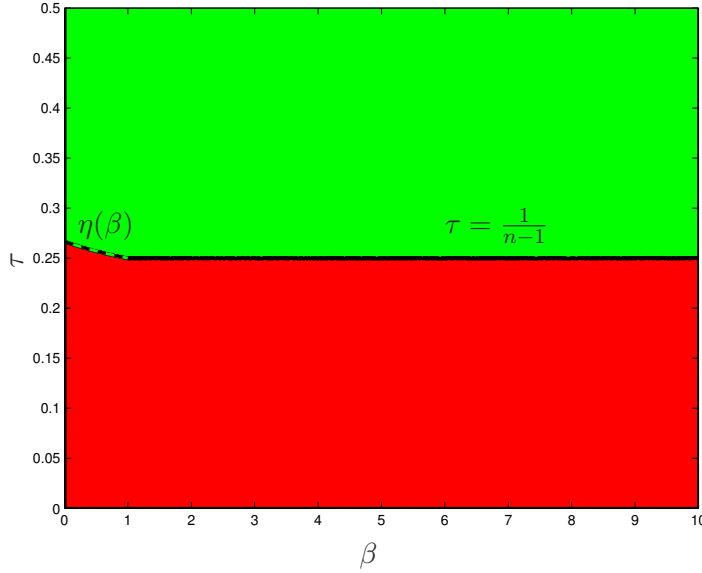
one gets  $a(1) = [0, 1, \dots, 1]'$  and, after long but straightforward manipulations,

$$p_1(2) \geq \theta_1(2), \quad p_2(2) \geq \theta_2(2).$$

Therefore,  $a(2) = \mathbb{1}$  and hence  $a_\infty = 1$  for every  $\tau \leq \frac{\beta+n-1}{n(\beta+n-2)} = \eta(\beta)$ .  $\square$

**Corollary 3.3.** *System (5),(6), with  $F$  defined as in (14),  $G$  given by (25) and initial condition (8), satisfies*

$$- a(t) = \mathbb{1} \text{ for all } t \geq 1, \text{ if and only if } \tau \leq \min \left\{ \frac{\beta+n-1}{n(n-1)}, \eta(\beta) \right\};$$



**Figure 3.** Complete graph, UAL setting,  $n = 5$ .

$$- a(t) = 0 \text{ for all } t \geq 1, \text{ if and only if } \tau > \max \left\{ \frac{\beta+n-1}{n(n-1)}, \eta(\beta) \right\}.$$

Figures 3 and 4 show the curve  $\eta(\beta)$  in the  $\tau - \beta$  plane (dashed), along with the horizontal line  $\tau = \frac{1}{n-1}$  (solid), for  $n = 5$  and  $n = 20$  respectively. Notice that in the latter case, the two lines are almost coincident. Also in this setting, the area in which all the agents end up to be inactive grows with  $n$  (notice the scale on  $\tau$ ), while the region in which all the agents become active tends to shrink.

It is worth observing that when  $\beta = 1$ , matrix  $G$  is the same in both the WAL and the UAL setting, so that conditions in Theorems 3.1 and 3.2 coincide. In this case, the scenario addressed in (Liu et al., 2012) is recovered. In particular, from Corollaries 3.2 and 3.3 it turns out that only two situations occur: either  $a(1) = 1$  if  $\tau \leq \frac{1}{n-1}$ , or  $a(1) = 0$  otherwise. Hence, the steady state behaviour is always achieved in one step. The introduction of the parameter  $\beta$ , accounting for the relative confidence of each agent on its own opinion, has significantly enriched the picture of possible asymptotic behaviours of the system. For  $\beta > 1$ , three new different situations appear in the WAL setting: all the agents eventually become active; all the agents eventually become inactive; the initial situation is maintained indefinitely. As  $\beta$  increases, the latter situation occurs for a larger range of values of the initial threshold  $\tau$ . This corresponds to the fact that in a network whose agents are more self-confident, it is more difficult to persuade them to change their status. Conversely, for  $\beta < 1$ , this behaviour disappears and either 1 is reached (in one or two steps) or all the agents become inactive in one step.

Another interesting observation concerns the differences between the WAL and UAL scenarios. The same five behaviours described above for the WAL setting, are present also in setting UAL, but the condition in which  $a(t) = a(0)$ ,  $\forall t$ , occurs only if  $\tau$  is exactly equal to  $\frac{1}{n-1}$ , which is clearly a singular condition.

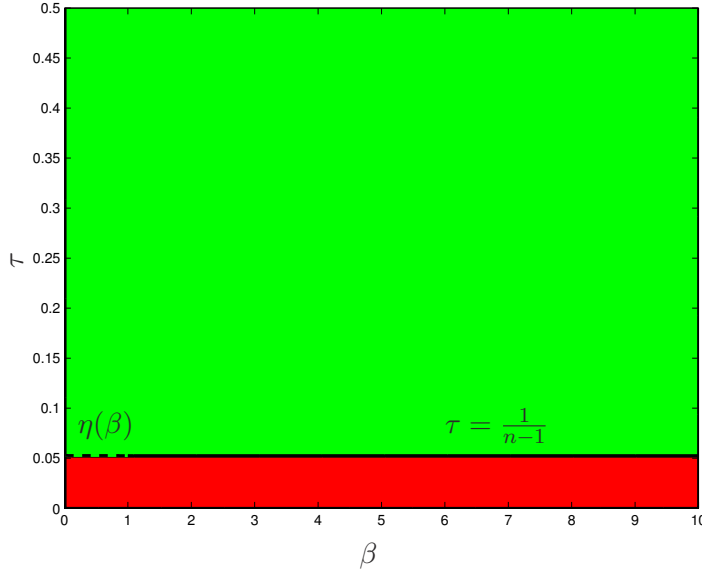


Figure 4. Complete graph, UAL setting,  $n = 20$ .

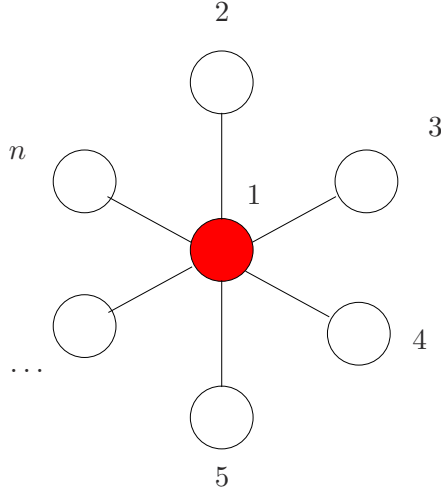
#### 4. Star graph

In this section we analyse the asymptotic behaviour of system (5),(6) when the graph has a star structure, as depicted in Figure 5. In the star graph, the radical agent ( $i = 1$ ) is the center of the graph, while the remaining  $n - 1$  ordinary agents are connected only to the radical. This leads to a matrix  $F$  of the form

$$F = \begin{bmatrix} \frac{\beta}{\beta+n-1} & \frac{1}{\beta+n-1} & \cdots & \cdots & \frac{1}{\beta+n-1} \\ \frac{1}{\beta+1} & \frac{\beta}{\beta+1} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ \frac{1}{\beta+1} & 0 & \cdots & \cdots & \frac{\beta}{\beta+1} \end{bmatrix} \quad (28)$$

We consider scenarios WAL and UAL, defined as in Section 3. While in the former  $G = F$ , in the latter one has

$$G = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ \frac{1}{2} & 0 & \cdots & \cdots & \frac{1}{2} \end{bmatrix} \quad (29)$$



**Figure 5.** Star graph with radical agent in the center.

Let us define

$$r = \frac{(n-1)(\beta+1)}{\beta+n-1}, \quad (30)$$

$$\lambda_s = \frac{\beta(2\beta+n)}{(\beta+1)(\beta+n-1)} - 1. \quad (31)$$

The following technical results are instrumental to the asymptotic analysis of the star graph interconnection.

**Lemma 4.1.** *Consider the dynamic model (5), with matrix  $F$  given by (28). If  $\theta(0) = [0, \tau, \dots, \tau]'$ ,  $0 < \tau < 1$ , then*

$$\theta_1(t) = \frac{r}{1+r} \tau (1 - \lambda_s^t), \quad (32)$$

$$\theta_i(t) = \frac{r}{1+r} \tau \left( 1 + \frac{1}{r} \lambda_s^t \right), \quad i = 2, \dots, n, \quad (33)$$

where  $r$  and  $\lambda_s$  are given by (30) and (31), respectively.

**Proof.** Let us first observe that, due to the structure of  $F$  in (28), one has that system (5) with the initial condition  $\theta(0) = [0, \tau, \dots, \tau]'$  satisfies  $\theta_2(t) = \theta_3(t) = \dots = \theta_n(t)$ ,  $\forall t \geq 0$ . This implies that one can analyse the behaviour of  $\theta(t)$  by considering the two-dimensional system

$$\begin{aligned} \theta_1(t+1) &= \frac{\beta}{\beta+n-1} \theta_1(t) + \frac{n-1}{\beta+n-1} \theta_2(t) \\ \theta_2(t+1) &= \frac{1}{\beta+1} \theta_1(t) + \frac{\beta}{\beta+1} \theta_2(t). \end{aligned}$$

It is easy to check that the eigenvalues of such system are 1 and  $\lambda_s$  in (31). Moreover, (32)-(33) readily follow from the system response to the initial condition  $\theta(0)$ .  $\square$

**Corollary 4.1.** Consider the dynamic model (5), with matrix  $F$  given by (28). If  $\theta(0) = [0, \tau, \dots, \tau]'$ ,  $0 < \tau < 1$ , then

$$\lim_{t \rightarrow \infty} \theta(t) = \frac{r}{1+r} \tau \mathbb{1}. \quad (34)$$

Moreover, if  $\beta > \sqrt{n-1}$ , then  $\lambda_s > 0$  and

$$\begin{aligned} \theta_1(t+1) &> \theta_1(t), \\ \theta_i(t+1) &< \theta_i(t), \quad i = 2, \dots, n, \end{aligned}$$

for all  $t \geq 0$ .

#### 4.1. Weighted activity level

Let us first, consider the WAL setting, i.e.,  $G = F$  as in (28). Define the functions of  $\beta$ :

$$\delta_1(\beta) = \frac{n\beta + 2(n-1)}{(n-1)(\beta+1)^2}, \quad (35)$$

$$\delta_2(\beta) = \frac{\beta(\beta+1)}{(\beta+n-1)} \delta_1(\beta), \quad (36)$$

$$\delta_3(\beta) = \beta \delta_1(\beta). \quad (37)$$

Such functions are shown in Figures 6-7 for  $n = 5$  and  $n = 20$ , respectively. The following result holds.

**Theorem 4.1.** System (5),(6), with  $G = F$  given by (28) and initial condition (8), exhibits the following asymptotic behaviours.

- i) For  $\beta > \sqrt{n-1}$ ,
  - if  $\tau < \delta_1(\beta)$ , then  $a_\infty = \mathbb{1}$ ;
  - if  $\tau > \delta_2(\beta)$ , then  $a_\infty = 0$ ;
  - if  $\delta_1(\beta) \leq \tau \leq \delta_2(\beta)$ , then  $a(t) = a(0)$ ,  $\forall t \geq 1$ .
- ii) For  $1 \leq \beta \leq \sqrt{n-1}$ , if

$$\left\lceil \log_{\lambda_s^2} \frac{1}{\lambda_s} \left( \frac{1}{\tau} \frac{r+1}{\beta+1} - r \right) \right\rceil \geq \left\lceil \log_{\lambda_s^2} \left( \frac{\beta}{\tau} \frac{r+1}{\beta+1} - r \right) \right\rceil \quad (38)$$

then  $a_\infty = \mathbb{1}$ . Otherwise,  $a_\infty = 0$ .

- iii) For  $\beta < 1$ ,
  - if  $\tau \leq \delta_3(\beta)$ , then  $a_\infty = \mathbb{1}$ ;
  - if  $\tau > \delta_1(\beta)$ , then  $a_\infty = 0$ ;
  - if  $\delta_3(\beta) < \tau \leq \delta_1(\beta)$ , then  $a(t)$  oscillates indefinitely between  $[1, 0, \dots, 0]'$  and  $[0, 1, \dots, 1]'$ .

**Proof.** By using the same argument as in Lemma 4.1, one has  $\theta_2(t) = \theta_3(t) = \dots = \theta_n(t)$ , and, being  $a(0)$  as in (8), also  $a_2(t) = a_3(t) = \dots = a_n(t)$ ,  $\forall t \geq 0$ . Hence, in the

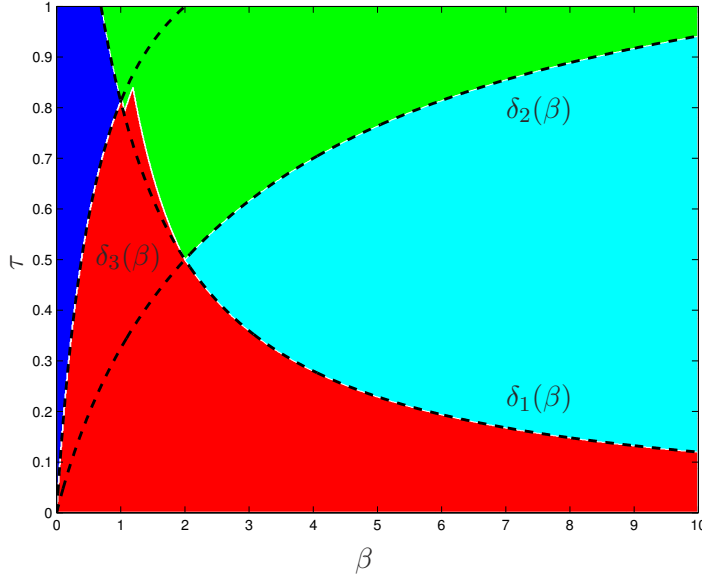


Figure 6. Star graph, WAL setting,  $n = 5$ .

sequel we will refer only to  $\theta_2(t)$  and  $a_2(t)$ .

i) Since  $p(1) = Ga(0)$ , one has

$$p_1(1) = \frac{\beta}{\beta + n - 1},$$

$$p_2(1) = \frac{1}{\beta + 1}.$$

Therefore  $a(1) = \mathbb{1}$  if and only if the following conditions are satisfied

$$\frac{\beta}{\beta + n - 1} \geq \theta_1(1) = \frac{n - 1}{\beta + n - 1} \tau, \quad (39)$$

$$\frac{1}{\beta + 1} \geq \theta_2(1) = \frac{\beta}{\beta + 1} \tau, \quad (40)$$

which are equivalent to

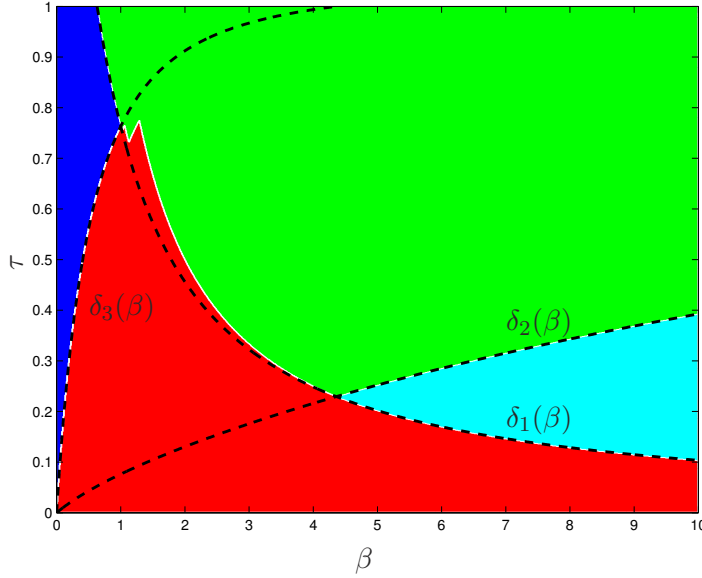
$$\tau \leq \min \left\{ \frac{1}{\beta}, \frac{\beta}{n - 1} \right\}.$$

Conversely,  $a(1) = 0$  if and only if

$$\tau > \max \left\{ \frac{1}{\beta}, \frac{\beta}{n - 1} \right\}.$$

Clearly, if  $\frac{1}{\beta} < \tau \leq \frac{\beta}{n-1}$ , one has  $a(1) = a(0)$ . This can occur only if  $\beta > \sqrt{n-1}$ . Then, according to Corollary 4.1, one has that  $\theta_1(t)$  is monotonically increasing, while  $\theta_2(t)$  is decreasing. Therefore, one will have  $a(t) = a(0)$  until either  $p_1(t) < \theta_1(t)$  or  $p_2(t) \geq \theta_2(t)$ . From (34) and (30), both conditions will never occur if the following





**Figure 7.** Star graph, WAL setting,  $n = 20$ .

inequalities hold

$$\frac{\beta}{\beta + n - 1} \geq \frac{(n-1)(\beta+1)}{n\beta + 2(n-1)} \tau, \quad (41)$$

$$\frac{1}{\beta + 1} \leq \frac{(n-1)(\beta+1)}{n\beta + 2(n-1)} \tau, \quad (42)$$

which correspond to  $\delta_1(\beta) \leq \tau \leq \delta_2(\beta)$ . If (42) is violated, i.e.  $\tau < \delta_1(\beta)$ , one eventually has  $a(\bar{t}) = \mathbb{1}$  for some  $\bar{t}$ . Similarly, if (41) is violated, i.e.  $\tau > \delta_2(\beta)$ , one will have  $a(\bar{t}) = 0$  for some  $\bar{t}$ .

ii) Let  $1 \leq \beta \leq \sqrt{n-1}$ . From the discussion in item i), it remains to analyse the situation in which  $\frac{\beta}{n-1} < \tau \leq \frac{1}{\beta}$ . By comparing with (39)-(40), this assumption implies  $a_1(1) = 0$ ,  $a_2(1) = 1$  and  $p(2) = Ga(1)$  so that

$$p_1(2) = \frac{n-1}{\beta + n - 1},$$

$$p_2(2) = \frac{\beta}{\beta + 1}.$$

Through straightforward manipulations, it is easy to show that for all  $\beta \leq \sqrt{n-1}$  it holds

$$\theta_1(2) = \left( \frac{\beta(n-1)}{(\beta + n - 1)^2} + \frac{\beta(n-1)}{(\beta + n - 1)(\beta + 1)} \right) \tau < \frac{n-1}{\beta + n - 1} \quad (43)$$

which leads to  $a_1(2) = 1$ . On the other hand, if  $\theta_2(2) > \frac{\beta}{\beta+1}$ , one will have  $a_2(2) = 0$  and hence  $a(2) = a(0)$ . Being  $-1 < \lambda_s \leq 0$  (see Corollary 4.1), from (43) and the

violation of the inequality in (39), one has

$$\frac{\beta}{\beta + n - 1} < \theta_1(t) < \frac{n - 1}{\beta + n - 1}$$

for all  $t \geq 1$ . This means that  $a(t)$  will keep oscillating between  $[1, 0, \dots, 0]'$  and  $[0, 1, \dots, 1]'$ , until one of the following conditions is violated

$$\theta_2(t) > \frac{\beta}{\beta + 1} \quad \text{for even } t; \quad (44)$$

$$\theta_2(t) \leq \frac{1}{\beta + 1} \quad \text{for odd } t. \quad (45)$$

Since  $\beta \geq 1$  and  $\theta_2(t)$  converges, it is apparent that both conditions cannot hold indefinitely: therefore,  $a(t)$  will eventually be equal either to  $\mathbb{1}$  or to  $0$ , depending on which condition is violated first. By using (33), it is easy to show that (44) is violated for  $t = 2k$  if

$$k \geq \left\lceil \log_{\lambda_s^2} \left( \frac{\beta}{\tau} \frac{r + 1}{\beta + 1} - r \right) \right\rceil$$

while (45) is violated for  $t = 2k + 1$  if

$$k \geq \left\lceil \log_{\lambda_s^2} \frac{1}{\lambda_s} \left( \frac{1}{\tau} \frac{r + 1}{\beta + 1} - r \right) \right\rceil$$

which leads to condition (38).

iii) If  $\beta < 1$ , by following the same reasoning as in the previous item, one has that (44) and (45) can hold simultaneously for all  $t$ , provided that

$$\frac{\beta}{\beta + 1} < \lim_{t \rightarrow +\infty} \theta_2(t) \leq \frac{1}{\beta + 1}. \quad (46)$$

By using (34), this corresponds to

$$\delta_3(\beta) < \tau \leq \delta_1(\beta).$$

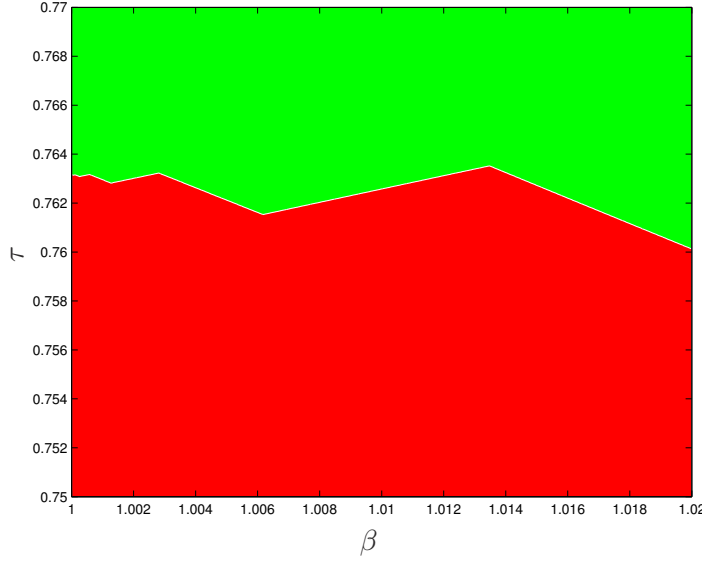
Conversely, if the leftmost inequality in (46) is violated one has  $a_\infty = \mathbb{1}$ , while violation of the rightmost inequality in (46) leads to  $a_\infty = 0$ .  $\square$

The next Corollary, which stems directly from the proof of Theorem 4.1, singles out the cases in which the asymptotic behaviour is achieved in one step.

**Corollary 4.2.** *System (5),(6), with  $G = F$  given by (28) and initial condition (8), satisfies*

- $a(t) = \mathbb{1}$  for all  $t \geq 1$ , if and only if  $\tau \leq \min \left\{ \frac{1}{\beta}, \frac{\beta}{n-1} \right\}$ ;
- $a(t) = 0$  for all  $t \geq 1$ , if and only if  $\tau > \max \left\{ \frac{1}{\beta}, \frac{\beta}{n-1} \right\}$ .

Figures 6-7 show the asymptotic behaviours achieved in scenario WAL for different values of  $\beta$  (relative confidence parameter) and  $\tau$  (initial threshold of the ordinary



**Figure 8.** A detail of the boundary defined by condition (38), for  $n = 20$ .

agents), in the cases of 5 and 20 agents, respectively. The different regions correspond to:  $a_\infty = 1$  (red);  $a_\infty = 0$  (green);  $a(t) = a(0)$ ,  $\forall t$  (light blue);  $a(t)$  switching between  $[1 \ 0 \ \dots \ 0]'$  and  $[0 \ 1 \ \dots \ 1]'$  (blue). The dashed curves represent the functions  $\delta_i(\beta)$  defined in (35)-(37). Notice that  $\delta_1(\beta)$  and  $\delta_2(\beta)$  intersect at  $\beta = \sqrt{n-1}$ , while  $\delta_1(\beta)$  and  $\delta_3(\beta)$  intersect at  $\beta = 1$ : these are the values that distinguish the three different asymptotic scenarios described by Theorem 4.1.

It is worth remarking the particular structure of the boundary defined by (38), which separates the regions in which  $a_\infty = 1$  and  $a_\infty = 0$ , when  $1 \leq \beta \leq \sqrt{n-1}$ . It can be observed that this curve changes its slope an infinite number of times in any interval  $\beta \in (1, 1 + \epsilon)$ , with  $\epsilon$  arbitrarily small. A detail of this behaviour is shown in Figure 8, for  $n = 20$ .

#### 4.2. Uniform activity level

In the UAL setting with the star graph interconnection, one has  $F$  as in (28) and  $G$  given by (29). Let  $r$  and  $\lambda_s$  be given by (30)-(31), and define the function of  $\beta$ :

$$\mu(\beta) = \frac{n\beta + 2(n-1)}{2(n-1)(\beta+1)}. \quad (47)$$

Before proving the main result, let us introduce the following technical lemma.

**Lemma 4.2.** *Let  $n \geq 2$  and  $0 < \beta < \sqrt{n-1}$ . Then,*

$$\frac{\beta + n - 1}{n(n-1)} < \frac{\beta + 1}{2\beta}.$$

**Proof.** One has

$$\frac{\beta + n - 1}{n(n-1)} < \frac{\sqrt{n-1} + n - 1}{n(n-1)} = \frac{1 + \sqrt{n-1}}{n\sqrt{n-1}} \leq \frac{1 + \sqrt{n-1}}{2\sqrt{n-1}} < \frac{\beta + 1}{2\beta}$$

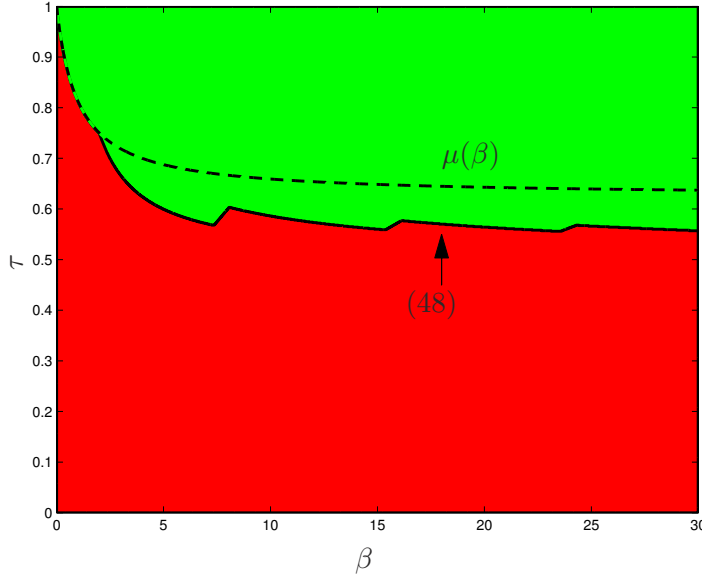


Figure 9. Star graph, UAL setting,  $n = 5$ .

where the latter inequality comes from the fact that  $\frac{\beta+1}{2\beta}$  is a strictly decreasing function of  $\beta$  in the interval  $0 < \beta < \sqrt{n-1}$ .  $\square$

**Theorem 4.2.** *System (5),(6), with  $F$  and  $G$  given by (28) and (29), respectively, and initial condition (8), exhibits the following asymptotic behaviours.*

i) For  $\beta \geq \sqrt{n-1}$ , if

$$\left\lceil \log_{\lambda_s} \left( 1 - \frac{r+1}{r n \tau} \right) \right\rceil \geq \left\lceil \log_{\lambda_s} \left( \frac{1+r}{2\tau} - r \right) \right\rceil \quad (48)$$

then  $a_\infty = \mathbb{1}$ . Otherwise,  $a_\infty = 0$ .

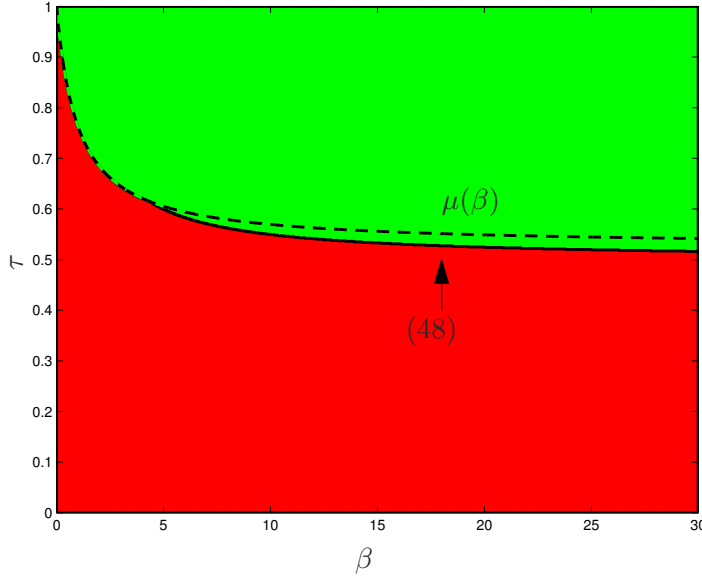
ii) For  $\beta < \sqrt{n-1}$ ,

- if  $\tau < \mu(\beta)$ , then  $a_\infty = \mathbb{1}$ ;
- if  $\tau > \mu(\beta)$ , then  $a_\infty = 0$ ;
- if  $\tau = \mu(\beta)$ , then  $a(t)$  oscillates indefinitely between  $[1, 0, \dots, 0]'$  and  $[0, 1, \dots, 1]'$ .

**Proof.** As in Theorem 4.1, it holds  $\theta_2(t) = \theta_3(t) = \dots = \theta_n(t)$  and  $a_2(t) = a_3(t) = \dots = a_n(t)$ ,  $\forall t \geq 0$ . Hence, we can refer only to  $\theta_2(t)$  and  $a_2(t)$ .

i) Being  $p(1) = Ga(0)$ , one has  $p_1(1) = \frac{1}{n}$  and  $p_2(1) = \frac{1}{2}$ . Therefore  $a(1) = \mathbb{1}$  if and only if the following conditions are satisfied

$$\begin{aligned} \frac{1}{n} &\geq \theta_1(1) = \frac{n-1}{\beta+n-1} \tau, \\ \frac{1}{2} &\geq \theta_2(1) = \frac{\beta}{\beta+1} \tau, \end{aligned}$$



**Figure 10.** Star graph, UAL setting,  $n = 20$ .

which are equivalent to

$$\tau \leq \min \left\{ \frac{\beta + n - 1}{n(n-1)}, \frac{\beta + 1}{2\beta} \right\}.$$

Conversely,  $a(1) = 0$  if and only if

$$\tau > \max \left\{ \frac{\beta + n - 1}{n(n-1)}, \frac{\beta + 1}{2\beta} \right\}.$$

If  $\frac{\beta+1}{2\beta} < \tau \leq \frac{\beta+n-1}{n(n-1)}$ , one has  $a(1) = a(0)$ . Notice that, due to Lemma 4.2, such a  $\tau$  exists only if  $\beta \geq \sqrt{n-1}$ . Hence, according to Corollary 4.1, the thresholds  $\theta_1(t)$  and  $\theta_2(t)$  are, respectively, monotonically increasing and decreasing. Therefore, one will have  $a(t) = a(0)$  until either  $p_1(t) < \theta_1(t)$  or  $p_2(t) \geq \theta_2(t)$ , which, according to (32)-(33), correspond to

$$\frac{1}{n} < \frac{r}{1+r} \tau (1 - \lambda_s^t), \quad (49)$$

$$\frac{1}{2} \geq \frac{1}{1+r} \tau (r + \lambda_s^t), \quad (50)$$

where  $r$  and  $\lambda_s$  are given by (30) and (31). Through straightforward manipulations, (49)-(50) lead respectively to

$$t \geq t_0 \triangleq \left\lceil \log_{\lambda_s} \left( 1 - \frac{r+1}{r n \tau} \right) \right\rceil,$$

$$t \geq t_1 \triangleq \left\lceil \log_{\lambda_s} \left( \frac{1+r}{2\tau} - r \right) \right\rceil.$$

Clearly, if  $t_0 > t_{\perp}$  one has  $a_{\infty} = 1$ , while if  $t_0 < t_{\perp}$ ,  $a_{\infty} = 0$ . In case that  $t_0 = t_{\perp}$ , one gets  $a(t_0) = a(t_{\perp}) = [0 \ 1 \ 1 \ \dots \ 1]'$ , which leads to  $p_1(t_0 + 1) = \frac{n-1}{n}$  and  $p_2(t_0 + 1) = \frac{1}{2}$ . Since  $\theta_2(t)$  is decreasing,  $\theta_2(t_0 + 1) < \theta_2(1) \leq \frac{1}{2}$ . On the other hand  $\theta_1(t)$  is increasing and hence

$$\theta_1(t_0 + 1) < \lim_{t \rightarrow +\infty} \theta_1(t) = \frac{r}{1+r} \tau = \frac{(n-1)(\beta+1)}{n\beta+2(n-1)} \tau \leq \frac{n-1}{n} \quad (51)$$

where it is easy to show that the latter inequality holds for all  $n \geq 2$ , being  $\tau \leq 1$ . Hence  $a(t_0 + 1) = 1$  and  $a_{\infty} = 1$ . This proves condition (48).

ii) Let  $\beta < \sqrt{n-1}$ . Then, due to Lemma 4.2,  $\frac{\beta+n-1}{n(n-1)} < \frac{\beta+1}{2\beta}$ . Therefore, following the reasoning in item i),  $a(1)$  cannot be equal to  $a(0)$ . In particular,  $a(t)$  will keep oscillating between  $[1, 0, \dots, 0]'$  and  $[0, 1, \dots, 1]'$  if the following conditions are satisfied

$$\theta_1(t) > \frac{1}{n}, \quad \theta_2(t) \leq \frac{1}{2} \quad \text{for odd } t; \quad (52)$$

$$\theta_1(t) \leq \frac{n-1}{n}, \quad \theta_2(t) > \frac{1}{2} \quad \text{for even } t. \quad (53)$$

Corollary 4.1 states that both  $\theta_1(t)$  and  $\theta_2(t)$  converge asymptotically to  $\frac{r}{1+r} \tau$ . Moreover, due to (32),  $\theta_1(t)$  is increasing for all even  $t$ . Thanks to (51), one can conclude that the first condition in (53) will hold indefinitely. On the other hand, the second condition in (52) will be eventually violated if and only if

$$\frac{r}{1+r} \tau > \frac{1}{2}$$

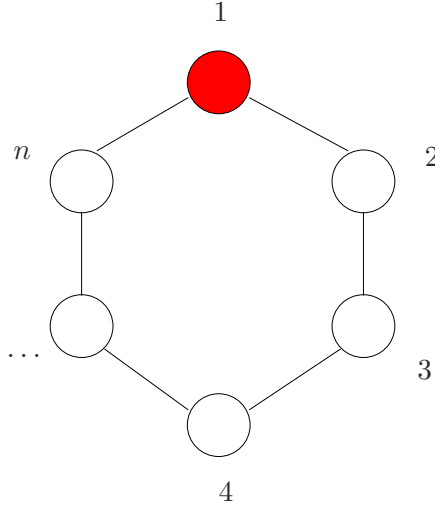
which corresponds to  $\tau > \mu(\beta)$ . This leads to  $a_{\infty} = 0$ . Conversely, if  $\tau < \mu(\beta)$ , either the first condition in (52) or the second condition in (53) will eventually be violated, thus leading to  $a_{\infty} = 1$ . Finally, when  $\tau = \mu(\beta)$ , all conditions (52)-(53) will hold indefinitely, thus leading to oscillations of  $a(t)$  between  $[1, 0, \dots, 0]'$  and  $[0, 1, \dots, 1]'$ .  $\square$

**Corollary 4.3.** *System (5),(6), with  $F$  and  $G$  given by (28) and (29), respectively, and initial condition (8), satisfies*

- $a(t) = 1$  for all  $t \geq 1$ , if and only if  $\tau \leq \min \left\{ \frac{\beta+n-1}{n(n-1)}, \frac{\beta+1}{2\beta} \right\}$ ;
- $a(t) = 0$  for all  $t \geq 1$ , if and only if  $\tau > \max \left\{ \frac{\beta+n-1}{n(n-1)}, \frac{\beta+1}{2\beta} \right\}$ .

Figures 9-10 show the asymptotic behaviours achieved in scenario UAL for different values of  $\beta$  and  $\tau$ , for  $n = 5$  and  $n = 20$ , respectively. Colours have the same meaning as in Figures 6-7. The dashed line represents the function  $\mu(\beta)$  defined in (47), while the solid line corresponds to condition (48).

As for the complete graph, also for the star graph it can be observed that the WAL scenario shows a wider variety of asymptotic behaviours with respect to the UAL scenario. In particular, in the latter the initial activity pattern  $a(0)$  is never maintained indefinitely and the persistent oscillations for  $\beta < 1$  occur only under the singular condition  $\tau = \mu(\beta)$  (while in the WAL scenario they show up for the entire range of  $\tau$  values). Notice that these oscillations are due to the shyness of the agents, which are less confident in their own opinion than in that of their neighbours, thus



**Figure 11.** Ring graph.

leading to persistent switchings between activity and inactivity. Moreover, it can be shown that in the UAL scenario, one has  $a_\infty = \mathbb{1}$  whenever  $\tau \leq \frac{1}{2}$ , irrespective of the number of agents  $n$  and of the confidence parameter  $\beta$ . Conversely, in the WAL scenario, the region in which  $a_\infty = \mathbb{1}$  tends to shrink as either  $n$  or  $\beta$  grow.

## 5. Ring graph

In this section we analyse the asymptotic behaviour of system (5)-(6) when the graph has a ring structure. In the ring graph, agent  $i$  has as neighbours agents  $i - 1$  and  $i + 1$ , with the convention  $n + 1 = 1$  (see Figure 11). We still assume that agent 1 is a radical while the others are ordinary, and we analyse the asymptotic behaviour of the system starting from the initial condition (8). The matrix  $F$  is now given by

$$F = \begin{bmatrix} \frac{\beta}{\beta+2} & \frac{1}{\beta+2} & 0 & \dots & 0 & \frac{1}{\beta+2} \\ \frac{1}{\beta+2} & \frac{\beta}{\beta+2} & \frac{1}{\beta+2} & 0 & \dots & 0 \\ 0 & \frac{1}{\beta+2} & \ddots & \ddots & \dots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & \ddots & \ddots & \frac{1}{\beta+2} \\ \frac{1}{\beta+2} & 0 & \dots & \dots & \frac{1}{\beta+2} & \frac{\beta}{\beta+2} \end{bmatrix} \quad (54)$$

while matrix  $G$  in scenario UAL is

$$G = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \dots & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \dots & 0 \\ 0 & \frac{1}{3} & \ddots & \ddots & \dots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & \ddots & \ddots & \frac{1}{3} \\ \frac{1}{3} & 0 & \dots & \dots & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad (55)$$

In order to simplify the treatment, hereafter only the case in which  $n$  is odd (i.e., the number of ordinary agents is even) will be considered. Moreover, let  $h = \frac{n-1}{2}$ . The following technical result, relying on the properties of circulant matrices (Davis, 1979), provides the analytic expression for the threshold evolution according to equation (5).

**Lemma 5.1.** *Consider the dynamic model (5), with matrix  $F$  given by (54),  $\theta(0) = [0, \tau, \dots, \tau]'$ ,  $0 < \tau < 1$  and odd  $n$ . Then,*

$$\theta(t) = \frac{n-1}{n} \tau \mathbf{1} - \tau \sum_{k=1}^h (\lambda_k)^t v_k, \quad (56)$$

where

$$\lambda_k = \frac{1}{\beta+2} \left( \beta + 2 \cos \left( k \frac{2\pi}{n} \right) \right), \quad (57)$$

$$v_k = \frac{2}{n} \left[ 1 \quad \cos \left( k \frac{2\pi}{n} \right) \quad \dots \quad \cos \left( (n-1)k \frac{2\pi}{n} \right) \right]', \quad (58)$$

for  $k = 1, \dots, h$ , and  $h = \frac{n-1}{2}$ .

**Proof.** Matrix  $F$  in (54) is a circulant matrix, i.e. it has the form

$$F = \begin{pmatrix} f_0 & f_1 & f_2 & \dots & f_{n-1} \\ f_{n-1} & f_0 & f_1 & \dots & f_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_1 & f_2 & f_3 & \dots & f_0 \end{pmatrix},$$

where  $f_0 = \frac{\beta}{\beta+2}$ ,  $f_1 = f_{n-1} = \frac{1}{\beta+2}$  and  $f_i = 0$ , for  $i \neq 0, 1, n-1$ . The eigenvalues and eigenvectors of a circulant matrix can be computed analytically (e.g., see Davis, 1979). Let  $\omega_k = e^{k \frac{2\pi}{n} j}$ , where  $j = \sqrt{-1}$ . The eigenvalues of  $F$  are given by

$$\lambda_k = \sum_{i=0}^{n-1} f_i \omega_k^i = \frac{1}{\beta+2} \left( \beta + 2 \cos \left( k \frac{2\pi}{n} \right) \right), \quad k = 0, \dots, n-1.$$

The eigenvectors of  $F$  are given by

$$u_k = \frac{1}{\sqrt{n}} [1 \quad \omega_k \quad \dots \quad \omega_k^{n-1}]', \quad k = 0, \dots, n-1. \quad (59)$$

and form an orthonormal basis. A circulant matrix can always be diagonalised. Let  $U = [u_0 \ u_1 \ \dots \ u_{n-1}]$ , then  $F = U \Lambda U^*$ , where  $\Lambda = \text{diag}(\lambda_0, \dots, \lambda_{n-1})$  and  $U^*$  is the conjugate transpose of  $U$  (Th. 3.2.1 in Davis, 1979). Observing that  $\lambda_0 = 1$ ,  $\lambda_k = \lambda_{n-k}$ ,



$k = 1, \dots, h$ , and  $u_0 = \frac{1}{\sqrt{n}}\mathbb{1}$ , the evolution of the thresholds  $\theta(t)$  can be written as

$$\begin{aligned}\theta(t) &= U\Lambda^t U^* \theta(0) = \frac{n-1}{n} \tau \mathbb{1} + \sum_{k=1}^h \lambda_k^t (u_k u_k^* + u_{n-k} u_{n-k}^*) \theta(0) \\ &= \frac{n-1}{n} \tau \mathbb{1} - \frac{\tau}{\sqrt{n}} \sum_{k=1}^h \lambda_k^t (u_k + u_{n-k}),\end{aligned}$$

where the last equality comes from  $u_k^* \theta(0) = -\frac{\tau}{\sqrt{n}}$ , since  $u_k^* \mathbb{1} = 0$ ,  $k = 1, \dots, n-1$ . The thesis (56) easily follows by noting that the entries of  $u_k$  in (59) are such that  $\omega_k^l + \omega_{n-k}^l = 2 \cos(l \frac{2\pi}{n})$ , for  $l = 0, \dots, n-1$  and  $k = 1, \dots, h$ .  $\square$

From (56) and (58) it can be checked that  $\theta_{i+1}(t) = \theta_{n-i+1}(t)$ , for  $i = 1, \dots, h$  and for all  $t$ . Hence, due to the structure of matrices  $F$  and  $G$  resulting from the ring interconnection, one will also have  $a_{i+1}(t) = a_{n-i+1}(t)$  and  $p_{i+1}(t) = p_{n-i+1}(t)$ , for  $i = 1, \dots, h$ . The following lemma gives other useful properties that will be instrumental to proving the main result.

**Lemma 5.2.** *Consider the same assumptions as in Lemma 5.1 and let  $\beta \geq 1$ . Then for all  $t \geq 1$  the following statements hold*

- i)  $\theta_i(t) \leq \theta_{i+1}(t)$ , for  $i = 1, \dots, h$ ;
- ii)  $\theta_i(t-1) \leq \theta_{i+1}(t)$ , for  $i = 1, \dots, h$ .

**Proof.** i) By (8), the statement is true at  $t = 0$ . Now, let the statement hold at time  $t$ . Then, recalling that  $\theta_{h+1}(t) = \theta_{h+2}(t)$  for all  $t$ , one has for  $i = 2, \dots, h$

$$\begin{aligned}\theta_{i+1}(t+1) - \theta_i(t+1) &= \frac{1}{\beta+2} \theta_i(t) + \frac{\beta}{\beta+2} \theta_{i+1}(t) + \frac{1}{\beta+2} \theta_{i+2}(t) \\ &\quad - \left( \frac{1}{\beta+2} \theta_{i-1}(t) + \frac{\beta}{\beta+2} \theta_i(t) + \frac{1}{\beta+2} \theta_{i+1}(t) \right) \geq 0,\end{aligned}$$

while, being  $\beta \geq 1$ ,

$$\begin{aligned}\theta_2(t+1) - \theta_1(t+1) &= \frac{1}{\beta+2} \theta_1(t) + \frac{\beta}{\beta+2} \theta_2(t) + \frac{1}{\beta+2} \theta_3(t) - \frac{\beta}{\beta+2} \theta_1(t) \\ &\quad - \frac{2}{\beta+2} \theta_2(t) \geq \frac{\beta-1}{\beta+2} (\theta_2(t) - \theta_1(t)) \geq 0.\end{aligned}$$

Therefore, the claim holds by induction.

ii) By applying the result in item i), for all  $i = 1, \dots, h$ , one has

$$\begin{aligned}\theta_{i+1}(t) &= \frac{1}{\beta+2} \theta_i(t-1) + \frac{\beta}{\beta+2} \theta_{i+1}(t-1) + \frac{1}{\beta+2} \theta_{i+2}(t-1) \\ &\geq \left( \frac{1}{\beta+2} + \frac{\beta}{\beta+2} + \frac{1}{\beta+2} \right) \theta_i(t-1) = \theta_i(t-1).\end{aligned}$$

$\square$

### 5.1. Weighted activity level

Now, we are ready to characterise the asymptotic behaviour in the WAL scenario, for the ring graph interconnection. In order to streamline the presentation, only the case  $\beta \geq 1$  will be treated.

**Theorem 5.1.** *Let  $\mathcal{E}$  be a ring graph,  $F = G$  given by (54) as in the WAL setting,  $n$  odd and  $\beta \geq 1$ . Define*

$$q_j(\beta) = \inf_t \left\{ \frac{n-1}{n} - \frac{2}{n} \sum_{k=1}^h (\lambda_k)^t \cos \left( jk \frac{2\pi}{n} \right) \right\}, \quad j = 1, \dots, h, \quad (60)$$

$$q_0(\beta) = \frac{n-1}{n\beta} \quad (61)$$

where  $\lambda_k$  is given by (57), and set  $\hbar = \lfloor \frac{h}{2} \rfloor$ . Then, system (5),(6), with initial condition (8), exhibits the following asymptotic behaviours:

- i) If  $\tau \leq \frac{n}{n-1} \frac{1}{(\beta+2)}$ , then  $a_\infty = \mathbb{1}$ .
- ii) If

$$\frac{1}{(\beta+2)q_{j+1}(\beta)} \leq \tau \leq \frac{1}{(\beta+2)q_j(\beta)}, \quad (62)$$

for some  $j \in \{0, 1, \dots, \hbar\}$ , then  $a_\infty = \alpha_j$ , given by

$$\alpha_j = \left[ \underbrace{1 \dots 1}_{j+1} \quad 0 \dots 0 \quad \underbrace{1 \dots 1}_j \right]', \quad j = 1, \dots, \hbar. \quad (63)$$

- iii) If  $\tau > \max \left\{ \frac{1}{(\beta+2)q_0(\beta)}, \frac{1}{(\beta+2)q_1(\beta)} \right\}$ , then  $a_\infty = 0$ .

**Proof.** Before proving the single items, let us show by induction that  $p_i(t) \geq p_{i+1}(t)$ , for  $i = 1, \dots, h$  and for all  $t \geq 0$ . Being

$$p(0) = \left[ \frac{\beta}{\beta+2} \quad \frac{1}{\beta+2} \quad 0 \quad \dots \quad 0 \quad \frac{1}{\beta+2} \right]'$$

and  $\beta \geq 1$ , the statement is true at  $t = 0$ . Let  $p_i(t) \geq p_{i+1}(t)$ ,  $i = 1, \dots, h$ , at time  $t$ . Then, being  $\theta_i(t) \leq \theta_{i+1}(t)$  according to item i) in Lemma 5.2, one has that  $a(t) = \alpha_j$  in (63) for some  $j$ , which in turn gives

$$p(t+1) = G\alpha_j = \left[ \underbrace{1 \dots 1}_j \quad \frac{\beta+1}{\beta+2} \quad \frac{1}{\beta+2} \quad \underbrace{0 \dots 0}_{n-2j-3} \quad \frac{1}{\beta+2} \quad \frac{\beta+1}{\beta+2} \quad \underbrace{1 \dots 1}_{j-1} \right]' \quad (64)$$

and hence  $p_i(t+1) \geq p_{i+1}(t+1)$ ,  $i = 1, \dots, h$ .

- i) According to the previous discussion,  $a(t)$  is either equal to  $\mathbb{1}$  or 0, or it takes values  $\alpha_j$  in (63), which then lead to  $p(t+1)$  as in (64). Being  $\lim_{t \rightarrow +\infty} \theta(t) = \frac{n-1}{n}\tau$ , if  $\frac{n-1}{n}\tau < \frac{1}{\beta+2}$ , one has that eventually  $a(t)$  will switch from  $\alpha_{j-1}$  to  $\alpha_j$ , for all

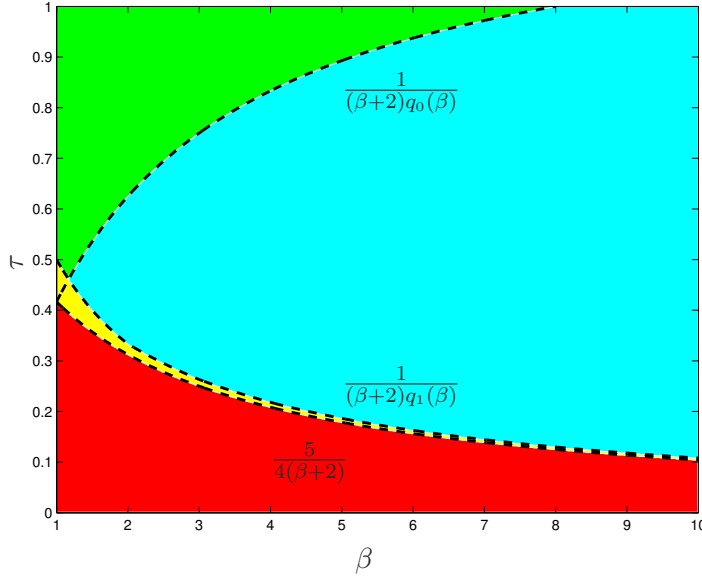


Figure 12. Ring graph, WAL setting,  $n = 5$ .

$j = 1, \dots, h$ , thus leading to  $a_\infty = \alpha_h = \mathbb{1}$ .

ii) Let  $j \in \{1, 2, \dots, h\}$ . From (56) and (60), one gets

$$\inf_t \theta_{j+1}(t) = \tau q_j(\beta).$$

Through some tedious manipulations, it is possible to show that the infimum in (60) is equal to  $\frac{n-1}{n}$  for  $j = h+1, \dots, h$ , and it is approached asymptotically as  $t$  grows to infinity. Conversely, for  $j = 1, \dots, h$ , the infimum satisfies  $q_j(\beta) < \frac{n-1}{n}$  and is attained at some finite  $t_j^*$ . Moreover, being  $\theta_{j+1}(t) = \tau$  for all  $t \leq j-1$ , one obviously has that  $t_j^* \geq j$ . By following the same reasoning as in item i), the switch from  $\alpha_j$  to  $\alpha_{j+1}$  will never occur if  $\tau q_j(\beta) > \frac{1}{\beta+2}$ . Now, let (62) hold. From the rightmost inequality, by applying item ii) in Lemma 5.2, one gets

$$\frac{1}{\beta+2} \geq \theta_{j+1}(t_j^*) \geq \theta_j(t_j^* - 1) \geq \dots \geq \theta_2(t_j^* - j + 1).$$

Being  $t_j^* \geq j$ , one has  $t_j^* - j + 1 \geq 1$ . Hence, all the thresholds  $\theta_i$ ,  $i = 2, \dots, j$  will take a value below  $\frac{1}{\beta+2}$  in successive times, thus guaranteeing that all the switchings from  $\alpha_{i-1}$  to  $\alpha_i$  will eventually occur. This proves that condition (62) leads to  $a_\infty = \alpha_j$ .

iii) If  $\tau > \frac{1}{(\beta+2)q_1(\beta)}$ , one has that the switching from  $a(0) = \alpha_0$  to  $\alpha_1$  never occurs. In such a case, one gets

$$p(t) = Ga(0) = \left[ \frac{\beta}{\beta+2} \quad \frac{1}{\beta+2} \quad 0 \quad \dots \quad 0 \quad \frac{1}{\beta+2} \right]'$$

and  $a(t) = \alpha_0$  indefinitely, unless  $\lim_{t \rightarrow +\infty} \theta_1(t) > \frac{\beta}{\beta+2}$ , for some  $t$ . The latter condition corresponds to  $\frac{n-1}{n}\tau > \frac{\beta}{\beta+2}$ , i.e.,  $\tau > \frac{1}{(\beta+2)q_0(\beta)}$ .  $\square$

Figures 12-13 show the asymptotic behaviours achieved in scenario WAL for dif-

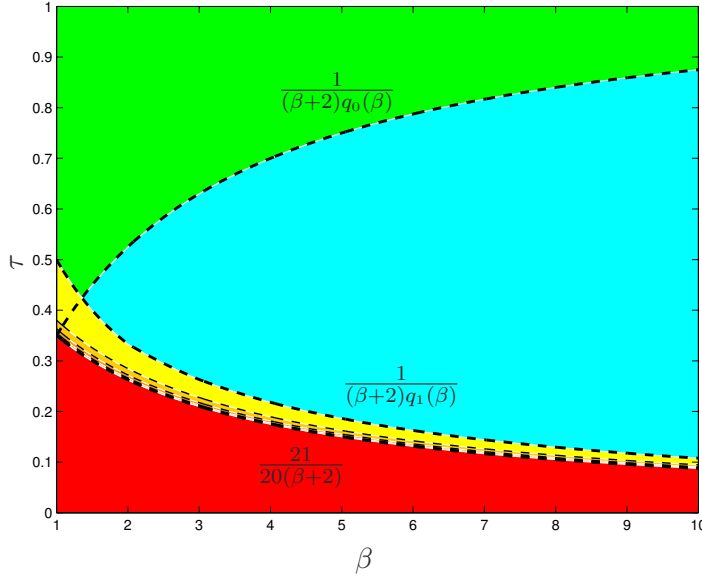


Figure 13. Ring graph, WAL setting,  $n = 21$ .

ferent values of  $\beta$  and  $\tau$ , in the cases of 5 and 21 agents, respectively. The different regions correspond to:  $a_\infty = \mathbb{1}$  (red);  $a_\infty = 0$  (green);  $a(t) = a(0), \forall t$  (light blue);  $a_\infty = \alpha_j$  (different shades of yellow). In particular, for the case  $n = 5$  only  $a_\infty = \alpha_1$  is present, while for  $n = 21$  one can observe five regions corresponding to  $a_\infty = \alpha_j$ ,  $j = 1, \dots, 5$ , the lightest (and largest) one corresponding to  $\alpha_1$ . The dashed curves represent the boundaries defined by Theorem 5.1 as functions of  $\tau$  and  $\beta$ .

In Figure 13, it can be noticed that the regions in which  $a_\infty = \alpha_j$  tend to shrink as  $j$  grows, while they approach the region in which  $a_\infty = \mathbb{1}$ . Moreover, these regions also shrink as  $\beta$  grows. On the other hand, their number increases with the number of agents, being proportional to  $\frac{n}{4}$ . Apart from these regions, the other asymptotic patterns are the same as for the star graph, but the region in which  $a_\infty = a(0)$  is much larger in the ring network, while those in which 0 or  $\mathbb{1}$  are achieved are significantly reduced (compare, e.g., Figures 7 and 13).

## 5.2. Uniform activity level

Let us consider the UAL setting. The asymptotic behaviour for  $\beta \geq 1$  is described by the next result.

**Theorem 5.2.** *Let  $\mathcal{E}$  be a ring graph,  $F$  and  $G$  given by (54) and (55), respectively,  $n$  odd and  $\beta \geq 1$ . Let the functions  $q_j(\beta)$ ,  $j = 1, \dots, h$ , be defined as in (60). Then, system (5),(6), with initial condition (8), exhibits the following asymptotic behaviours:*

- i) If  $\tau \leq \frac{n}{3(n-1)}$ , then  $a_\infty = \mathbb{1}$ .
- ii) If

$$\frac{1}{3q_{j+1}(\beta)} \leq \tau \leq \frac{1}{3q_j(\beta)},$$

for some  $j \in \{1, 2, \dots, h\}$ , then  $a_\infty = \alpha_j$ , given by (63).

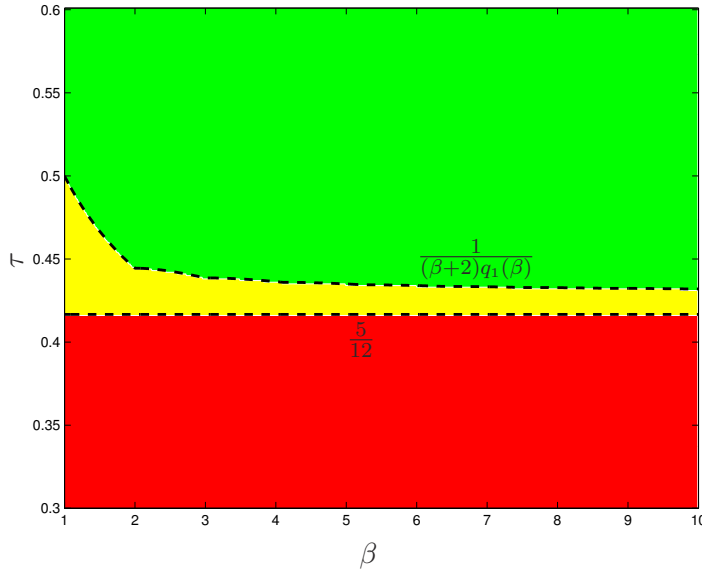


Figure 14. Ring graph, UAL setting,  $n = 5$ .

iii) If  $\tau > \frac{n-1}{3n}$ , then  $a_\infty = 0$ .

**Proof.** By adopting the same argument as in the proof of Theorem 5.1, it can be shown that  $a(t)$  can take only the values  $\mathbb{1}$ ,  $0$ , or  $\alpha_j$  in (63). In the latter case, one gets

$$p(t+1) = G\alpha_j = \left[ \underbrace{1 \dots 1}_j \quad \frac{2}{3} \quad \frac{1}{3} \quad \underbrace{0 \dots 0}_{n-2j-3} \quad \frac{1}{3} \quad \frac{2}{3} \quad \underbrace{1 \dots 1}_{j-1} \right]'$$

Then, item i) and item ii) for  $j = 1, \dots, h$  can be proven by following the same reasoning as in the proof of Theorem 5.1.

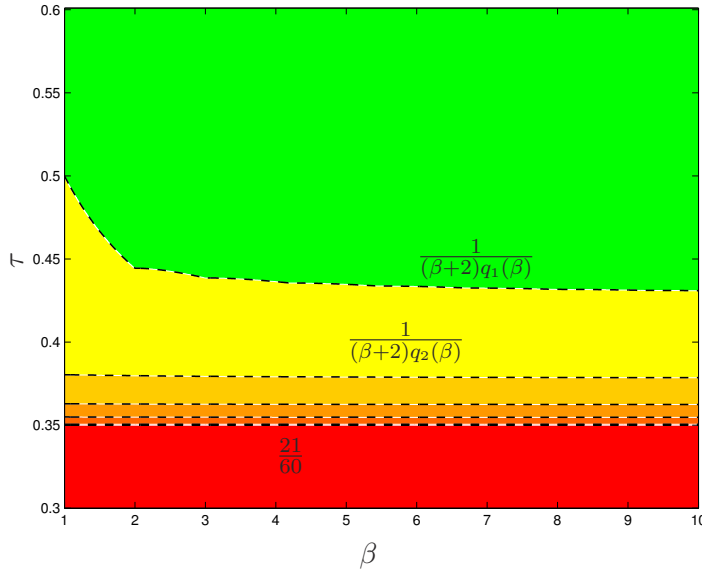
Concerning item iii), let us first observe that if  $\tau > \frac{1}{3q_1(\beta)}$ , the switching from  $a(0) = \alpha_0$  to  $\alpha_1$  never occurs. Being

$$p(1) = G\alpha_0 = \left[ \frac{1}{3} \quad \frac{1}{3} \quad 0 \quad \dots \quad 0 \quad \frac{1}{3} \right]'$$

and  $q_1(\beta) < \frac{n}{n-1}$ , one has  $\tau \frac{n-1}{n} > \frac{1}{3}$  and hence  $\lim_{t \rightarrow +\infty} \theta_1(t) > \frac{1}{3}$ . This means that eventually one will have  $\theta_1(t) > p_1(t)$  and therefore  $a_\infty = 0$ .  $\square$

Figures 14-15 show the asymptotic behaviours achieved in scenario UAL for different values of  $\beta$  and  $\tau$ , for  $n = 5$  and  $n = 21$ , respectively. Colours have the same meaning as in Figures 12-13. The range of  $\tau$  is reduced to highlight the presence of the regions in which  $a_\infty = \alpha_j$ . The dashed line represents the boundaries defined by Theorem 5.2 as function of  $\tau$  and  $\beta$ .

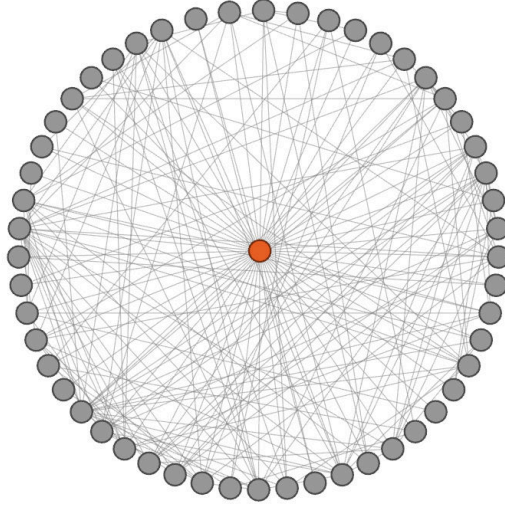
According to Theorem 5.2, in the UAL scenario it never occurs that  $a_\infty = a(0)$ , i.e., the initial condition cannot be maintained indefinitely. Once again, this is a major difference with respect to what happens in the WAL setting (compare Figures 14-15 with Figures 12-13). Moreover, a larger value of the self-confidence parameter  $\beta$



**Figure 15.** Ring graph, UAL setting,  $n = 21$ .

enlarges the gap between the extreme asymptotic behaviours  $a_\infty = 1$  and  $a_\infty = 0$  in the WAL scenario, while this is not the case in the UAL case.

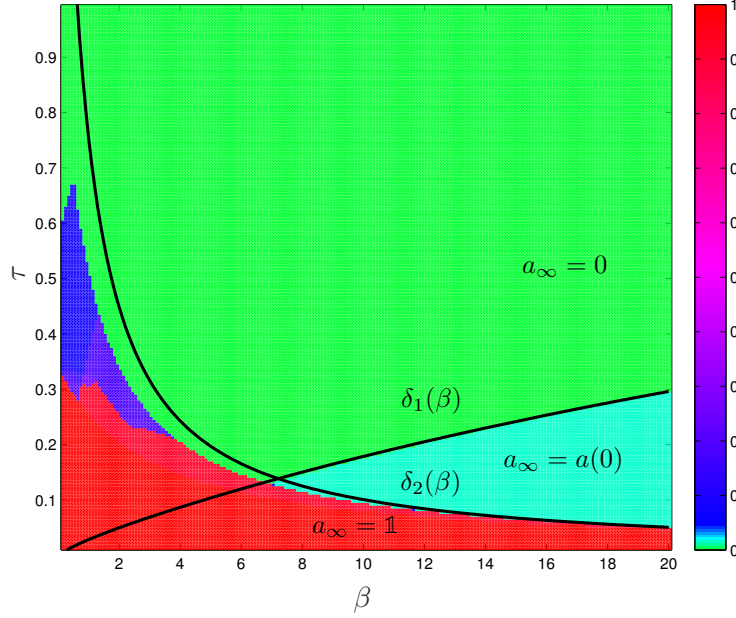
By comparing the results obtained in the two considered scenarios, it can be concluded that the choice of the matrix  $G$  plays a key role in defining the pattern of asymptotic behaviours in all the interconnection topologies considered in the paper.



**Figure 16.** Ego network used for simulation, including  $n = 53$  nodes and 198 edges (the red node is the ego node).

## 6. Experimental results with ego networks

In this section, we evaluate to what extent the analytic results presented so far, and obtained in the case of simple graph topologies, apply to more realistic social networks.



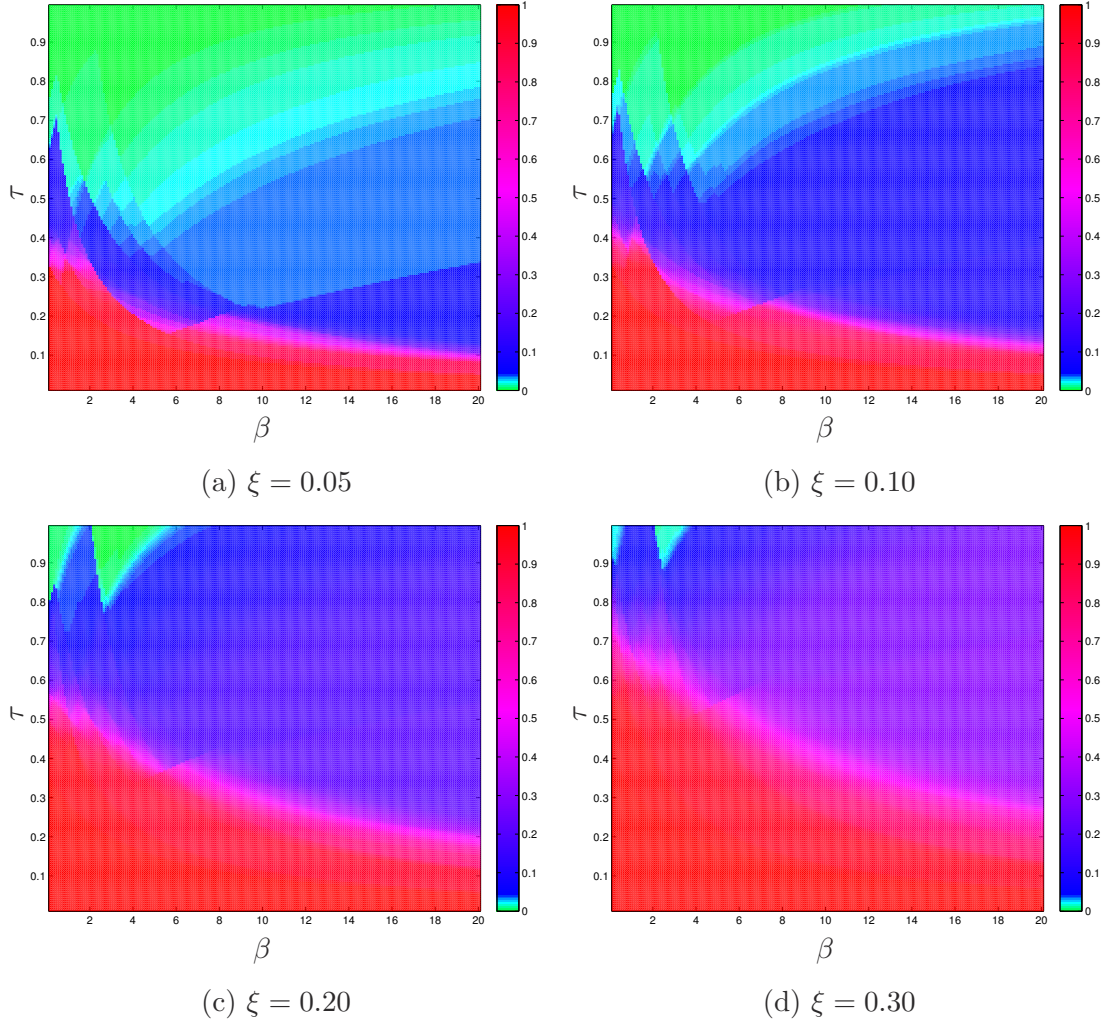
**Figure 17.** Final fraction of radical agents when the only initial radical agent is placed at the ego node.

To this end, we have carried out extensive simulations on a real ego network and observed how the asymptotic behaviours depend on the model parameters.

The test network used in this study is extracted from the data set described in (Leskovec and Mcauley, 2012), consisting of ten ego networks taken from Facebook. The network selected has  $n = 53$  nodes, including the ego node, and 198 edges, resulting in an average degree of 7.47 edges per node (see Fig. 16). The highest degree node is clearly the ego node, which, by definition, is connected to all the other nodes. The node with the second highest degree has 19 incident edges, whereas 10 nodes are connected to the ego node only. Dynamic system (5),(6) has been simulated with such an ego network constituting the underlying communication infrastructure. The WAL setting has been considered, i.e.  $F = G$  is assumed in throughout this section, with the entries of  $F$  given by (7). Different combinations of the initial threshold  $\tau$  and the self-confidence weight  $\beta$  have been simulated. Parameter  $\tau$  ranges from 0.01 to 0.99, whereas  $\beta$  varies from 0.1 to 20.

In the first scenario considered, the ego node is the only agent initially active. The fraction number of final active agents, as a function of  $\tau$  and  $\beta$ , is shown in Fig. 17. Among the graphs studied in Sec. 3-5, the network topology more similar to the ego network under consideration is clearly the star network. Although a non negligible number of additional edges between the non-central nodes are now present, the asymptotic behaviours of the system are very similar to those analytically derived in Sec. 4.1 (e.g., compare Fig. 7 to Fig. 17). For large values of the self-confidence weight, i.e. for  $\beta > \sqrt{n-1} \simeq 7.21$ , the asymptotic behaviour of the test network is in very good agreement with that predicted by Theorem 4.1 (point (i)), with  $a_\infty$  switching from  $\mathbb{1}$  to  $a(0)$  to 0 as  $\tau$  crosses the functions  $\delta_1(\beta)$  and  $\delta_2(\beta)$  (see Fig. 17). For smaller values of  $\beta$  a richer variety of asymptotic behaviours are now observed. In contrast to what happens for a truly star network, in this case 13 different values of  $a_\infty$  are found. Notice, however, that three stationary asymptotic behaviours predicted by Theorem 4.1, namely  $a_\infty \in \{0, a(0), \mathbb{1}\}$ , cover more than 95% of the simulation





**Figure 18.** Final fraction of radical agents, averaged over 100 different initial conditions, for different initial fractions  $\xi$  of radical agents.

runs.

A second set of simulations has been carried out to analyse how the presence of more than one initial radical agent modifies the final distribution of active agents. To this end, the simulations have been initialised to a number of radicals equal to  $\text{round}(\xi n)$ , where  $\xi$  denotes the fraction of initial radical agents. The identity of the initial radicals (i.e., the node where initial radicals are placed within the network) may have an impact on their ability to persuade a larger number of neighbours. Intuitively, more central or more connected nodes (in a sense, more “popular” agents) are able to mobilise a higher number of individuals. To mitigate such an effect, simulation results are averaged over 100 different randomly generated identities of the initially active agents. For consistency with the theoretical analysis, the ego node is always initially active. The final fraction of active agents is reported in Fig. 18, for four different values of  $\xi$  ranging from 0.05 to 0.30. It can be noticed that the smaller the number of initial radicals, the sharper the transition between regions corresponding to different asymptotic behaviours. Although somehow blurred by the averaging process, separating curves similar to those shown in Fig. 8 can be observed. For values of  $\beta$



close to one, the transition between the regions with all active and all inactive final agents is very irregular and spiked. This suggests that the fractal boundary found in Theorem 4.1 (point (ii)), and shown in Fig. 8, is revealing of a phenomenon which can be experienced also in actual ego networks.

## 7. Conclusions

This paper has presented a class of dynamic threshold models which can be used to analyse collective actions in social networks. The main feature of this model class is that the threshold is time-varying, as it evolves according to a dynamic opinion model. The asymptotic activity pattern of the network clearly depends not only on the graph topology, but also on the level of self-confidence of the agents. Moreover, a crucial role is played by the selected mechanism for the computation of the neighbours' activity level, which determines how the agents decide to become active or not.

The analytic results obtained so far support the thesis proposed in (Hassanpour, 2014), and based on empirical evidence, according to which *“in the presence of a risk-averse majority and a radical minority, adding more links among the majority does not necessarily help mobilisation.”* By looking at the regions corresponding to all agents becoming eventually active (e.g., red regions in Figs. 2, 7 and 13), it can be clearly seen that achieving full mobilisation in a highly connected network (e.g., a complete graph) can be harder than doing it in a less connected topology (e.g., star and ring graphs).

The study presented in this paper can be extended along different research directions, some of which are highlighted below.

- **Network topology.** One limitation of the results obtained so far is that they hold under quite strong assumptions on the topology of the underlying communication network (complete, star or ring graphs). This has allowed us to analytically characterise the asymptotic behaviours of the network and to identify interesting phenomena observed in more complex structures as well, as confirmed by numerical simulations. Undertaking a similar analysis for more realistic social networks (e.g., ego networks or power-law networks) would represent a major improvement of this work.
- **Initial conditions.** A further enrichment of the analysis is the consideration of a broader class of initial conditions. In this respect, interesting questions concern the effect that the number of initial radical agents, or their centrality within the network, have on the asymptotic behaviour of the system. Similarly, an additional aspect worth investigation is the presence of groups of ordinary agents having different initial thresholds (e.g., modelling two parties with different initial opinions about the action to be undertaken).
- **Individual model.** One key feature of the proposed model over existing ones is its ability to reproduce the behaviour of individuals exhibiting a wide range of self-confidence levels, by suitably tuning the parameter  $\beta$ . So far, only homogeneous communities have been studied, i.e. all the agents are supposed to behave according to the same value of  $\beta$ . However, the flexibility of the model allows one to study also networks composed of groups of agents with different self-confidence levels, thus analysing which types of dynamics arise between confident and hesitant agents.
- **Opinion dynamics model.** A major factor influencing the network evolution

is the dynamic model used to represent the time evolution of the individuals' threshold. A variation of the widely used DeGroot opinion model has been used in this work. However, alternative opinion dynamics models can also be considered, such as models with time-varying or state-dependent weights (like in the Hegselmann-Krause model, Blondel et al., 2009), models including antagonistic interactions (Altafini and Lini, 2015; Proskurnikov et al., 2016), or stochastic models frequently used in the social learning literature (Acemoglu et al., 2011a).

## References

- Acemoglu, D., Como, G., Fagnani, F., and Ozdaglar, A. (2013). Opinion fluctuations and disagreement in social networks. *Mathematics of Operations Research*, 38(1):1–27.
- Acemoglu, D., Dahleh, M., Lobel, I., and Ozdaglar, A. (2011a). Bayesian learning in social networks. *The Review of Economic Studies*, 78(4):1201–1236.
- Acemoglu, D., Ozdaglar, A., and ParandehGheibi, A. (2010). Spread of (mis)information in social networks. *Games and Economic Behavior*, 70(2):194–227.
- Acemoglu, D., Ozdaglar, A., and Yildiz, E. (2011b). Diffusion of innovations in social networks. In *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference*, pages 2329–2334.
- Adam, E., Dahleh, M., and Ozdaglar, A. (2012). On the behavior of threshold models over finite networks. In *Proceedings of the 51st IEEE Conference on Decision and Control*, pages 2672–2677.
- Altafini, C. and Lini, G. (2015). Predictable dynamics of opinion forming for networks with antagonistic interactions. *IEEE Transactions on Automatic Control*, 60(2):342–357.
- Blondel, V., Hendrickx, J., and Tsitsiklis, J. (2009). On Krause’s multi-agent consensus model with state-dependent connectivity. *IEEE Transactions on Automatic Control*, 54(11):2586–2597.
- Centola, D. (2010). The spread of behavior in an online social network experiment. *Science*, 329(5996):1194–1197.
- Dahleh, M., Tahbaz-Salehi, A., Tsitsiklis, J. N., and Zoumpoulis, S. I. (2013). Coordination with local information. In *ACM Performance Evaluation Review, Proceedings of the joint Workshop on Pricing and Incentives in Networks and Systems, ACM Sigmetrics*.
- Davis, P. J. (1979). *Circulant matrices*. American Mathematical Soc.
- DeGroot, M. H. (1974). Reaching a consensus. *Journal of the American Statistical Association*, 69(345):118–121.
- Etesami, S. and Basar, T. (2015). Game-theoretic analysis of the Hegselmann-Krause model for opinion dynamics in finite dimensions. *IEEE Transactions on Automatic Control*, 60(7):1886–1897.
- Fardad, M., Zhang, X., Lin, F., and Jovanovic, M. (2012). On the optimal dissemination of information in social networks. In *Proceedings of the 51st IEEE Conference on Decision and Control*, pages 2539–2544.
- Friedkin, N. (2015). The problem of social control and coordination of complex systems in sociology: A look at the community cleavage problem. *IEEE Control Systems Magazine*, 35(3):40–51.
- Friedkin, N. E. (2010). The attitude-behavior linkage in behavioral cascades. *Social Psychology Quarterly*, 73(2):196–213.
- Garulli, A., Giannitrapani, A., and Valentini, M. (2015). Analysis of threshold models for collective actions in social networks. In *Proceedings of the 14th European Control Conference*, pages 211–216, Linz (AT).
- Ghaderi, J. and Srikant, R. (2014). Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate. *Automatica*, 50(12):3209–3215.
- Granovetter, M. (1978). Threshold models of collective behavior. *American Journal of Sociology*, 83(3):1420–1430.

- ology, 83(6):1420–1443.
- Hassanpour, N. (2014). Media disruption and revolutionary unrest: Evidence from Mubarak’s quasi-experiment. *Political Communication*, 31(1):1–24.
- Jing, G., Zheng, Y., and Wang, L. (2014). Flocking of multi-agent systems with multiple groups. *International Journal of Control*, 87(12):2573–2582.
- Leskovec, J. and McAuley, J. J. (2012). Learning to discover social circles in ego networks. In Pereira, F., Burges, C. J. C., Bottou, L., and Weinberger, K. Q., editors, *Advances in Neural Information Processing Systems 25*, pages 539–547.
- Liu, J., Hassanpour, N., Tatikonda, S., and Morse, A. (2012). Dynamic threshold models of collective action in social networks. In *Proceedings of the 51st IEEE Conference on Decision and Control*, pages 3991–3996.
- Montanari, A. and Saberi, A. (2010). The spread of innovations in social networks. *Proceedings of the National Academy of Sciences*, 107(47):20196–20201.
- Moreau, L. (2005). Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50(2):169–182.
- Morris, S. and Shin, H. S. (2003). Global games: theory and applications. In *Advances in Economics and Econometrics. Proceedings of the Eight World Congress of the Econometric Society*. Cambridge University Press.
- Olfati-Saber, R., Fax, A., and Murray, R. M. (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233.
- Proskurnikov, A. V., Matveev, A. S., and Cao, M. (2016). Opinion dynamics in social networks with hostile camps: Consensus vs. polarization. *IEEE Transactions on Automatic Control*, 61(6):1524–1536.
- Ravazzi, C., Frasca, P., Tempo, R., and Ishii, H. (2015). Ergodic randomized algorithms and dynamics over networks. *IEEE Transactions on Control of Network Systems*, 2(1):78–87.
- Ren, W. and Beard, R. W. (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5):655–661.
- Rosa, D. and Giua, A. (2013). A non-progressive model of innovation diffusion in social networks. In *Proceedings of 52nd IEEE Conference on Decision and Control*, pages 6202–6207.
- Siegel, D. (2009). Social networks and collective action. *American Journal of Political Science*, 53(1):122–138.
- Touri, B. and Shamma, J. (2014). Global games with noisy sharing of information. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, pages 4473–4478.
- Xiao, F. and Wang, L. (2006). State consensus for multi-agent systems with switching topologies and time-varying delays. *International Journal of Control*, 79(10):1277–1284.
- Yang, Y., Dimarogonas, D. V., and Hu, X. (2014). Opinion consensus of modified Hegselmann–Krause models. *Automatica*, 50(2):622–627.