Distributed set membership estimation with time-varying graph topology

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***Abstract*— Distributed estimation schemes are increasingly popular these days. A distributed algorithm, specifically tailored to recursive set membership estimation problems, was recently proposed and analyzed for networks featuring a static topology. It was shown that the agents' estimates asymptotically converge to a common point lying in the intersection of all the agents' feasible sets. In this paper, by building on recent results on constrained consensus, we prove convergence in the more challenging scenario of networks with time-varying topology. It is shown that convergence is guaranteed if the sequence of graphs is jointly strongly connected over finite-length time intervals. Moreover, an asynchronous version of the proposed algorithm is presented, whose convergence can be deduced from the previously obtained results.**

I. Introduction

The set membership framework is a popular alternative to tackle estimation problems in the presence of bounded uncertainty [1], [2]. One distinctive feature of such a framework is the possibility to completely characterize in terms of feasible sets all the possible values of the unknown quantities that are compatible with the available information. Such a feature makes set membership approaches particularly appealing for robust estimation problems (e.g., [3], [4]) or applications requiring guaranteed estimation errors, such as mobile robotics problems [5], [6] or automotive applications [7], [8], [9]. Distributed versions of set membership estimation techniques, able to deal with a network of sensors, have been proposed as well (e.g., [10], [11]).

Recently, a consensus-based algorithm has been presented in [12] for distributing the set membership estimation process over a network of agents. In that work, each agent stores: i) a local feasible set which is recursively updated when new measurements are available; ii) a local estimate that is updated by computing an average of the neighbors' estimates and then projecting it on the local feasible set. In case of static topology of the communication graph, it was shown that the estimates of all the agents converge to a common point laying in the intersection of the feasible sets of all the nodes. Estimation algorithms having such a property are called asymptotic interpolatory algorithms in the set membership framework [13].

In this work, we extend the results of [12] to the more challenging scenario of communication graphs having a time-varying topology. By building on recent results obtained for the constrained consensus problem [14], [15], [16], we show that the convergence property is preserved under mild assumptions on the time evolution of the network topology. In particular, convergence results apply to sequence of graphs jointly strongly connected over finite-length time intervals (not necessarily strongly connected at any time instant), and to directed and unbalanced networks as well. It is worth stressing that the major difference between the framework considered in this paper and that of references [14], [15], [16] is that in our case projections are made over an infinite sequence of sets. This is a consequence of the recursive estimation scheme, in which at each time instant a new measurement is taken and a new feasible set becomes available. On the contrary, in typical constrained consensus problems, each agent repeatedly projects over the same feasible set.

The paper is organized as follows. In Section II, the considered distributed set membership estimation algorithm is introduced. Its convergence analysis is presented in Section III. In Section IV it is shown that an asynchronous version of the proposed estimator can be seen as a special instance of the algorithm previously introduced, from which convergence can be deduced. A numerical example is reported in Section V, whereas some conclusions are drawn in Section VI.

Notation: Given a matrix A, we denote by $[A]_{ij}$ its ij-th entry. Given a point $p \in \mathbb{R}^n$ and a closed set $Z \subset \mathbb{R}^n$, we denote by $P_Z[p]$ the projection of p on Z, defined as $P_Z[p] = \arg\min_{z\in Z} \|p - z\|$, where $\|\cdot\|$ denotes the 2-norm in $\mathbb{R}^n$.

II. Distributed set membership estimation

The considered setup consists of a network of N agents cooperating to estimate an unknown parameter $x \in \mathbb{R}^n$. The network is modelled as a directed graph with time-varying topology, $\mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k))$, where $\mathcal{V} = \{1, \dots, N\}$ denotes the set of agents and $\mathcal{E}(k) \subseteq \mathcal{V} \times \mathcal{V}$ represent the interaction among the agents at time k. In particular, an edge $(j, i) \in \mathcal{E}(k)$ if and only at time k agent j sends its current estimate to agent i. In this case, we say that j is a neighbor of i. At each time instant k, agent i takes a noisy measurement of a known function of x. The measurement noise is supposed to be unknown-but-bounded (UBB). Under the UBB assumption, a *feasible measurement set* $M_i(k)$ can be associated to agent i at time k. Such a set contains all the possible values of the unknown parameter x that are compatible with the current measurement and the noise bound. Besides $M_i(k)$, each agent also stores its *local feasible parameter set* $X_i(k)$ containing the values of x that are compatible with all the

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measurements taken by node i *up to* time k. Clearly, the local feasible parameter set corresponds to the intersection of all the feasible measurement sets collected up to time k, i.e.

$$X_i(k) = \bigcap_{h=1}^{k} M_i(h), \quad i = 1, \dots, N. \tag{1}$$

By construction, $\{X_i(k)\}$ is a *nonincreasing sequence of sets*, i.e. $X_i(k+1) \subseteq X_i(k)$ for all k. Hence, according to the definition of limit of a sequence of sets [17], it converges to a set

$$X_i = \lim_{k\to\infty} X_i(k). \tag{2}$$

As a result, it is possible to define the *global asymptotic feasible parameter set* as

$$X = \bigcap_{i=1}^{N} X_i. \tag{3}$$

We are interested in finding a distributed set membership estimator that returns a sequence of estimates eventually converging to the global asymptotic feasible parameter set X, i.e., an *asymptotically interpolatory* estimator. A possible solution is provided by the distributed estimation algorithm described below.

An agent updates its estimate in two steps. First, it computes a weighted average of its estimate and those of its neighbors. Then, it projects such an average on the current local feasible parameter set. Let $x_i(k)$ be the estimate computed at time k by node i. For $i = 1, \dots, N$ and $k = 0, 1, 2, \dots$, the distributed estimation algorithm can be written as

$$X_i(k+1) = X_i(k) \cap M_i(k+1), \tag{4}$$

$$z_i(k) = \sum_{j=1}^{N} a_{ij}(k) x_j(k), \tag{5}$$

$$x_i(k+1) = P_{X_i(k+1)}[z_i(k)], \tag{6}$$

with the initial conditions $x_i(0)$ and $X_i(0) = \mathbb{R}^n$. Clearly, the weights $a_{ij}(k)$ must comply with the topology of the communication graph, i.e. $a_{ij}(k) = 0$ if $(j, i) \notin \mathcal{E}(k)$.

III. Convergence analysis

In this section, the convergence properties of Algorithm (4)-(6) are analyzed. In particular, it is shown that it is indeed an asymptotically estimator.

The following assumptions are enforced throughout the paper.

Assumption 1 (Sets): The feasible measurement sets $M_i(h)$, $h = 1, 2, \dots$, in (1) are closed convex sets. Moreover, the global asymptotic feasible parameter set X in (3) is not empty.

Assumption 2 (Network): There exist $T > 0$ and an infinite sequence of time instants $\{k_l\}$, $l = 0, 1, \dots$, such that for all l:

i) $k_{l+1} - k_l < T$;
ii) the union of graphs $\mathcal{G}(k_l)$, $\mathcal{G}(k_l+1), \dots, \mathcal{G}(k_{l+1}-1)$ is strongly connected.

Moreover, for all $i, j = 1, \dots, N$, and for all k, the weights $a_{ij}(k) \geq 0$ in (5) satisfy:

iii) $a_{ii}(k) > 0$;
iv) if $i \neq j$, $a_{ij}(k) > 0$ if and only if $(j, i) \in \mathcal{E}(k)$;
v) $\sum_{j=1}^{N} a_{ij}(k) = 1$;
vi) if $a_{ij}(k) > 0$, then $a_{ij}(k) > \eta$, for some $\eta > 0$. □

Under Assumption 1, the local feasible parameter sets $X_i(k)$, as well as their limit sets X_i, are closed convex sets, being the intersection of (possibly infinitely many) closed convex sets. The nonemptiness of the asymtpotic feasible set X is guaranteed if the measurement noise satisfies the UBB constraints. Assumptions 2 is typically made to ensure consensus achievement in the presence of time-varying graph topologies (e.g., see [15], [16]).

Let us recall the following results, concerning infinite sequences of closed convex sets. The proofs can be found in [12].

Lemma 1: Let $\{Z(k)\}$, $k = 1, 2, \dots$, be a nonincreasing sequence of closed convex subsets of $\mathbb{R}^n$ and denote by Z its limit. Consider an arbitrary point $p \in \mathbb{R}^n$ and let $q(k) = P_{Z(k)}[p]$ be its projection on $Z(k)$. Then, for any $z \in Z$ and $k = 1, 2, \dots$, it holds

$$(p - q(k))^\top (z - q(k)) \leq 0. \tag{7}$$

□

Lemma 2: Let $\{Z(k)\}$, $k = 1, 2, \dots$, be a nonincreasing sequence of closed convex subsets of $\mathbb{R}^n$ and denote by Z its limit. Let $\{z(k) \in Z(k)\}$ be a sequence admitting a convergent subsequence $\{z(k_j)\}$, $j = 1, 2, \dots$, to a point $\hat{z}$. Then, $\hat{z} \in Z$. □

In order to prove the convergence of the estimates generated by Algorithm (4)-(6), we need to exploit several properties of networks with time-varying graph topology, which essentially stem from Assumption 2. We briefly summarize them hereafter; the reader interested in the details of the derivations can find them in the referenced papers.

Denote by $A(k) \in \mathbb{R}^{N\times N}$ the matrix whose ij-th entry is $a_{ij}(k)$. Let

$$\Psi(t, s) = A(t-1) \dots A(s+1) A(s), \quad t > s \tag{8}$$

be the state transition matrix from time s to time t, with the convention $\Psi(t, t) = I$. By defining

$$u_i(k) = P_{X_i(k+1)}[z_i(k)] - z_i(k), \tag{9}$$

expressions (5)-(6) can be written as

$$x_i(k+1) = \sum_{i=1}^{N} a_{ij}(k) x_j(k) + u_i(k), \tag{10}$$

for $i = 1, \dots, N$. Let $\{k_l\}$ be a sequence of time instants defined as in Assumption 2. Then, the evolution of the estimates $x(k)$ at time instants k_l is given by

$$x(k_{l+1}) = \Psi(k_{l+1}, k_l) x(k_l) + \sum_{r=k_l+1}^{k_{l+1}} \Psi(k_{l+1}, r) u(r-1). \tag{11}$$

Under Assumptions 2, $\Psi(k_{l+1}, k_l)$ is a row-stochastic, irreducible and aperiodic matrix, for all k_l. Hence, $\{\Psi(k_{l+1}, k_l)\}$, $l = 0, 1, \ldots$, is a sequence of matrices admitting an absolute probability sequence $\{v(k_l)\}$

$$v^\top(k_l) = v^\top(k_{l+1})\Psi(k_{l+1}, k_l) \tag{12}$$

satisfying $v_i(k_l) > \delta$, for all $i = 1, \ldots, N$ and all k_l, for some $\delta > 0$ [16], [18]. An absolute probability sequence with such a property is said to be uniformly bounded away from zero. Therefore, summing up, there always exist a scalar $\delta > 0$ and an absolute probability sequence $\{v(k_l)\}$ such that for $i = 1, \ldots, N$ and all k_l:

$$v_i(k_l) > \delta \tag{13}$$

$$\sum_{i=1}^{N} v_i(k_l) = 1 \tag{14}$$

$$\sum_{i=1}^{N} v_i(k_{l+1}) \left[\Psi(k_{l+1}, k_l)\right]_{ij} = v_j(k_l) \tag{15}$$

with $\Psi(k_{l+1}, k_l)$ defined as in (8).

Let $u_i(k)$, $i = 1, \ldots, N$, $k = 1, 2, \ldots$, be defined as in (9), where $z_i(k)$ is computed according to (4)-(6). Then, by exploiting properties (13)-(15) and Lemma 1, it is possible to show that

$$\lim_{k\to\infty} \|u_i(k)\| = 0, \quad i = 1, \ldots, N. \tag{16}$$

The proof of (16) relies on arguments similar to those used in [16] and is omitted due to space constraints.

Now, let $\{x_i(k)\}$, $i = 1, \ldots, N$, $k = 1, 2, \ldots$, be the sequences of estimates computed according to (4)-(6). Define

$$y(k) = \frac{1}{N} \sum_{i=1}^{N} x_i(k). \tag{17}$$

Then, it is possible to show that

$$\lim_{k\to\infty} \|y(k) - x_i(k)\| = 0 \ , \quad i = 1, \ldots, N. \tag{18}$$

In order to prove (18), one can adopt a procedure analogous to that employed for Lemma 9 in [15]. The main argument consists essentially in studying the evolution of system (10) when driven by a vanishing input, as guaranteed by (16).

We are now ready to prove the main result of the paper.

Theorem 1: Let Assumptions 1-2 hold. Let $\{x_i(k)\}$, $i = 1, \ldots, N$, $k = 1, 2, \ldots$, be the sequences of estimates computed according to (4)-(6). Then, there exists a point $\hat{x} \in X$, with X given by (1)-(3), such that

$$\lim_{k\to\infty} x_i(k) = \hat{x}, \quad i = 1, \ldots, N.$$

Proof: Let $y(k)$ be given by (17) and define

$$q_i(k) = P_{X_i(k)}[y(k)]. \tag{19}$$

Since $x_i(k) \in X_i(k)$ for $i = 1, 2, \ldots, N$ and all k, we have that

$$\sum_{i=1}^{N} \|y(k) - q_i(k)\| \leq \sum_{i=1}^{N} \|y(k) - x_i(k)\| . \tag{20}$$

By exploiting (18), (20) implies that

$$\lim_{k\to\infty} \sum_{i=1}^{N} \|y(k) - q_i(k)\| = 0$$

and thus

$$\lim_{k\to\infty} \|y(k) - q_i(k)\| = 0 \tag{21}$$

for $i = 1, \ldots, N$. From (21), we can conclude that

$$\lim_{k\to\infty} \|q_i(k) - q_j(k)\| = 0 \tag{22}$$

for $i, j = 1, \ldots, N$.

Now, let $\bar{x}$ be any point in X. From (5),(6), one has that for $i = 1, \ldots, N$

$$\begin{aligned} \|x_i(k) - \bar{x}\|^2 &\leq \|z_i(k-1) - \bar{x}\|^2 \\ &\leq \sum_{j=1}^{N} a_{ij}(k-1) \|x_j(k-1) - \bar{x}\|^2 \end{aligned} \tag{23}$$

where the first inequality comes from the properties of projections on convex sets and the second one from Jensen's inequality. By iterating the inequalities above backwards in time and recalling the definition of the transition matrix $\Psi(\cdot, \cdot)$ in (8), one gets

$$\begin{aligned} \|x_i(k) - \bar{x}\|^2 &\leq \sum_{j=1}^{N} \left[\Psi(k, 0)\right]_{ij} \|x_j(0) - \bar{x}\|^2 \\ &\leq \max_j \|x_j(0) - \bar{x}\|^2 , \end{aligned} \tag{24}$$

where the last inequality comes from the row-stochasticity of matrix $\Psi(k, 0)$. From (24), one has that all the sequences $\{x_i(k)\}$ are bounded, and therefore so is the sequence $\{y(k)\}$. Consequently, from (21), the sequences $\{q_i(k)\}$ are bounded too and hence they admit a converging subsequence $\{q_i(k_h)\}$, i.e.

$$\lim_{h\to\infty} q_i(k_h) = \hat{x}_i. \tag{25}$$

By Lemma 2, the limit points $\hat{x}_i \in X_i$. But, from (22) one has that $\hat{x}_1 = \cdots = \hat{x}_N \triangleq \hat{x} \in X$. From (18) and (21), it follows

$$\lim_{h\to\infty} x_i(k_h) = \hat{x} \tag{26}$$

for $i = 1, \ldots, N$. By the definition of limit, for any $\epsilon > 0$ there exists a $\hat{h}$ such that $\|x_i(k_h) - \hat{x}\| < \epsilon$ for all $h \geq \hat{h}$ and $i = 1, \ldots, N$.

Using the same arguments as in (23)-(24), one has that for $i = 1, \ldots, N$ and all $k \geq k_{\hat{h}}$

$$\begin{aligned} \|x_i(k) - \hat{x}\|^2 &\leq \|z_i(k-1) - \hat{x}\|^2 \\ &\leq \sum_{j=1}^{N} \left[\Psi(k, k_{\hat{h}})\right]_{ij} \left\|x_j(k_{\hat{h}}) - \hat{x}\right\|^2 \\ &\leq \max_j \left\|x_j(k_{\hat{h}}) - \hat{x}\right\|^2 \leq \epsilon^2, \end{aligned}$$

where the third inequality comes from the row-stochasticity of matrices $\Psi(k, k_{\hat{h}})$. Hence, the thesis follows. □

Remark 1: Theorem 1 can be seen as an extension of the convergence results presented in [15], [16] on constrained consensus algorithms in the presence of time-varying topology. A major difference with respect to those works is that in the recursive estimation problem considered here, the local feasible set $X_i(k)$ is updated at each time instant, rather than being constant. As a result, an agent projects the current estimate each time on a different set. Theorem 1 shows that convergence is guaranteed as long as the sequence of local feasible sets is nonincreasing. □

Remark 2: Assumption 2 about the time-varying topology of the network is similar to those adopted in related work such as [15], [16]. In particular, Assumption 2 does not require that the communication graphs be strongly connected at any time instant. It is sufficient that the union of graphs over uniformly bounded time intervals is jointly strongly connected. Moreover, unbalanced networks are allowed, resulting in weight matrices not necessarily doubly stochastic. As a result, this framework can be used to study the convergence of asynchronous implementations of Algorithm (4)-(6), as explained in the next section. The presence of time delays can also be accounted for, by adopting a framework similar to the one in [15], where an extended model is studied by defining a state vector that includes delayed copies of the agent states taken at different time instants.

Remark 3: At time instant $k+1$, agent i updates its local feasible set $X_i(k+1)$ by computing the intersection of the current local feasible set $X_i(k)$ with the set $M_i(k+1)$ associated to the new measurement available (see (4)). Hence, the local feasible sets $X_i(k)$ form a nonincreasing sequence of sets by construction. It is easy to see that when such a property does not hold, the convergence of all the estimates $x_i(k)$ to a common point belonging to the global feasible set X cannot be guaranteed anymore. At the same time, this limits the applicability of the proposed distributed algorithm to scenarios in which the structure of the measurement sets is such that the intersection in (4) can be efficiently computed.

IV. Asynchronous implementation

An asynchronous version of the considered distributed set membership estimator can be recovered as a special instance of Algorithm (4)-(6). Consider N nodes connected over a network with static topology, modelled as a strongly connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Nodes interact according to a communication setup in which each node wakes up when a local timer triggers. Similarly to the scenario considered in [19], when a node is awake it gathers local estimates from its neighbors and performs steps (4)-(6). Then, it switches to an idle mode in which it can only send its estimate x_i upon request of other nodes.

From a global perspective this asynchronous model can be treated as a synchronous one in which, at each iteration k, only a subset of edges $\mathcal{E}(k) = \{(j,i) \mid i \in \mathcal{V}(k), j \in \mathcal{N}_i^{in}\} \subseteq \mathcal{E}$ is active, with $\mathcal{V}(k) \subseteq \mathcal{V}$ being the set of nodes that are awake at iteration k and $\mathcal{N}_i^{in}$ denoting the set of neighbors of node i in the communication graph $\mathcal{G}$. If we assume that the time between two consecutive triggering events of any node is uniformly bounded by a constant T, the graph sequence $\mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k))$ clearly satisfies Assumption 2 with constant T. If at iteration k node i is idle (i.e., $i \notin \mathcal{V}(k)$), then its measurement set in (1) is taken as $M_i(k) = \mathbb{R}^n$ and its weights satisfy $a_{ii}(k) = 1$ and $a_{ij}(k) = 0$ for all $j \neq i$. This asynchronous setup fits Algorithm (4)-(6). A graphical illustration is depicted in Fig. 1 for a simple network. Clearly, the above discussion can be further extended to asynchronous communication protocols over time-varying graphs with a similar reasoning.

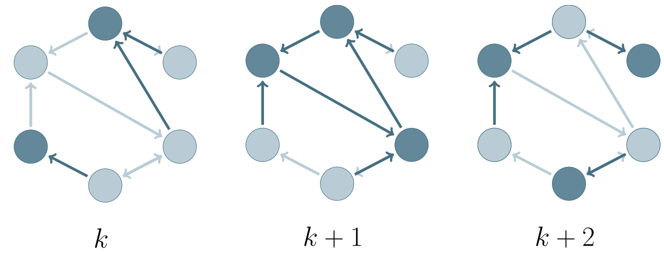

Fig. 1. Asynchronous algorithm. At each iteration k, nodes in $\mathcal{V}(k)$ and edges in $\mathcal{E}(k)$ are depicted in blue, while inactive ones are shaded. After 3 iterations, the union of the graphs is strongly connected.

V. Numerical example

In order to illustrate how the proposed distributed set membership estimator works, in this section we tackle two numerical problems: linear regression and target localization in a sensor network. In particular, we tested algorithm (4)-(6) with two different communication protocols, as described below.

SDIA: Synchronous Distributed Interpolatory Algorithm. The topology of the communication graph is static (i.e., $\mathcal{E}(k) = \mathcal{E}$) and it is used to simulate a synchronous communication protocol, similar to the setup considered in [12].

ADIA: Asynchronous Distributed Interpolatory Algorithm. The topology of the communication graph is time-varying (i.e., $\mathcal{E}(k_1) \neq \mathcal{E}(k_2)$) and it is used to simulate the asynchronous implementation of (4)-(6) described in the previous section.

A. Linear regression

At iteration k, the node $i \in \mathcal{V}(k)$ takes a measurement $\psi_i(k) \in \mathbb{R}$ of an unknown parameter $\theta^\star \in \mathbb{R}^n$, according to

$$\psi_i(k) = \phi_i^\top \theta^\star + \xi_i(k). \tag{27}$$

In (27), $\phi_i \in \mathbb{R}^n$ is a regressor vector associated to node i and $\xi_i(k) \in \mathbb{R}$ is an UBB noise satisfying

$$|\xi_i(k)| \leq \epsilon_i, \quad \forall k, \tag{28}$$

with $\epsilon_i \geq 0$. Without loss of generality, assume $\|\phi_i\| = 1$, for all i. From (27) and (28), each measurement set is a strip in the parameter space, i.e.

$$M_i(k) = \{\theta \in \mathbb{R}^n : |\phi_i^\top \theta - \psi_i(k)| \leq \epsilon_i\}. \tag{29}$$

From (4) and (29), the local feasible set for node i at time k can be easily computed as

$$X_i(k) = \{\theta \in \mathbb{R}^n : \max_{h \le k}\{\psi_i(h)\} - \epsilon_i \le \phi_i^\top \theta \le \min_{h \le k}\{\psi_i(h)\} + \epsilon_i\}, \tag{30}$$

which requires just to store the maximum and minimum value of the measurements $\psi_i(k)$.

A Monte Carlo simulation has been carried out for the case $n = 5$. For different values of the number of nodes N ranging from 7 to 200, 1000 simulation runs have been performed. The initial estimates $x_i(0)$ and the regressors ϕ_i in (27) are randomly generated at each run. The measurement errors $\xi_i(k)$ are uniformly distributed in $[-\epsilon_i,\ \epsilon_i]$, with $\epsilon_i = 0.1$. Each simulation run is stopped when the distance of the estimates of all the nodes to X is smaller than a desired accuracy $\delta = 10^{-3}$. A static complete graph has been used for the SDIA. The same complete graph has been used as underlying communication graph for the ADIA. In this case, the asynchronous awakening of the nodes has been randomly generated so that the resulting sequence of communication graphs $\mathcal{G}(k)$ satisfies Assumption 2 with $T = 2$. The average number of iterations required to reach the desired accuracy is reported in Fig. 2. It can be seen that the larger the network the smaller the number of iterations. Moreover, as expected, the asynchronous version of the algorithm exhibits a slower convergence rate when compared to its synchronous counterpart.

A second set of simulations has been carried out in order to highlight the role of the bound T. The ADIA has been tested on two networks of $N = 10$ and $N = 20$ agents, respectively. In both cases, the underlying communication graph has a complete topology, as before. Different random sequences of node awakenings have been simulated, resulting in a sequence of graphs $\mathcal{G}(k)$ satisfying Assumption 2 with different values of T. For each $T \in \{2, 3, \ldots, 20\}$, 1000 simulation runs have been performed. The average number of iterations is reported in Fig. 3. As expected, the convergence rate of Algorithm (4)-(6) heavily depends on T since it influences the rate at which the information is spread through the network. The number of iterations roughly grows linearly with T, with a growth rate that decreases as the network becomes larger.

B. Target localization

In this example, the objective is to estimate the unknown position $p^\star \in \mathbb{R}^2$ of a target in a sensor network. Similarly to the problem considered in [20], we assume that at iteration k each node i, located at position c_i, takes a measurement of the target bearing angle as

$$\psi_i(k) = \text{atan2}(d_{i,2}, d_{i,1}) + \xi_i(k), \tag{31}$$

where $d_i = [d_{i,1},\ d_{i,2}]^\top = p^\star - c_i$ and $\xi_i(k)$ is a UBB noise uniformly distributed in the range $[-\epsilon_i,\ \epsilon_i]$ rad with $\epsilon_i = 0.8$. In this case, both $M_i(k)$ and the feasible parameter sets $X_i(k)$, take on the form of sectors of $\mathbb{R}^2$ originating from

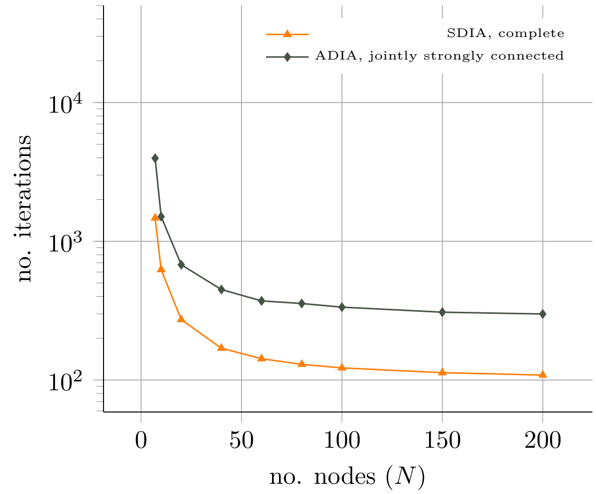

Fig. 2. Comparison of SDIA and ADIA over an underlying complete graph.

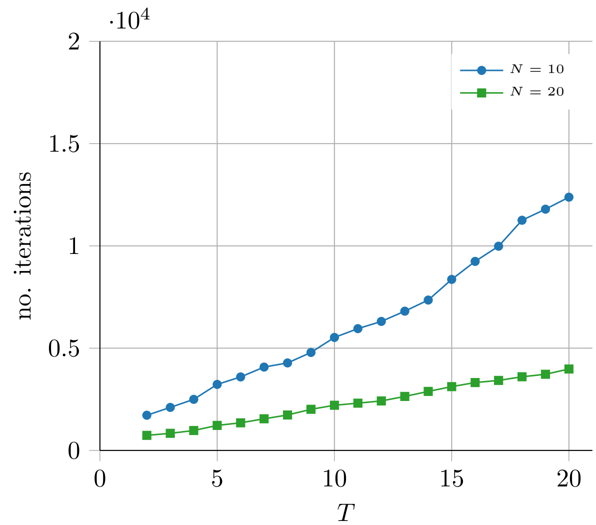

Fig. 3. Effect of T.

the sensor locations c_i. For example, assume $N = 10$ sensors deployed as shown in Fig. 4. Let the actual target position be $p^\star = [0,\ 0]^\top$ and the initial estimates $x_i(0)$ be uniformly distributed in $[-4,\ 4]^2$. In this example, the underlying communication graph has been randomly generated as an Erdős-Rényi graph with probability $p = 0.5$. At each time k, each node is awake with probability 0.4 and it is idle with probability 0.6. The ADIA algorithm has been run for $\bar{k} = 100$ iterations. Figure 4 depicts the feasible parameter set at time $k = 50$.

Define the overall feasible set at iteration k as

$$X(k) = \bigcap_{i=1}^{N} X_i(k).$$

It is the set of all the target locations compatible with the information collected by all the nodes up to time k. Clearly, $X(\bar{k})$ represents an approximation of the asymptotic feasible set X. Figure 5 shows the distances $\text{dist}(x_i(k), X(\bar{k}))$ of the

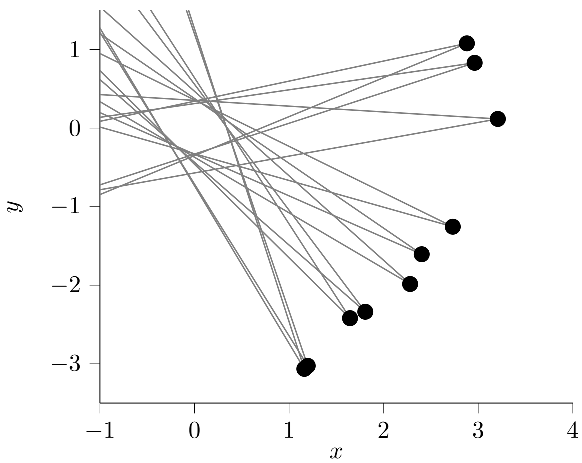

Fig. 4. Target localization: feasible parameter sets (grey sectors) after 50 iterations.

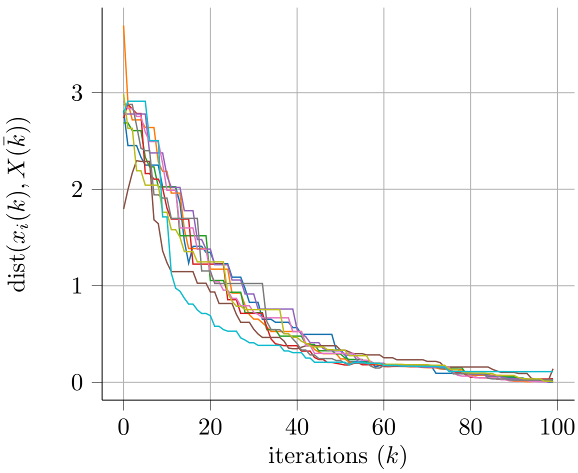

Fig. 5. Target localization: distance of $x_i(k)$ from $X(\bar{k})$ for all i.

sequence of estimates $\{x_i(k)\}$ from the set $X(\bar{k})$. It can be seen that the distances $\mathrm{dist}(x_i(k), X(\bar{k}))$ are not necessarily monotonically non increasing. This phenomenon is due to the consensus step in (5) which can drive an agent estimate faraway from the feasible set. However, in the long run all the distances decrease and eventually tend to zero, according to Theorem 1. Moreover it can be observed that for some i the distance $\mathrm{dist}(x_i(k), X(\bar{k}))$ remains unchanged for some consecutive iterations. What typically happens during these periods is that agent i is not awake, so that $\mathcal{N}_i^{in}(k) = \emptyset$ and the estimate $x_i(k)$ remains the same.

VI. Conclusions

In this paper we have studied the properties of a distributed set membership estimation algorithm, in the presence of time-varying network topology. At each iteration, each agent alternates a consensus step and a projection step over the local feasible set in order to generate a sequence of estimates. It is shown that when the union of communication graphs is jointly strongly connected over finite-length time intervals, convergence of the agents' estimates to a common feasible point is guaranteed. As a side contribution, it is shown how to cast an asynchronous version of the considered algorithm as an equivalent synchronous one, for which the convergence analysis previously carried out applies.

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