Nonlinear Orbit Control with Longitude Tracking

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***Abstract*— The growing level of autonomy of unmanned space missions has attracted a significant amount of research in the aerospace field towards feedback orbit control. Existing Lyapunov-based controllers can be used to to transfer a spacecraft between two elliptic orbits of given size and orientation, but do not consider the stabilization of the spacecraft phase angle along the orbit, which is a key requirement for application to formation flying missions. This paper presents a control law based on the orbital element parametrization, which is able to track a given true longitude (i.e. a reference phase angle), in addition to the parameters describing the reference orbit shape and orientation. A numerical simulation of an orbital rendezvous demonstrates the effectiveness of the proposed approach.**

I. Introduction

A fundamental research topic in astrodynamics deals with transferring a spacecraft between two elliptic orbits. Historically, this problem has been tackled by using optimization techniques [1], [2], [3], [4], or feedback stabilization methods [5], [6], [7], [8]. In the former approach, no closed-form solution is available in general and a two-point boundary value problem is solved numerically to get the optimal open-loop thrusting profile. The related computations are lengthy, thus making this approach not suitable for applications requiring on-line computation of the control signals, such as formation flying or rendez-vous. Existing Lyapunov-based stabilization methods, on the other hand, provide simple feedback controllers, but usually do not consider the transfer time and the injection point on the final orbit.

Most of the literature available on the orbit stabilization problem describes the trajectory of an orbiting body either in terms of cartesian position and velocity or by an equivalent set of variables introduced by Kepler, known as the orbital elements. The latter parametrization is useful because it captures the constants of the orbital motion. References [5], [7], [8] developed nonlinear orbital element feedback schemes based on the Jurdjevic-Quinn conditions [9]. Similar results have been derived in [6], by using the cartesian coordinate representation of the orbital elements. Such techniques have proven to be effective in low-thrust applications [10], [11], [12], but do not address the stabilization of the spacecraft phase angle along the orbit. This angle is often referred to as the true longitude.

Motivated by the increasing number of distributed space missions, the orbit control problem is also widely discussed in the formation flying literature. The interested reader is referred to [13], [14] for a survey on recent results. While the coordination of multiple spacecraft does indeed require to track the (relative) true longitude, the techniques developed to this purpose often rely on linearization assumptions [15], [16], [17], [18]. Therefore, their applicability is limited to spacecraft separated by a very short distance.

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A unified approach has been presented in [19]. By using backstepping and forwarding techniques (see, e.g., [20]), the authors derived a passivity-based controller able to track a given true longitude, in addition to five modified equinoctial orbital elements describing the orbit shape and orientation. Nevertheless, the obtained results are limited to the case of perfectly circular reference orbits, which leaves out many scenarios of theoretical and practical interest. It is known, for instance, that low altitude orbits cannot have zero eccentricity, due to the asymmetry of the Earth's gravity field.

In this paper, a nonlinear control law is proposed which asymptotically stabilizes the six modified equinoctial elements, including the true longitude, of any closed orbit. The solution is arrived at by using a design procedure inspired by backstepping and damping control techniques. The simple structure of the controller makes it suitable for a number of space missions involving orbit reconfiguration and formation flying maneuvers. A numerical simulation of a rendez-vous maneuver is performed to illustrate the proposed approach, and to validate the obtained theoretical results.

The paper is organized as follows. In Section II, a brief introduction to the orbital element parametrization is given and the considered orbit control problem is formulated. Section III is devoted to the design of the controller, which is demonstrated by the numerical simulation in Section IV. Some concluding remarks are outlined in Section V.

Notation

The notation is fairly standard. $\mathbb{R}^n$ is the real $n-$space; for a real vector or matrix v, v^T denotes its transpose. To save space, $\cos(\cdot)$, $\sin(\cdot)$ are abbreviated with $\mathrm{c}(\cdot)$ and $\mathrm{s}(\cdot)$, respectively. Moreover,

$$R(\phi) = \begin{bmatrix} \mathrm{c}(\phi) & -\mathrm{s}(\phi) \\ \mathrm{s}(\phi) & \mathrm{c}(\phi) \end{bmatrix}$$

is the counter-clockwise rotation operator by an angle ϕ in $\mathbb{R}^2$. The continuous time index is denoted as $t \in \mathbb{R}^+$.

II. Problem formulation

Classical orbital elements are commonly used as a parametrization of the position $r \in \mathbb{R}^3$ and velocity $\dot{r} \in \mathbb{R}^3$ of an orbiting body, since they provide a clear physical insight of the body motion. The semi-major axis $a > 0$ and eccentricity $e \in [0, 1]$ define the shape of the orbit.

The inclination $i \in [0, \pi]$, longitude of the ascending node $\Omega \in [0, 2\pi]$ and argument of perigee $\omega \in [0, 2\pi]$ define the orientation of the orbital plane with respect to a given inertial, right-handed reference frame centered at the central body (e.g., the Earth). The true anomaly $\nu(t) \in [0, 2\pi]$ defines the instantaneous angle at which the spacecraft is located relative to the perigee position, as illustrated in Fig. 1.

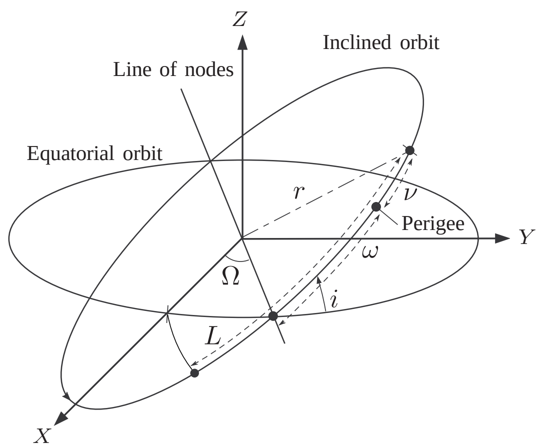

Fig. 1. Orbital elements.

It is well known that ω is indeterminate for circular orbits (i.e., when $e = 0$) and Ω is indeterminate for equatorial orbits (i.e., when $i = 0$). These singularities can be avoided by adopting a different parameterization of the orbit using the modified equinoctial elements $\psi = [\psi_1 \ \dots \ \psi_6]^T$, defined as [21]

$$\begin{array}{lllll} \psi_1 & = & L & = & \Omega + \omega + \nu \\ \psi_2 & = & p & = & a(1 - e^2) \\ \psi_3 & = & e_X & = & e \cdot \mathsf{c}(\Omega + \omega) \\ \psi_4 & = & e_Y & = & e \cdot \mathsf{s}(\Omega + \omega) \\ \psi_5 & = & h_X & = & \tan(i/2)\,\mathsf{c}(\Omega) \\ \psi_6 & = & h_Y & = & \tan(i/2)\,\mathsf{s}(\Omega). \end{array} \quad (1)$$

In this parameterization, L is the true longitude shown in Fig. 1, p is the orbit semi-parameter, e_X, e_Y are the components of the eccentricity vector, and h_X, h_Y are the components of the inclination vector. Notice that any closed Keplerian orbit is such that $\psi_2 = p > 0$. Moreover, the escape to parabolic orbits (i.e., $e = 1$) is not possible with continuous feedback [22], which is the case considered in this paper. Therefore, in the following we restrict our attention to the case $e < 1$. Hence, the state vector ψ must belong to the set

$$\Psi = \{\psi \in \mathbb{R}^6 : \ \psi_2 > 0, \quad \psi_3^2 + \psi_4^2 < 1\}.$$

The dynamics of the orbital elements ψ in (1), in the presence of forcing inputs, are described by Gauss's variational equations. Let us introduce the control input vector $u = [u_\theta \ u_r \ u_h]^T$, where u_θ, u_r and u_h denote the along-track, radial and cross-track components of the acceleration, respectively. The resulting dynamics can be expressed as [23]:

$$\dot{\psi} = f(\psi) + g(\psi)u, \quad (2)$$

where the vector fields $f(\psi)$ and $g(\psi)$ are given by

$$f(\psi) = \left[\ \sqrt{\frac{\mu}{\psi_2^3}}(1 + \zeta_X)^2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \ \right]^T, \quad (3)$$

$$g(\psi) = \begin{bmatrix} 0 & 0 & \sqrt{\frac{\psi_2}{\mu}}\frac{\eta}{1+\zeta_X} \\ \sqrt{\frac{\psi_2^3}{\mu}}\frac{2}{1+\zeta_X} & 0 & 0 \\ \sqrt{\frac{\psi_2}{\mu}}\frac{q_X}{1+\zeta_X} & \sqrt{\frac{\psi_2}{\mu}}\,\mathsf{s}(\psi_1) & -\sqrt{\frac{\psi_2}{\mu}}\frac{\psi_4 \eta}{1+\zeta_X} \\ \sqrt{\frac{\psi_2}{\mu}}\frac{q_Y}{1+\zeta_X} & -\sqrt{\frac{\psi_2}{\mu}}\,\mathsf{c}(\psi_1) & \sqrt{\frac{\psi_2}{\mu}}\frac{\psi_3 \eta}{1+\zeta_X} \\ 0 & 0 & \sqrt{\frac{\psi_2}{\mu}}\frac{1+h^2}{2(1+\zeta_X)}\,\mathsf{c}(\psi_1) \\ 0 & 0 & \sqrt{\frac{\psi_2}{\mu}}\frac{1+h^2}{2(1+\zeta_X)}\,\mathsf{s}(\psi_1) \end{bmatrix}. \quad (4)$$

In (3)-(4),

$$\begin{aligned} \begin{bmatrix} \zeta_X \\ \zeta_Y \end{bmatrix} &= R(\psi_1) \begin{bmatrix} \psi_3 \\ -\psi_4 \end{bmatrix} \\ q_X &= \psi_3 + (2 + \zeta_X)\,\mathsf{c}(\psi_1) \\ q_Y &= \psi_4 + (2 + \zeta_X)\,\mathsf{s}(\psi_1) \\ \eta &= \psi_5\,\mathsf{s}(\psi_1) - \psi_6\,\mathsf{c}(\psi_1) \\ h^2 &= \psi_5^2 + \psi_6^2, \end{aligned}$$

and μ is the gravitational parameter of the central body. Notice that ζ_Y does not affect the system dynamics.

The control objective is to track the reference trajectory specified by the orbital elements

$$\psi^r(t) = [\psi_1^r(t), \psi_2^r, \psi_3^r, \psi_4^r, \psi_5^r, \psi_6^r]^T,$$

which are the solution to equation (2) with $u = 0$, i.e.,

$$\dot{\psi}^r = f(\psi^r), \quad (5)$$

corresponding to the given initial condition

$$\psi^r(0) = [L^r(0), \ p^r, \ e_X^r, \ e_Y^r, \ h_X^r, \ h_Y^r]^T \in \Psi.$$

Let $\tilde{\psi} = \psi - \psi^r$ denote the tracking error. Then, the error dynamics evolves according to the time-varying system

$$\dot{\tilde{\psi}} = \tilde{f}(\tilde{\psi}; \psi^r) + g(\tilde{\psi} + \psi^r)u, \quad (6)$$

where $\tilde{f}(\tilde{\psi}; \psi^r) = f(\tilde{\psi} + \psi^r) - f(\psi^r)$. The orbit control problem considered in this paper can be formulated as follows.

Problem 1: Find a continuous state feedback control law $u = u(\tilde{\psi}; \psi^r)$ such that the error system (6) is globally asymptotically stable, which in turn guarantees that

$$\lim_{t \to \infty} \tilde{\psi}(t) = 0$$

for any initial condition $\psi(0), \psi^r(0) \in \Psi$.

III. Control system design

In order to derive a solution to Problem 1, we first introduce a diffeomorphic coordinate transformation $x = x(\tilde{\psi};\psi^r)$ in system (6), defined as follows

$$x_1 = \tilde{\psi}_1 \tag{7}$$

$$x_2 = \sqrt{1+\frac{\tilde{\psi}_2}{\psi_2^r}}-1 \tag{8}$$

$$\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{\psi_2^r}{\tilde{\psi}_2+\psi_2^r} & 0 \\ 0 & \sqrt{\frac{\psi_2^r}{\tilde{\psi}_2+\psi_2^r}} \end{bmatrix} R(\tilde{\psi}_1+\psi_1^r) \begin{bmatrix} \tilde{\psi}_3+\psi_3^r \\ -\tilde{\psi}_4-\psi_4^r \end{bmatrix} + \begin{bmatrix} -\frac{\tilde{\psi}_2}{\tilde{\psi}_2+\psi_2^r} \\ 0 \end{bmatrix} - \begin{bmatrix} \zeta_X^r \\ \zeta_Y^r \end{bmatrix}, \tag{9}$$

$$x_5 = \tilde{\psi}_5 \tag{10}$$

$$x_6 = \tilde{\psi}_6, \tag{11}$$

where

$$\begin{bmatrix} \zeta_X^r \\ \zeta_Y^r \end{bmatrix} = R(\psi_1^r) \begin{bmatrix} \psi_3^r \\ -\psi_4^r \end{bmatrix}.$$

A similar transformation is used in [19] for the case of circular reference orbits (i.e., $\zeta_X^r = \zeta_Y^r = 0$). System (6) in the new coordinate set has the form

$$\dot{x} = \begin{bmatrix} F(\chi;\psi^r) \\ 0_{2\times1} \end{bmatrix} + \begin{bmatrix} G(\chi;\psi^r) \\ 0_{2\times2} \end{bmatrix} \begin{bmatrix} u_\theta \\ u_r \end{bmatrix} + H(x;\psi^r)\, u_h\,, \tag{12}$$

where $\chi = [x_1 \dots x_4]^T$ and

$$F(\chi,\psi_r) = \begin{bmatrix} 0 & F_{12} & F_{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -F_{33} & -F_{12} \\ 0 & F_{42} & F_{12}+F_{43} & 0 \end{bmatrix} \chi$$

$$G(\chi;\psi^r) = \begin{bmatrix} 0 & 0 \\ G_{21} & 0 \\ 0 & 0 \\ 0 & G_{42} \end{bmatrix},$$

being

$$F_{12} = \sqrt{\frac{\mu}{{\psi_2^r}^3}}\,(x_3+1+\zeta_X^r)^2$$

$$F_{13} = \sqrt{\frac{\mu}{{\psi_2^r}^3}}\,(x_3+2+2\zeta_X^r)$$

$$F_{42} = \sqrt{\frac{\mu}{{\psi_2^r}^3}}\,(x_2+2)\,(x_3+1+\zeta_X^r)^3$$

$$F_{33} = F_{13}\,\zeta_Y^r$$

$$F_{43} = F_{13}\,\zeta_X^r$$

$$G_{21} = \sqrt{\frac{\psi_2^r}{\mu}}\frac{1}{(x_3+1+\zeta_X^r)}$$

$$G_{42} = \sqrt{\frac{\psi_2^r}{\mu}}.$$

Note that, by virtue of the properties of the considered problem, all F_{ij} and G_{ij} are positive functions of χ and ψ^r, except for F_{33} and F_{43}. The vector $H(x;\psi^r)$ in (12) can be computed as

$$H(x;\psi^r) = \frac{\partial x}{\partial \tilde{\psi}}\, g_h(\tilde{\psi}+\psi^r), \tag{13}$$

where $g_h(\cdot)$ denotes the third column of $g(\cdot)$ in (4), and the right hand side of (13) can be expressed in terms of χ and ψ^r by inverting the coordinate transformation (7)-(11). The expression of $\partial x/\partial\tilde{\psi}$ is reported in Appendix (a).

The structure of system (12) allows one to tackle the control design problem in a two-step procedure.

Step 1

In the first step, we assume $u_h = 0$ and derive a feedback stabilizer for the fourth order subsystem

$$\dot{\chi} = F(\chi;\psi^r) + G(\chi;\psi^r)\begin{bmatrix} u_\theta \\ u_r \end{bmatrix} \tag{14}$$

using the inputs u_θ and u_r. The stabilization of the full system will be addressed afterwards, using the input u_h.

For system (14), let us first introduce the following non-singular transformation in the input variables

$$u_\theta = \frac{1}{G_{21}}\,(v - F_{12}k_1x_1) \tag{15}$$

$$u_r = \frac{1}{G_{42}}\,(w - F_{43}x_3)\,, \tag{16}$$

where v and w are the new input variables and k_1 is any given positive constant, to be treated as a design parameter. Hence, system (14) becomes

$$\dot{x}_1 = F_{12}x_2 + F_{13}x_3 \tag{17}$$

$$\dot{x}_2 = -F_{12}k_1x_1 + v \tag{18}$$

$$\dot{x}_3 = -F_{33}x_3 - F_{12}x_4 \tag{19}$$

$$\dot{x}_4 = F_{42}x_2 + F_{12}x_3 + w. \tag{20}$$

Let

$$x_4^* = \frac{1}{F_{12}}\left(F_{13}k_1k_2^{-1}x_1 - F_{33}x_3 + \lambda_3\right), \tag{21}$$

where $k_2 > 0$ is constant and $\lambda_3(x_1,x_2,x_3)$ is any given continuous function such that sgn(λ_3) = sgn(x_3). Similarly to backstepping control design, we use x_4^* as a virtual input for system (17)-(19), and consider the transformed state vector

$$z = [z_1\ \ z_2\ \ z_3\ \ z_4]^T = [x_1\ \ x_2\ \ x_3\ \ (x_4 - x_4^*)]^T. \tag{22}$$

In the new coordinates, equations (17)-(20) read

$$\dot{z}_1 = F_{12}z_2 + F_{13}z_3 \tag{23}$$

$$\dot{z}_2 = -F_{12}k_1z_1 + v \tag{24}$$

$$\dot{z}_3 = -F_{13}k_1k_2^{-1}z_1 - F_{12}z_4 - \lambda_3 \tag{25}$$

$$\dot{z}_4 = F_{42}z_2 + F_{12}z_3 + w - \dot{x}_4^*. \tag{26}$$

Consider the Lyapunov function candidate

$$V_1(z) = \frac{k_1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{k_2}{2}z_3^2 + \frac{k_2}{2}z_4^2. \tag{27}$$

The time derivative of (27), along the trajectories of (23)-(26), reads

$$\dot{V}_1(z;\psi^r) = (v+F_{42}k_2z_4)z_2 - k_2\lambda_3 z_3 + (w-\dot{x}_4^*)k_2z_4. \quad (28)$$

In order to render $\dot{V}_1$ in (28) negative semidefinite, we make the following choice for the inputs v and w

$$v = -F_{42}k_2z_4 - \lambda_2 \quad (29)$$

$$w = \dot{x}_4^* - \lambda_4, \quad (30)$$

where $\lambda_2(z)$ and $\lambda_4(z)$ are continuous functions such that $\text{sgn}(\lambda_2) = \text{sgn}(z_2)$ and $\text{sgn}(\lambda_4) = \text{sgn}(z_4)$, respectively. With this particular choice, (28) boils down to

$$\dot{V}_1(z) = -\lambda_2 z_2 - k_2\lambda_3 z_3 - k_2\lambda_4 z_4\,, \quad (31)$$

which is indeed negative semidefinite. Clearly, (31) vanishes in the set $\{z : z_2 = z_3 = z_4 = 0\}$. Notice from (23) that z_1 is constant in this set. Moreover, it follows from (24)-(25) that $\dot{z}_2 \neq 0$ and $\dot{z}_3 \neq 0$ for $z_1 \neq 0$. Hence, the largest invariant set in which $\dot{V}_1 = 0$ is $z = 0$. Since $\chi = 0$ for $z = 0$, one can conclude that the equilibrium point $\chi = 0$ is globally asymptotically stabilized by the proposed control law.

The final expression of the control inputs u_θ and u_r is obtained from (15)-(16), (22) and (29)-(30) as

$$u_\theta(\chi;\psi^r) = -\frac{1}{G_{21}}[F_{12}k_1x_1 + F_{42}k_2(x_4 - x_4^*)] - \frac{\lambda_2}{G_{21}} \quad (32)$$

$$u_r(\chi;\psi^r) = -\frac{1}{G_{42}}\left(F_{43}\,x_3 - \dot{x}_4^*\right) - \frac{\lambda_4}{G_{42}}, \quad (33)$$

with x_4^* given by (21).

Step 2

The stabilization of the full system can be tackled by introducing the previously derived control inputs u_θ, u_r in (12) and rewriting the resulting dynamics as

$$\dot{x} = \begin{bmatrix} F_{cl}(\chi;\psi^r) \\ 0_{2\times 1} \end{bmatrix} + H(x;\psi^r)\,u_h, \quad (34)$$

where

$$F_{cl}(\chi;\psi^r) = F(\chi;\psi^r) + G(\chi;\psi^r)\begin{bmatrix} u_\theta(\chi;\psi^r) \\ u_r(\chi;\psi^r) \end{bmatrix}. \quad (35)$$

Given the structure of (34), it turns out that the origin of the full system can be globally stabilized via a damping controller, as explained next. Consider the Lyapunov function

$$V_2(x;\psi^r) = V_1(z;\psi^r) + \frac{1}{2}x_5^2 + \frac{1}{2}x_6^2, \quad (36)$$

where z is defined by (22). The time derivative of (36), along the trajectories of (34), can be written as

$$\dot{V}_2(x;\psi^r) = \dot{V}_1(z) + \frac{\partial V_2(x;\psi^r)}{\partial x}H(x;\psi^r)u_h, \quad (37)$$

where $\dot{V}_1(z)$ is given by (31).

Let

$$u_h(x;\psi^r) = -d\,\frac{\partial V_2(x;\psi^r)}{\partial x}\,H(x;\psi^r), \quad (38)$$

where d is a given positive scalar continuous function,

$$\frac{\partial V_2(x;\psi^r)}{\partial x} = \left[\frac{\partial V_1(z;\psi^r)}{\partial \chi} \;\; x_5 \;\; x_6\right], \quad (39)$$

and the expression of $\partial V_1(z;\psi^r)/\partial\chi$ is reported in Appendix (b). With this particular choice, (37) is negative semidefinite and vanishes if and only if $z_2 = z_3 = z_4 = 0$ and $u_h = 0$. Then, according to the previous analysis, $\lim_{t\to\infty}\chi(t) = 0$. Using this fact and observing that x_5 and x_6 are constant for $u_h = 0$, it can be verified from (13), (34) and (38)-(39) that the largest invariant set in which $\dot{V}_2 = 0$ is the trivial one $x = 0$. Hence, the proposed control law renders the equilibrium point of the full system (34) globally asymptotically stable with the Lyapunov function (36).

Summarizing, the feedback control law defined by u_θ and u_r given by (32)-(33), combined with u_h as in (38), is a solution to Problem 1.

Remark 1: The proposed method for the solution of Problem 1 actually provides the parameterization of a class of stabilizing controllers via the tunable design parameters k_1, k_2, λ_2, λ_3, λ_4 and d. It can be foreseen that such parameters can be exploited to enforce further control specifications or to optimize suitable performance indices. One such specification concerns the magnitude of the controls in (32), (33) and (38). For circular reference orbits, i.e., for $F_{33} = F_{43} = 0$, it turns out that the magnitude of the control inputs can be made arbitrarily small locally by scaling the tuning parameters of the controller. This is not guaranteed, however, for eccentric reference orbits, since in this case the term $(F_{43}x_3 - \dot{x}_4^*)$ in (33) requires the exact cancellation of a part of the system dynamics (without further assumptions on λ_2, λ_3 and λ_4). A systematic method for exploiting the design parameters for performance is out of the scope of the present paper and is the subject of current investigation.

IV. Numerical simulation

Consider the problem of transferring a spacecraft from a near-circular equatorial orbit to an higher altitude elliptic orbit, with a prescribed longitude (i.e. phase) along the final orbit. This problem may occur, for instance, in formation flying applications, in which an actively controlled chaser spacecraft is required to intercept a passive target spacecraft. The orbital elements of the initial and the reference orbit are reported in Table I. The reference eccentricity and inclination vectors correspond to a target eccentricity of 0.5 and a target inclination of 35 deg.

TABLE I

Orbital elements of the initial and the reference orbit

Orbital element	Initial orbit	Reference orbit
True longitude	$L(0) = 2.44$ rad	$L^r(0) = 3.92$ rad
Semi-parameter	$p(0) = 6778$ km	$p^r = 19800$ km
Eccentricity vector	$e_X(0) = 9.4 \cdot 10^{-4}$	$e_X^r = -0.13$
	$e_Y(0) = 3.4 \cdot 10^{-4}$	$e_Y^r = 0.48$
Inclination vector	$h_X(0) = 0$	$h_X^r = 0.30$
	$h_Y(0) = 0$	$h_Y^r = 0.08$

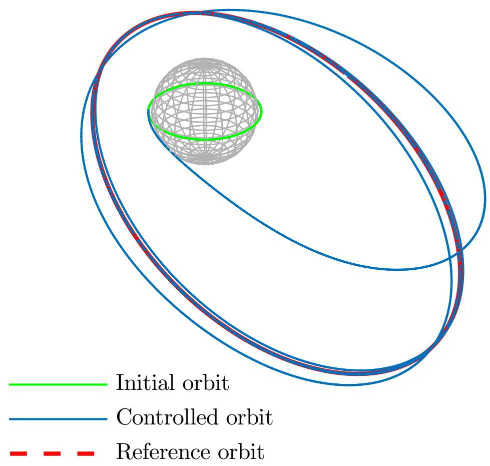

Fig. 2. Orbital transfer trajectory.

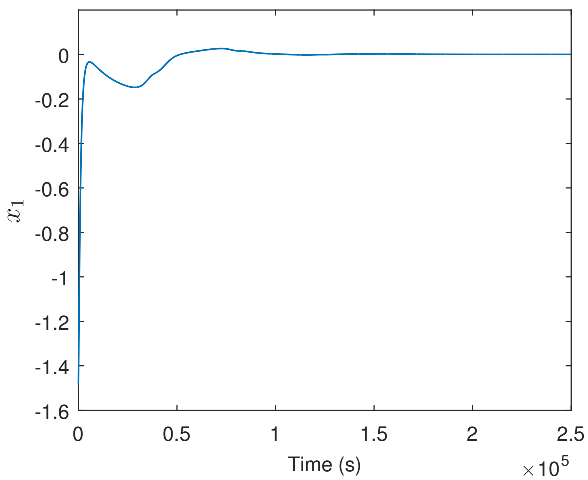

Fig. 3. True longitude tracking error.

A numerical simulation relying on the dynamic model (6) has been performed to demonstrate the application of the proposed approach to the considered rendez-vous problem, and to validate the obtained theoretical results. To ensure a fast convergence towards the reference orbit, the following tuning parameters have been empirically selected for the nonlinear controller: $k_1 = 10^{-5}$, $k_2 = 10^{-4}$, $\lambda_2 = 10^{-4}\, z_2$, $\lambda_3 = 7 \cdot 10^{-4}\, z_3$, $\lambda_4 = 7 \cdot 10^{-4}\, z_4$, and $d = 10^{-4}$.

The resulting trajectory is depicted in Fig. 4, in terms of cartesian coordinates. As expected, the asymptotic tracking of the reference trajectory is achieved. In Fig. 3, it can be seen that the true longitude tracking error is steered to zero after a short initial transient, so that the correct phase is acquired along the reference orbit. The control input signals are reported for completeness in Fig. 4.

V. Conclusions

This paper has studied the trajectory tracking problem for the class of reference trajectories consisting of unperturbed closed orbits about a central body. A feedback control law has been derived for this problem, which allows for the

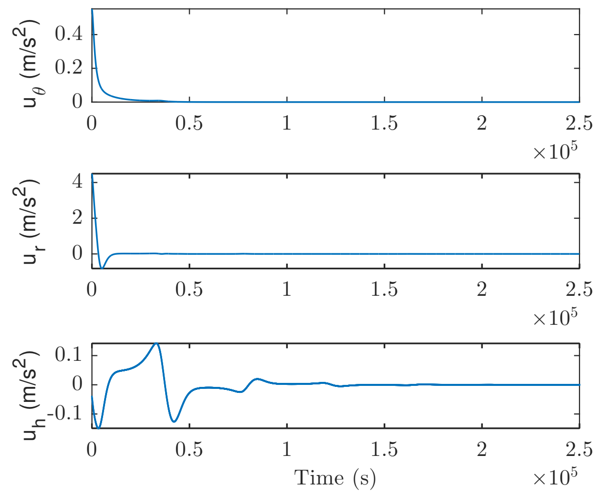

Fig. 4. Control input signals.

synchronization of the phase of the orbiting body along a reference orbit. The stability of the closed loop system has been proved by standard Lyapunov arguments, and the applicability of the proposed controller has been illustrated by numerical simulation of a spacecraft rendez-vous maneuver. Several performance-related aspects still remain to be investigated. Future work should be tailored to the evaluation of the controller performance with respect to metrics such as the settling time, the fuel consumption and the magnitude of the control inputs.

Appendix

The following expressions are used in the paper.

(a) The Jacobian of the coordinate transformation (7)-(11), with respect to the original coordinates $\tilde{\psi}$, in (13) is given by

$$\frac{\partial x}{\partial \tilde{\psi}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0_{1\times 2} \\ 0 & \frac{1}{2\psi_2^r\sqrt{\psi_2/\psi_2^r}} & 0 & 0 & 0_{1\times 2} \\ -\frac{\psi_2^r}{\psi_2}\zeta_Y & -\frac{\psi_2^r}{\psi_2^2}(1+\zeta_X) & \frac{\psi_2^r}{\psi_2}\mathrm{c}(\psi_1) & \frac{\psi_2^r}{\psi_2}\mathrm{s}(\psi_1) & 0_{1\times 2} \\ \sqrt{\frac{\psi_2^r}{\psi_2}}\zeta_X & \frac{-\psi_2^r}{2\psi_2^2\sqrt{\psi_2^r/\psi_2}}\zeta_Y & \sqrt{\frac{\psi_2^r}{\psi_2}}\mathrm{s}(\psi_1) & -\sqrt{\frac{\psi_2^r}{\psi_2}}\mathrm{c}(\psi_1) & 0_{1\times 2} \\ 0_{2\times 1} & 0_{2\times 1} & 0_{2\times 1} & 0_{2\times 1} & I_{2\times 2} \end{bmatrix}.$$

(b) The gradient $\partial V_1/\partial\chi$ of the Lyapunov function V_1 in (39) is given by

$$\frac{\partial V_1(z;\psi^r)}{\partial \chi}^T = \begin{bmatrix} k_1 z_1 - \dfrac{F_{13}}{F_{12}} k_1 z_4 - \dfrac{k_2}{F_{12}}\dfrac{\partial \lambda_3}{\partial z_1} z_4 \\ z_2 - \dfrac{k_2}{F_{12}}\dfrac{\partial \lambda_3}{\partial z_2} z_4 \\ k_2 z_3 + \gamma k_2 z_4 - \dfrac{k_2}{F_{12}}\dfrac{\partial \lambda_3}{\partial z_3} z_4 \\ k_2 z_4 \end{bmatrix},$$

where

$$\gamma = \frac{1}{F_{12}}\left[\left(\tau - \frac{\sqrt{\tau}F_{13}}{\sqrt{F_{12}}}\right)\frac{k_1}{k_2} z_1 + F_{33}\left(\frac{\sqrt{\tau}}{\sqrt{F_{12}}} z_3 - 1\right) - \tau\zeta_Y^r z_3\right]$$

and $[z_1 \;\; z_2 \;\; z_3 \;\; z_4]^T = [x_1 \;\; x_2 \;\; x_3 \;\; (x_4 - x_4^*)]^T$.

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