Analysis of threshold models for collective actions in social networks

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***Abstract*— In this paper, we study the asymptotic behaviors of threshold models used to describe the formation of collective actions in social networks. At each time instant, each agent of the network makes a choice between two possible actions. The decision is made on the basis of the actions chosen by its neighbors and the value of a dynamically updated threshold. The main novelty of the proposed model is the introduction of a parameter accounting for the level of self-confidence of the agents, which affects the dynamic evolution of the threshold and in turn the way the agents make their decision. The objective is to identify which are the possible limiting behaviors of the network and under which conditions each of them occurs. Three different network topologies are considered. In the case of complete graph, the asymptotic behaviors are analytically characterized, whereas for star and ring graphs an extensive numerical analysis is presented.**

I. Introduction

Since their introduction in [1], threshold models have been widely used to explain the collective behavior of a community of individuals in many different application domains, ranging from diffusion of innovations, to spread of riots or strikes, to opinion dynamics. Such models find a natural application in the analysis of social networks. Driven by their recent development and pervasiveness, several studies have explored the mechanisms underlying the spread of information or behaviors in such communities (see e.g. [2]-[4] and references therein). Recently there has been an increasing interest to the application of system theoretic methodologies to the analysis of dynamic models relevant to these problems [5]-[8].

In this context, threshold models are well suited to predict the occurrence of cascade effects, i.e. the possibility that a behavior or opinion adopted by a small number of influential agents will spread to a large part of the network [9]. In [10], threshold models are adopted to analyze how innovations spread into a network starting from a set of promoters. Such a model has been later generalized in [11] to account for the possibility for a member of the network to abandon a previously adopted innovation. Moreover the effect of the presence of a group of agents which maintain the innovation for a finite time despite the behavior of their neighbors is analyzed. Threshold models have been recently used in [12], [13] to analyze the mechanisms underlying the formation of a collective action taking place during political unrest or social revolutions. The objective is to identify under which conditions a single radical agent is able to eventually persuade all the individuals of the network to engage in the demonstration. A peculiarity of such a model is the adoption of two different classes of thresholds in order to capture the different behavior of radical and uncommitted members.

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In this paper, we generalize the models proposed in [12] along two directions. First, the assumption that each agent weights equally its own opinion and other members' opinions is relaxed. A parameter is introduced representing the relative weight that an agent assigns to its own opinion with respect to that of its neighbors. Moreover, two different mechanisms modelling the way in which an agent decides whether to become active or not are considered. Similarly to what is assumed in [9],[11],[12], a non progressive model is adopted, meaning that each agent can change its actions multiple times. At each time instant, each individual compares its current threshold value with an indicator of the average activity level of the neighbors. In the proposed model, such an indicator can be either the fraction of active neighbors (as in [12]) or a weighted average of the number of active neighbors which takes into account the self-confidence of each agent. The main contribution of the paper is the analysis of the asymptotic behavior of the network for different graph topologies. The case of complete network is analytically characterized, while a thorough simulation study is carried out for star and ring graphs. It turns out that in the proposed framework, a wider variety of collective limiting behaviors emerge. Moreover, the decision mechanism plays a key role in determining the possible asymptotic behaviors of the community.

The paper is organized as follows. In Section II, the addressed threshold models are introduced, together with the two different decision schemes considered. Section III presents the analytic results in the case of complete graph, whereas a numerical analysis for the case of star and ring topologies is presented in Section IV. Section V contains some conclusions and possible lines of future research. The proofs of the technical results can be found in [14].

II. Problem formulation

Let us consider a network of $n > 1$ agents described by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes the vertex set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. Two agents i and j are *neighbors* if $(i, j) \in \mathcal{E}$. Let $\mathcal{N}_i$ denote the set of neighbors of agent i and n_i be its cardinality. By convention, an agent is always considered a neighbor of itself, i.e. $(i, i) \in \mathcal{E}$, for all i. In this work, the network topology is assumed to be time-invariant.

In accordance with the model proposed in [12], two variables are associated to agent i, namely the threshold $\theta_i(t) \in [0,\ 1]$ and the action $a_i(t) \in \{0,\ 1\}$. The variable

a_i discriminates whether the corresponding agent is active at time t ($a_i(t) = 1$) or not ($a_i(t) = 0$). The threshold $\theta_i(t)$ is used to model the way an agent changes its action value.

The agent behavior is described by the time evolution of the threshold and the action variable. At each time step, an agent updates its threshold to a weighted average of its neighbors' threshold

$$\theta_i(t+1) = \sum_{j\in\mathcal{N}_i} f_{ij}\theta_j(t), \qquad i = 1, \dots, n, \tag{1}$$

where the weights are such that $0 < f_{ij} < 1$ and $\sum_h f_{ih} = 1$, $\forall i, j$. After updating the threshold, each agent computes the activity level of its neighbors as

$$p_i(t) = \sum_{j\in\mathcal{N}_i} g_{ij}a_j(t), \qquad i = 1, \dots, n, \tag{2}$$

where the weights are such that $0 < g_{ij} < 1$ and $\sum_h g_{ih} = 1$, $\forall i, j$. Finally, the new action value is computed by comparing the activity level $p_i(t)$ with the threshold $\theta_i(t+1)$ and setting

$$a_i(t+1) = \begin{cases} 1 & \text{if } p_i(t) \geq \theta_i(t+1) \\ 0 & \text{else} \end{cases}, \quad i = 1, \dots, n. \tag{3}$$

By setting $f_{ij} = g_{ij} = 0$ whenever $j \notin \mathcal{N}_i$, equations (1) and (2) can be rewritten in matrix form as

$$\theta(t+1) = F\theta(t), \tag{4}$$

$$p(t) = Ga(t), \tag{5}$$

where $\theta = [\theta_1, \dots, \theta_n]'$, $a = [a_i, \dots, a_n]'$, $p = [p_1, \dots, p_n]'$, and F and G are matrices whose ij-th entries are f_{ij} and g_{ij}, respectively. By introducing the function

$$\phi(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{else }, \end{cases}$$

and exploiting (5), equations (3) become

$$a(t+1) = \phi(Ga(t) - F\theta(t)), \tag{6}$$

where the function $\phi(\cdot)$ is to be intended componentwise. If the initial threshold values are $0 \leq \theta_i(0) \leq 1$ and not all zero, then it is easy to check that $a_e = 0$ and $a_e = \mathbb{1}$ (where $\mathbb{1}$ denotes a vector whose entries are all equal to 1) are always equilibrium points for system (6).

In this paper, the entries of matrix F are chosen as

$$f_{ij} = \begin{cases} \frac{\beta}{\beta+n_i-1} & \text{if } i = j, \\ \frac{1}{\beta+n_i-1} & \text{if } j \in \mathcal{N}_i,\ j \neq i, \\ 0 & \text{else }, \end{cases} \tag{7}$$

where $\beta > 0$ is the relative weight each agent assigns to its current threshold value compared to that of its neighbors. In other words, β can be interpreted as the relative confidence that each agents has on its own opinion, with respect to that of the other members of the network. As long as matrix G is concerned, we consider two different settings:

a) $G = F$, i.e. the same relative weight is adopted in computing the activity level of the neighbors;

b) $G = \{g_{ij}\}$, such that

$$g_{ij} = \begin{cases} \frac{1}{n_i} & \text{if } j \in \mathcal{N}_i, \\ 0 & \text{else }, \end{cases} \tag{8}$$

i.e., $p_i(t)$ in (2) represents the fraction of active neighbors of agent i.

Notice that in scenario b) each agent decides whether to become active or not by just "counting" the number of active neighbors. Conversely, in scenario a) an agent weights in a different way the fact that its neighbors are active with respect to its own activity status. This is consistent with the idea that a self-confident individual, weighting its own opinion β times that of its neighbors, will also consider in a different way its own behavior with respect to that of its neighbors.

Remark 1: Notice that when $\beta = 1$ the weights are $f_{ij} = g_{ij} = \frac{1}{n_i}$, $\forall i$, $\forall j \in \mathcal{N}_i$, and hence the framework considered in [12] is recovered. In this case, the threshold update rule (4) consists in computing the average of the neighbors' thresholds, and the activity level (5) is equal to the fraction of active neighbors.

We are interested in studying the asymptotic behavior of the system when at time $t = 0$ there is an active agent with threshold equal to zero (a *radical*), while the remaining agents are inactive and have all the same threshold equal to τ, with $0 < \tau < 1$ (these agents are called *ordinary*). This corresponds to the initial condition

$$\theta(0) = [0,\ \tau, \dots, \tau]', \quad a(0) = [1,\ 0, \dots, 0]'. \tag{9}$$

Clearly, for a given weighting scheme, the behavior of system (4)-(6) depends heavily on the topology of the interconnection network. A notable case is that in which the graph $\mathcal{G}$ is assumed to be complete, i.e., $(i, j) \in \mathcal{E}$, for all $i,\ j$. This case is thoroughly analyzed in the next section.

III. Complete graph

In order to study the asymptotic behavior of system (4)-(6), we need to introduce the following technical results.

Lemma 1: Consider the dynamic model (4), with the weights chosen as in (7). If the interconnection graph is complete and $\theta(0) = [0,\ \tau, \dots, \tau]'$, $0 < \tau < 1$, then

$$\theta_1(t) = \frac{n-1}{n}\tau\left(1 - \lambda^t\right), \tag{10}$$

$$\theta_i(t) = \frac{n-1}{n}\tau\left(1 + \frac{1}{n-1}\lambda^t\right),\ i = 2, \dots, n, \tag{11}$$

where $\lambda = \dfrac{\beta - 1}{\beta + n - 1}$.

Corollary 1: Consider the dynamic model (4), with the weights chosen as in (7). If the interconnection graph is complete and $\theta(0) = [0,\ \tau, \dots, \tau]'$, $0 < \tau < 1$, then

$$\lim_{t\to\infty} \theta(t) = \frac{n-1}{n}\tau\mathbb{1}. \tag{12}$$

Moreover, if $\beta > 1$, then

$$\theta_1(t+1) > \theta_1(t), \tag{13}$$

$$\theta_i(t+1) < \theta_i(t), \quad i = 2, \dots, n, \tag{14}$$

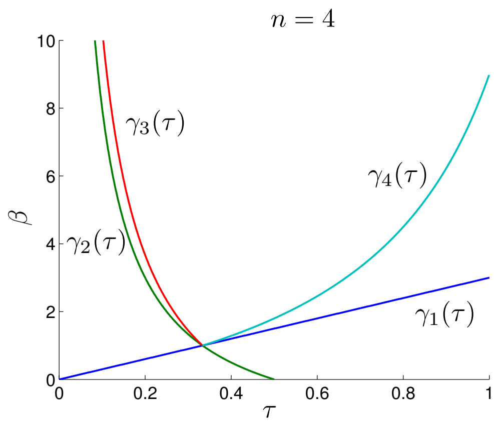

Fig. 1. Case a), $n = 4$.

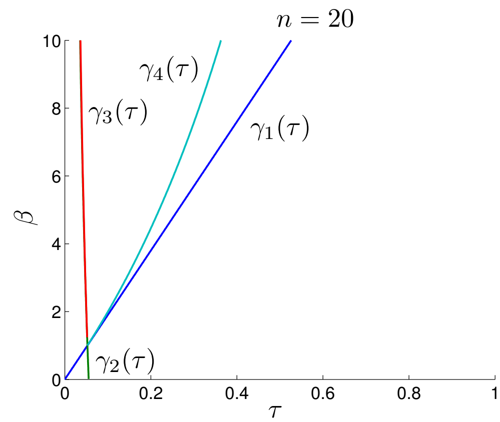

Fig. 2. Case a), $n = 20$.

for all $t \geq 0$.

A. Setting a)

Let us first, consider the setting a), i.e.,

$$G = F = \frac{\beta}{\beta + n - 1} I_n + \frac{1}{\beta + n - 1}(\mathbb{1}\mathbb{1}' - I_n). \quad (15)$$

Define the functions of $\tau \in (0, 1)$:

$$\gamma_1(\tau) = (n-1)\tau, \quad (16)$$

$$\gamma_2(\tau) = \frac{1}{\tau} - (n-2), \quad (17)$$

$$\gamma_3(\tau) = \frac{n}{n-1}\frac{1}{\tau} - (n-1), \quad (18)$$

$$\gamma_4(\tau) = \frac{(n-1)^2\tau}{n - (n-1)\tau}. \quad (19)$$

Such functions are shown in Figs. 1-2 for $n = 4$ and $n = 20$, respectively. The following result holds.

Theorem 1: System (4),(6), with $G = F$ given by (15) and initial condition (9), exhibits the following asymptotic behaviors:

i) if $\gamma_1(\tau) \leq \beta \leq \gamma_2(\tau)$, then $a(t) = \mathbb{1}$, $\forall t \geq 1$;

ii) if $\gamma_2(\tau) < \beta < \gamma_1(\tau)$, then $a(t) = 0$, $\forall t \geq 1$;

iii) if $\beta < \gamma_1(\tau)$ and $\beta \leq \gamma_2(\tau)$, then $a(t) = \mathbb{1}$, $\forall t \geq 2$;

iv) if $\beta \geq \max\{\gamma_3(\tau), \gamma_4(\tau)\}$, then $a(t) = a(0)$, $\forall t > 0$,

v) if $\gamma_1(\tau) \leq \beta < \gamma_4(\tau)$, then there exists a finite t_1 such that $a(t) = 0$, $\forall t \geq t_1$;

vi) if $\gamma_2(\tau) < \beta < \gamma_3(\tau)$, then there exists a finite t_2 such that $a(t) = \mathbb{1}$, $\forall t \geq t_2$.

Theorem 1 gives the complete characterization of the asymptotic behavior of system (4),(6), with initial condition (9). Notice that there are three possible asymptotic activity profiles: i) all the agents become active; ii) all the agents become inactive; iii) the situation remains always the same as in the initial condition (i.e., agent 1 is active and all the others are inactive). It should be also remarked that the first two cases occur in at most two steps, except for the following conditions:

- if $\gamma_1(\tau) \leq \beta < \gamma_4(\tau)$, then $a(t) = 0$ within a finite time t_1;
- if $\gamma_2(\tau) < \beta < \gamma_3(\tau)$, then $a(t) = \mathbb{1}$ within a finite time t_2.

In such cases, it is not possible to find an upper bound on t_1 or t_2. For example, if $n = 5$, $\tau = 0.99$ and $\beta = 15$, one has that $a(t) = 0$ only for $t \geq 19$. Similarly, if $n = 5$, $\tau = 0.01$ and $\beta = 118$, one has that $a(t) = \mathbb{1}$ only for $t \geq 56$. Figures 1-2 show the curves $\gamma_i(\tau)$ in the $\beta - \tau$ plane, for $n = 4$ and $n = 20$ respectively. It can be observed that in the latter case the curves $\gamma_2(\tau)$ and $\gamma_3(\tau)$ are almost indistinguishable. As expected, the area in which all the agents end up to be inactive grows with n, while the region in which all the agents become active tends to shrink, as well as that in which the initial condition $a(0)$ is maintained.

B. Setting b)

Now, let us consider setting b). When the interconnection graph is complete, this means that

$$G = \frac{\mathbb{1}\mathbb{1}'}{n}, \quad (20)$$

and F is given by (15). Let us define the functions of $\tau \in (0, 1)$:

$$\eta_1(\tau) = (n-1)(n\tau - 1), \quad (21)$$

$$\eta_2(\tau) = \frac{n - 1 - n(n-2)\tau}{n\tau - 1}. \quad (22)$$

Then, the following result holds.

Theorem 2: System (4),(6), with F defined as in (15), G given by (20) and initial condition (9), has the following asymptotic behaviors:

i) if $\eta_1(\tau) \leq \beta \leq \eta_2(\tau)$, then $a(t) = \mathbb{1}$, $\forall t \geq 1$;

ii) if $\eta_2(\tau) < \beta < \eta_1(\tau)$, then $a(t) = 0$, $\forall t \geq 1$;

iii) if $\beta < \eta_1(\tau)$ and $\beta \leq \eta_2(\tau)$, then $a(t) = \mathbb{1}$, $\forall t \geq 2$;

iv) if $\tau > \frac{1}{n-1}$, $\beta \geq \eta_1(\tau)$, then there exists a finite t_1 such that $a(t) = 0$, $\forall t \geq t_1$;

v) if $\tau < \frac{1}{n-1}$, $\beta > \eta_2(\tau)$, then there exists a finite t_2 such that $a(t) = \mathbb{1}$, $\forall t \geq t_2$;

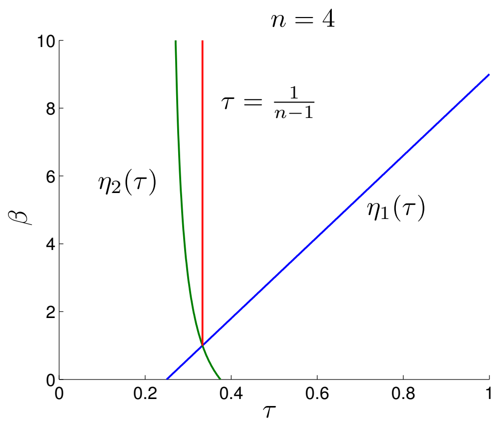

Fig. 3. Case b), $n = 4$.

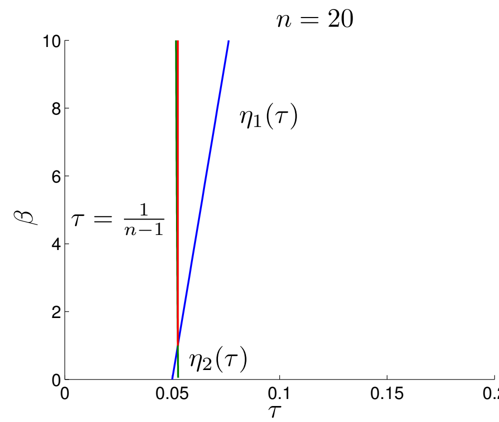

Fig. 4. Case b), $n = 20$.

vi) if $\tau = \frac{1}{n-1}$, $\beta > 1$, then $a(t) = a(0)$, $\forall t \geq 0$.

Figures 3-4 show the curves $\eta_i(\tau)$ in the $\beta - \tau$ plane, along with the vertical line $\tau = \frac{1}{n-1}$, for $n = 4$ and $n = 20$ respectively. In the latter case the curve $\eta_2(\tau)$ is almost coincident to the vertical line. Also in this case, the area in which all the agents end up to be inactive grows with n (notice the scale on τ), while the region in which all the agents become active tends to shrink.

C. Discussion

First, notice that when $\beta = 1$, matrix G is the same in both settings, so that conditions in Theorems 1 and 2 coincide. In this case, the considered scenario matches exactly that addressed in [12], and the results therein presented are recovered. In particular, it can be noticed from Figures 1-4 that only two situations occur: either $a(1) = \mathbb{1}$ if $\tau \leq \frac{1}{n-1}$, or $a(1) = 0$ otherwise. The introduction of the parameter β, accounting for the relative confidence of each agent on its own opinion, has significantly enriched the picture of possible asymptotic behaviors of the system. For $\beta > 1$, three new different situations appear in setting a): all the agents become active or inactive, but in more than one step; the initial situation is maintained indefinitely. As β increases, the latter situation occurs for a larger range of values of the initial threshold τ. This corresponds to the fact that in a network whose agents are more self-confident, it is more difficult to persuade them to change their activity status. Conversely, for $\beta < 1$, this behavior disappears and either $\mathbb{1}$ is reached (in one or two steps) or all the agents become inactive in one step.

Another interesting observation concerns the differences between the scenarios a) and b). The same five behaviors described above for setting a), are present also in setting b), but the condition in which $a(t) = a(0)$, $\forall t$, occurs only if τ is exactly equal to $\frac{1}{n-1}$, which is clearly a singular condition. This confirms the intuition that matrix G has a strong influence on the asymptotic behavior of the system.

IV. Star and Ring Graph

In this section we analyze other two graph structures considered in [12]: the star graph and the ring graph. Although a full theoretical characterization of these cases is beyond the scope of this paper, a detailed numerical analysis of the asymptotic behaviors is given next.

A. Star graph

In the star graph, the radical agent ($i = 1$) is the centre of the graph, while the remaining $n - 1$ ordinary agents are connected only to the radical. This leads to a matrix F of the form

$$F = \begin{pmatrix} \frac{\beta}{\beta+n-1} & \frac{1}{\beta+n-1} & \dots & \dots & \frac{1}{\beta+n-1} \\ \frac{1}{\beta+1} & \frac{\beta}{\beta+1} & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ \frac{1}{\beta+1} & 0 & \dots & \dots & \frac{\beta}{\beta+1} \end{pmatrix}.$$

We consider scenarios a) and b), defined as in Section III. While in the former $G = F$, in the latter one has

$$G = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \dots & \dots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ \frac{1}{2} & 0 & \dots & \dots & \frac{1}{2} \end{pmatrix}.$$

Figures 5-6 show the asymptotic behaviors achieved in scenario a) for different values of τ (initial threshold of the ordinary agents) and β (relative confidence parameter), for $n = 5$ and $n = 20$, respectively. The corresponding plots for scenario b) are shown in Figures 7-8. The different regions correspond to:

- $\exists \bar{t}$ such that $a(t) = \mathbb{1}$, $\forall t \geq \bar{t}$ (red);
- $\exists \bar{t}$ such that $a(t) = 0$, $\forall t \geq \bar{t}$ (green);
- $a(t) = a(0)$, $\forall t$ (cyan);
- $a(t)$ oscillates indefinitely between $[1\ 0\ \dots\ 0]'$ and $[0\ 1\ \dots\ 1]'$ (blue).

It can be observed that setting a) shows a wider variety of behaviors than setting b). As for the complete graph, the case $\beta = 1$ is the same in the two settings and coincides with that studied in [12]. By increasing the value of β, we first notice a larger range of τ values for which all the agents tend

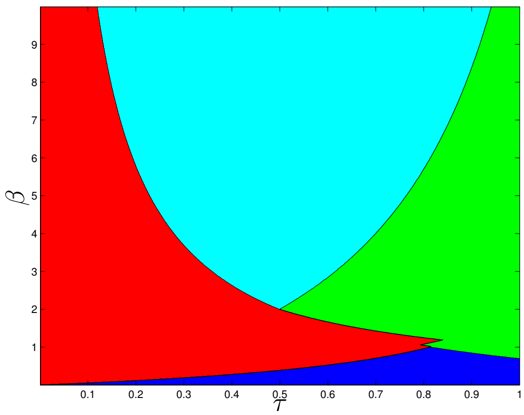

Fig. 5. Star graph, case a), $n = 5$.

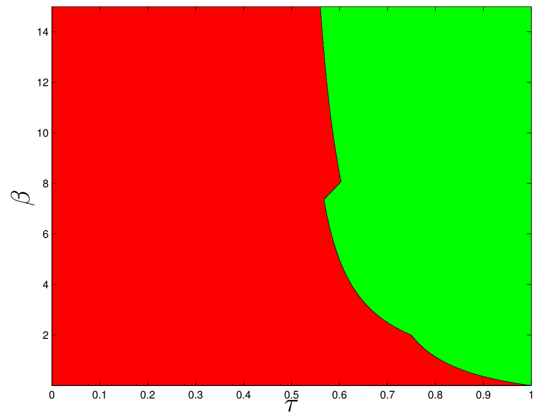

Fig. 7. Star graph, case b), $n = 5$.

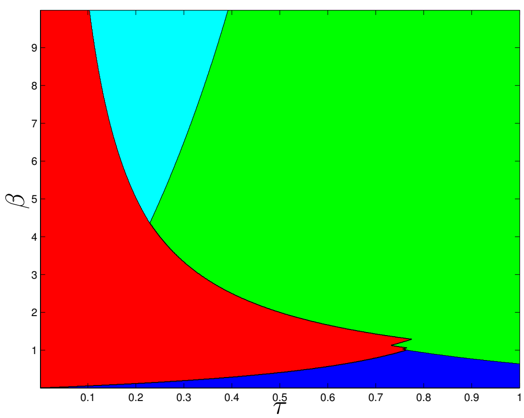

Fig. 6. Star graph, case a), $n = 20$.

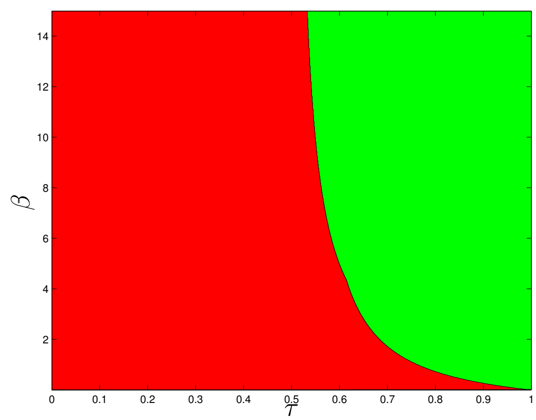

Fig. 8. Star graph, case b), $n = 20$.

to become inactive and successively, only in setting a), an area in which the initial condition is maintained indefinitely. However, this area tends to decrease for larger values of n and does not show up in setting b). The complexity of the overall system behavior is also testified by the non monotonicity of the curve separating the areas in which the asymptotic value of $a(t)$ is $\mathbb{1}$ and 0, respectively. In particular, it has been observed that in setting a), this curve seems to change its slope an infinite number of times in any interval $\beta \in (1, 1+\epsilon)$, with ϵ arbitrarily small.

B. Ring graph

In the ring graph, agent i has as neighbors agents $i-1$ and $i+1$ (with the convention $n+1 = 1$). We still assume that agent 1 is a radical while the others are ordinary, and we analyze the asymptotic behavior of the system starting from the initial condition (9). The matrix F is now given by

$$F = \begin{pmatrix} \frac{\beta}{\beta+2} & \frac{1}{\beta+2} & 0 & \dots & 0 & \frac{1}{\beta+2} \\ \frac{1}{\beta+2} & \frac{\beta}{\beta+2} & \frac{1}{\beta+2} & 0 & \dots & 0 \\ 0 & \frac{1}{\beta+2} & \ddots & \ddots & \dots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & \ddots & \ddots & \frac{1}{\beta+2} \\ \frac{1}{\beta+2} & 0 & \dots & \dots & \frac{1}{\beta+2} & \frac{\beta}{\beta+2} \end{pmatrix},$$

while matrix G in scenario b) is

$$G = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \dots & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \dots & 0 \\ 0 & \frac{1}{3} & \ddots & \ddots & \dots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & \ddots & \ddots & \frac{1}{3} \\ \frac{1}{3} & 0 & \dots & \dots & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

The asymptotic behaviors achieved in scenario a) and b) for different values of τ and β and $n = 20$, are shown in Figures 9 and 10, respectively. The red, green and cyan regions have the same meaning as for the star graph. The blue region corresponds to an indefinite oscillation between two vectors $a(t)$ in which only the even or the odd agents are active (we observed that this behavior is present only if the number of agents is even). Finally, the yellow regions correspond to cases in which there exists an integer r, $1 \leq r \leq \lfloor \frac{n}{2} \rfloor - 1$, and a time $\bar{t}$ such that for all $t \geq \bar{t}$,

$$\begin{aligned} a_i(t) &= 1 \quad i = 1, \dots, r+1 \text{ and } i = n-r+1, \dots, n \\ a_i(t) &= 0 \quad i = r+2, \dots, n-r. \end{aligned}$$

Different shades of yellow correspond to different values of r. For example, the largest yellow regions correspond to the case $r = 1$.

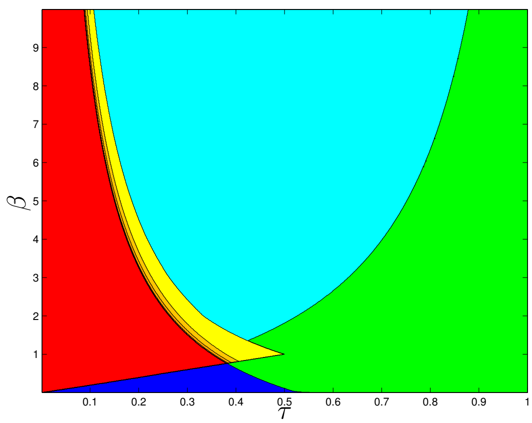

Fig. 9. Ring graph, case a), $n = 20$.

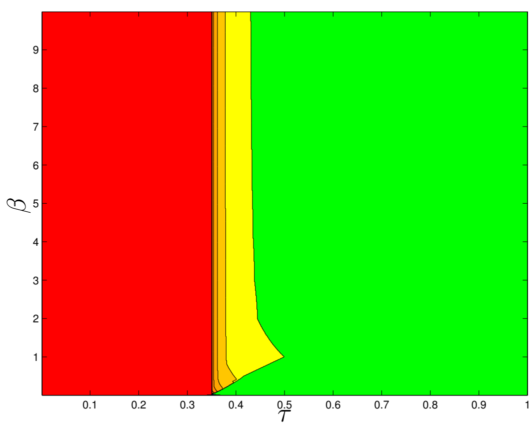

Fig. 10. Ring graph, case b), $n = 20$.

Once again, the case in which the initial condition $a(0)$ is maintained indefinitely shows up only in setting a). Moreover, it is apparent that the influence of parameter β on the system behavior is more remarkable in setting a) than in setting b). This confirms that a key aspect for this class of threshold models is the choice of the matrix G, which indeed plays a crucial role in the definition of the decision of the agents to become active or not.

V. CONCLUSIONS

A class of threshold models which can be used to describe the onset of collective actions in social networks, has been analyzed. It has been shown that the introduction of a parameter accounting for the level of self-confidence of the agents has significantly enriched the variety of asymptotic behaviors of the network. Moreover, it is apparent that the choice of the mechanism underlying the computation of the agents' activity levels has a strong influence on the network evolution.

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Clearly, the work is still at a preliminary stage and several developments can be foreseen. The derivation of analytic results for the star and ring graph structures is ongoing. It would be also interesting to evaluate the behavior of the network for other graph structures, or for different initial conditions. Another generalization, may concern the study of networks in which there are groups of agents with different self-confidence levels. Finally, a possible evolution of the proposed threshold models is to consider time-varying averaging schemes of the agent thresholds, inspired by works on opinion dynamics [15] or on consensus with adaptive weights [16], [17].

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