Maintaining connectivity among multiple agents in cyclic pursuit: a geometric approach

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***Abstract*— This paper studies the connectivity maintenance problem in linear and nonlinear cyclic pursuit, when different control gains are assigned to each agent. Feasibility/infeasibility conditions for this problem are established in both the linear and nonlinear scenarios using geometric arguments. These conditions elucidate the role played by the control gains, initial conditions and communication radius of the agents, on the connectivity of the robotic network.**

I. Introduction

Multi-agent systems and distributed control are nowadays topics of increasing popularity. As several recent surveys and books have witnessed [1]–[3], research on these themes is no more in its infancy, but it entered a maturity stage. Multiple factors, such as, e.g., the recent technological advances in wireless networks, processor design and sensor integration, have contributed to the vigorous growth of this multidisciplinary research area, that is at the intersection of system theory, robotics, telecommunications, computer science and biology.

Cyclic pursuit is a prototypal distributed control problem in which a group of agents are labeled from 1 to n, and agent i pursues agent $i+1$ modulo n. Different kinematic models for the agents have been investigated in the literature. By far, the most extensively studied is the *single-integrator*. In [4], the authors have considered single-integrator agents in cyclic pursuit as one of three possible strategies for rendezvous. The problem of accelerating the convergence rate has been addressed in [5] using a hierarchical scheme. In [6], the authors have proposed a generalization of the linear cyclic pursuit law by assigning different control gains to each agent and shown that the rendezvous point can be controlled by suitably tuning the gains. In [7], the classic cyclic pursuit strategy has been extended by letting each agent chases its leading neighbor along the line of sight rotated by a common offset angle. Recently, in [8], the authors have explored how interconnection topology influences symmetry in agents' trajectories and shown that a circulant communication structure preserves rotation, and in particular instances, dihedral group symmetries.

Besides the single-integrator, the *unicycle* model has also had considerable success in the literature. In the seminal paper [9], the possible equilibrium formations of unicycles in cyclic pursuit have been studied. Further results have been presented in [10] where each unicycle's forward velocity is proportional to the distance between its leading neighbor and itself. In [11] the authors have generalized the results in [9] to unicycles with different speeds and control gains, and established a necessary condition for the existence of equilibria. Other extensions have been recently proposed in [12], where the collective circular motion of a team of unicycles around a virtual reference beacon is studied, and in [13], where each vehicle's linear and angular velocity is chosen to be proportional to the projection of its prey's position on its forward and lateral direction, respectively.

An underlying assumption in all the previous works is that each vehicle can *always* communicate with its leading neighbor (from which it receives the position or angular information necessary to compute its control input). However, since real robots have a limited communication range (typically modelled as a disk of finite radius centered at them), this does not always occur in practice. Although the *connectivity maintenance problem* is well-known in the multi-agent systems literature and several original solutions (most of whom aim at controlling the algebraic connectivity of the underlying communication graph) have been recently proposed (see, e.g., [14]–[16] and the references therein), little attention has been devoted to the cyclic pursuit. In particular, a challenging problem consists in studying the role played by the control gains, initial conditions and communication radius of the agents, on the connectivity of the robotic network. A first step towards this direction has been taken in [17] for linear cyclic pursuit, where the solution of the corresponding dynamical system has been analyzed.

Following the general framework proposed in [6], [11], in this paper we assume different control gains for each agent and study the connectivity maintenance problem in linear and nonlinear cyclic pursuit. A detailed characterization of the single-integrator case is provided and feasibility/infeasibility conditions for the connectivity maintenance problem are established in both the linear and nonlinear scenarios, in terms of robots' control gains, initial conditions and communication radius, using *purely geometric arguments*.

The rest of the paper is organized as follows. In Sect. II some background material is presented. The main body of the article is divided into two parts: we study the connectivity maintenance problem for single-integrators in Sect. III and for unicycles in Sect. IV. Simulation results are discussed in Sect. V. Finally, in Sect. VI, conclusions are drawn and future research directions are highlighted.

II. Mathematical background

A. Positive, nonnegative and irreducible matrices

The following definitions recall the notions of *positive*, *nonnegative*, *essentially nonnegative* and *irreducible matrix* [18, pp. 26, 146, 27].

***Definition** 1 (Positive and nonnegative matrices):* A matrix $\mathbf{B} \in \mathbb{R}^{n\times n}$ is *positive* if all its entries are positive, and *nonnegative* if all its entries are nonnegative. ◇

***Definition** 2 (Essentially nonnegative matrices):* Let $\mathbf{I}$ be the $n\times n$ identity matrix. A matrix $\mathbf{B} \in \mathbb{R}^{n\times n}$ is *essentially nonnegative* if $\mathbf{B}+\sigma\mathbf{I}$ is nonnegative for all real σ sufficiently large. ◇

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Definition 3 (Irreducible matrices): A matrix $\mathbf{B} \in \mathbb{R}^{n\times n}$ is *cogredient* to a matrix $\mathbf{E}$ if for some permutation matrix $\mathbf{P}$, $\mathbf{P}\,\mathbf{B}\,\mathbf{P}^{\mathrm{T}} = \mathbf{E}$. $\mathbf{B}$ is *reducible* if it is cogredient to $\mathbf{E} = \left[\begin{smallmatrix}\mathbf{C} & \mathbf{0}\\ \mathbf{F} & \mathbf{D}\end{smallmatrix}\right]$, where $\mathbf{C}$ and $\mathbf{D}$ are square matrices, or if $n = 1$ and $\mathbf{B} = 0$. Otherwise, $\mathbf{B}$ is *irreducible*. ◇

The following lemma is drawn from [18, p. 146, Th. 3.12].

Lemma 1: Let $\mathbf{B} \in \mathbb{R}^{n\times n}$ be an essentially nonnegative matrix. Then, for all $t \geq 0$, $e^{\mathbf{B}t}$ is nonnegative. Moreover, $e^{\mathbf{B}t}$ is positive for some (and hence all) $t > 0$ if and only if $\mathbf{B}$ is irreducible.

B. Positively invariant sets

The material of this section is drawn from [19, Ch. 4]. Consider the following autonomous system,

$$\dot{\boldsymbol{\xi}}(t) = \mathbf{f}(\boldsymbol{\xi}(t)). \tag{1}$$

Suppose that (1) is defined in a proper open set $\mathcal{O} \subseteq \mathbb{R}^n$ and there exists a globally defined solution (i.e., for all $t \geq 0$) for every initial condition $\boldsymbol{\xi}(0) \in \mathcal{O}$.

Definition 4 (Positive invariance): The set $\mathcal{S} \subseteq \mathcal{O}$ is said to be *positively invariant* with respect to (1), if every solution of (1) with initial condition $\boldsymbol{\xi}(0) \in \mathcal{S}$ is such that $\boldsymbol{\xi}(t) \in \mathcal{S}$ for $t > 0$. ◇

Definition 5 (Bouligand's tangent cone): Given a closed set $\mathcal{S}$, the tangent cone to $\mathcal{S}$ at $\boldsymbol{\xi}$ is defined as $\mathcal{T}_{\mathcal{S}}(\boldsymbol{\xi}) = \{\mathbf{z} \in \mathbb{R}^n \mid \liminf_{\tau\to 0} \frac{1}{\tau}\,\mathrm{dist}(\boldsymbol{\xi} + \tau\,\mathbf{z}, \mathcal{S}) = 0\}$ where $\mathrm{dist}(\boldsymbol{\xi} + \tau\,\mathbf{z}, \mathcal{S}) \triangleq \inf_{\mathbf{y}\in\mathcal{S}} \|\boldsymbol{\xi} + \tau\,\mathbf{z} - \mathbf{y}\|$. ◇

Note that if $\mathcal{S}$ is a polyhedral set and its description in terms of hyperplanes $\mathcal{S} = \{\mathbf{z} \in \mathbb{R}^n \mid \mathbf{w}_i^{\mathrm{T}}\mathbf{z} \leq \zeta_i, \mathbf{w}_i \in \mathbb{R}^n, i \in \{1, 2, \ldots, \ell\}\}$, is available, then the tangent cone to $\mathcal{S}$ at $\boldsymbol{\xi}$ is simply given by $\mathcal{T}_{\mathcal{S}}(\boldsymbol{\xi}) = \{\mathbf{z} \in \mathbb{R}^n \mid \mathbf{w}_i^{\mathrm{T}}\mathbf{z} \leq 0 \text{ for all } i \text{ such that } \mathbf{w}_i^{\mathrm{T}}\boldsymbol{\xi} = \zeta_i\}$.

Theorem 1 (Nagumo): Consider system (1) and assume that for each initial condition $\boldsymbol{\xi}(0)$ in an open set $\mathcal{O}$, it admits a unique solution defined for all $t \geq 0$. Let $\mathcal{S} \subset \mathcal{O}$ be a closed set. Then $\mathcal{S}$ is positively invariant for system (1) if and only if the velocity vector satisfies the following condition:

$$\mathbf{f}(\boldsymbol{\xi}) \in \mathcal{T}_{\mathcal{S}}(\boldsymbol{\xi}),\ \forall\,\boldsymbol{\xi} \in \mathcal{S}. \tag{2}$$

Note that condition (2) is meaningful only for $\boldsymbol{\xi} \in \partial\mathcal{S}$, since for $\boldsymbol{\xi} \in \mathrm{int}\{\mathcal{S}\}$, $\mathcal{T}_{\mathcal{S}}(\boldsymbol{\xi}) \equiv \mathbb{R}^n$, ($\partial\mathcal{S}$ and $\mathrm{int}\{\mathcal{S}\}$ denote respectively the *boundary* and the *interior* of the set $\mathcal{S}$).

III. Linear cyclic pursuit

Consider $n > 2$ mobile robots R_i, $i \in \{1, 2, \ldots, n\}$ in the plane, and suppose that robot i pursues the next $i + 1$ modulo n (all robot indices $i + 1$ are henceforth evaluated modulo n). We will assume that the motion of robot i is modelled by the dynamics,

$$\dot{\mathbf{p}}_i(t) = \mathbf{u}_i(t),\quad i \in \{1, 2, \ldots, n\}, \tag{3}$$

where $\mathbf{p}_i(t) = [x_i(t),\, y_i(t)]^{\mathrm{T}}$ denotes the position of robot i at time t and $\mathbf{u}_i(t)$ its control input. Following the extension to the classic cyclic pursuit proposed in [6], we will suppose that,

$$\mathbf{u}_i(t) = k_i(\mathbf{p}_{i+1}(t) - \mathbf{p}_i(t)), \tag{4}$$

where the scalar $k_i \neq 0$ is the control gain of robot i. Note that a special case of the control law (4) was considered in [9, Sect. 3], where the gains of the robots are assumed to be positive and all identical. The x_i- and y_i-coordinates of robot i evolve independently, hence they can be decoupled into two identical linear systems of the form:

$$\dot{\mathbf{q}}(t) = \mathbf{A}\,\mathbf{q}(t), \tag{5}$$

where

$$\mathbf{A} = \begin{bmatrix} -k_1 & k_1 & 0 & \cdots & 0 & 0\\ 0 & -k_2 & k_2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -k_{n-1} & k_{n-1}\\ k_n & 0 & 0 & \cdots & 0 & -k_n \end{bmatrix},$$

and $\mathbf{q} = [x_1,\, x_2, \ldots, x_n]^{\mathrm{T}}$ or $\mathbf{q} = [y_1,\, y_2, \ldots, y_n]^{\mathrm{T}}$.

Before addressing the connectivity maintenance problem, that is the main focus of this paper, we present a suite of results that are useful to gain some insight into the geometric properties of system (5). The next proposition has been proved in [6, Th. 4].

Proposition 1: Consider system (3) with control input (4). Then we have,

$$\sum_{i=1}^{n} \frac{1}{k_i}\,\mathbf{p}_i(t) = \sum_{i=1}^{n} \frac{1}{k_i}\,\mathbf{p}_i(0),\ \ \forall\, t > 0. \tag{6}$$

If the control gains are all *strictly positive*, as we henceforth suppose, it follows immediately from (6) that the robots eventually rendezvous at the point (see Fig. 1),

$$\mathbf{p}_{\text{ren}} = \sum_{i=1}^{n} \frac{1}{k_i \sum_{j=1}^{n} \frac{1}{k_j}}\,\mathbf{p}_i(0).$$

Note that $\mathbf{p}_{\text{ren}}$ represents the *weighted mean* of the initial positions $\mathbf{p}_i(0)$ of the robots with weights $W_i = \prod_{j=1, j\neq i}^{n} k_j$, $i \in \{1, 2, \ldots, n\}$.

It has also been shown in [6, Th. 5] that if $k_i > 0$ for all i, a point $\mathbf{p}^*$ is a rendezvous point if and only if $\mathbf{p}^* \in \mathrm{co}\{\mathcal{D}\}$ where $\mathcal{D} \triangleq \{\mathbf{p}_1(0), \mathbf{p}_2(0), \ldots, \mathbf{p}_n(0)\}$ and $\mathrm{co}\{\mathcal{D}\}$ denotes the convex hull of the set $\mathcal{D}$. In the next proposition we shed further light into this property by characterizing the locus of trajectories of the n robots.

Proposition 2 (Locus of trajectories): Consider system (3) with control input (4) and let $k_i > 0$ for all i. Then, we have:

$$\mathbf{p}_i(t) \in \mathrm{int}\{\mathrm{co}\{\mathcal{D}\}\},\ \forall\, t > 0,\ i \in \{1, 2, \ldots, n\}. \tag{7}$$

Proof: Condition (7) is verified if and only if $e^{\mathbf{A}t}$ satisfies,

$$e^{\mathbf{A}t}\,\mathbb{1} = \mathbb{1},\ \forall\, t > 0, \tag{8}$$

and

$$e^{\mathbf{A}t} \text{ is positive},\ \ \forall\, t > 0, \tag{9}$$

where $\mathbb{1}$ denotes the vector of n ones. In fact, note that

$$\mathbf{p}_i(t) = \sum_{j=1}^{n} (e^{\mathbf{A}t})_{ij}\,\mathbf{p}_j(0),\ \forall\, t > 0,\ i = \{1, 2, \ldots, n\}, \tag{10}$$

where $(e^{\mathbf{A}t})_{ij}$ denotes the (i, j)-th entry of matrix $e^{\mathbf{A}t}$. To prove condition (8), it is sufficient to rewrite $e^{\mathbf{A}t}$ as its power series expansion and to note that $\mathbf{A}\mathbb{1} = \mathbf{0}$. Condition (9) easily follows from Lemma 1: in fact, matrix $\mathbf{A}$ is irreducible, and essentially nonnegative if $k_i > 0$ for all $i \in \{1, 2, \ldots, n\}$. ■

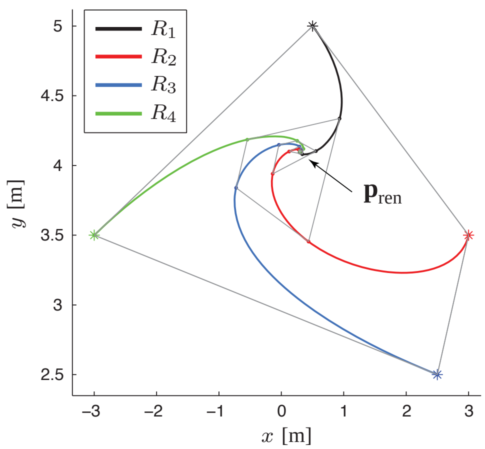

Fig. 1. Trajectory of 4 robots with gains $k_1 = 1$, $k_2 = 3$, $k_3 = 4$ and $k_4 = 2$, rendezvousing at $\mathbf{p}_{\rm ren}$. The sides of the quadrilaterals with vertices $\mathbf{p}_1(t)$, $\mathbf{p}_2(t)$, $\mathbf{p}_3(t)$, $\mathbf{p}_4(t)$ are drawn every 0.5 seconds (gray).

The following Prop. 3 characterizes the time evolution of the *area* of the convex hull of the positions of the n robots, and the time evolution of the *perimeter* of the polygon whose vertices are represented by the positions of the n vehicles (see Fig. 1). The next lemma is instrumental in proving Prop. 3 and shows that if the robots are not collinear at the initial time, they will never be collinear.

***Lemma** 2 (Collinearity of the robots):* If $\mathbf{p}_1(0)$, $\mathbf{p}_2(0)$, …, $\mathbf{p}_n(0)$ are not collinear, then $\mathbf{p}_1(t)$, $\mathbf{p}_2(t)$, …, $\mathbf{p}_n(t)$ are not collinear for every $t > 0$.

Proof: By contradiction, suppose that there is a time instant $\bar{t} > 0$ at which the robots are collinear. This means that there exists a vector $\mathbf{b} \in \mathbb{R}^2$ and a scalar $c \in \mathbb{R}$ such that,

$$\mathbf{p}_i(\bar{t})^{\rm T}\,\mathbf{b} = c, \quad i \in \{1, 2, \ldots, n\}. \tag{11}$$

Define $s_i \triangleq \mathbf{p}_i(0)^{\rm T}\,\mathbf{b}$ and $\mathbf{s} \triangleq [s_1, s_2, \ldots, s_n]^T$. Owing to (10), the equations in (11) can be rewritten in matrix form as,

$$e^{\mathbf{A}\bar{t}}\,\mathbf{s} = c\,\mathbb{1}. \tag{12}$$

Since a matrix exponential $e^{\mathbf{X}}$ is always invertible and $\left(e^{\mathbf{X}}\right)^{-1} = e^{-\mathbf{X}}$, the unique solution of (12) is,

$$\mathbf{s} = c\,e^{-\mathbf{A}\bar{t}}\,\mathbb{1} = c\,\mathbb{1}, \tag{13}$$

where we have exploited the fact that $e^{-\mathbf{A}\bar{t}}\,\mathbb{1} = \mathbb{1}$. Recalling the definition of $\mathbf{s}$, equation (13) means that $\mathbf{p}_i(0)^{\rm T}\,\mathbf{b} = c$, $i \in \{1, 2, \ldots, n\}$. Hence, the robots are initially collinear, thus contradicting the hypothesis. ■

Note that the assumption of strictly positive control gains is not mandatory for Lemma 2. In the next proposition we will use the symbol $\mathsf{A}(\mathcal{V})$ to denote the area of the convex polygon $\mathcal{V} \subset \mathbb{R}^2$.

***Proposition** 3:* Let us suppose that $\mathbf{p}_1(0)$, $\mathbf{p}_2(0), \ldots, \mathbf{p}_n(0)$ are not collinear and let $k_i > 0$ for all i. Then, the following properties hold for $t \to +\infty$:

i) $\mathsf{A}(\mathrm{co}\{\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t)\})$ monotonically decreases to zero.

ii) The perimeter $\mathsf{P}(t)$ of the polygon with vertices $\mathbf{p}_1(t)$, $\mathbf{p}_2(t)$, …, $\mathbf{p}_n(t)$ monotonically decreases to zero.

Proof: Owing to Prop. 2, at a generic time instant $t_1 > 0$ robot i lies in the interior of the convex hull of the initial positions of the n agents, i.e., $\mathbf{p}_i(t_1) \in \mathrm{int}\{\mathrm{co}\{\mathcal{D}\}\}$, $\forall\, t_1 > 0$, $i \in \{1, 2, \ldots, n\}$. This means that $\mathrm{co}\{\mathbf{p}_1(t_1), \ldots, \mathbf{p}_n(t_1)\} \subset \mathrm{co}\{\mathcal{D}\}$, $\forall\, t_1 > 0$. As a consequence, if we consider a second time instant t_2, with $t_2 > t_1$, it easily follows that,

$$\mathrm{co}\{\mathbf{p}_1(t_2), \ldots, \mathbf{p}_n(t_2)\} \subset \mathrm{co}\{\mathbf{p}_1(t_1), \ldots, \mathbf{p}_n(t_1)\}, \forall\, t_2 > t_1. \tag{14}$$

Since by hypothesis $\mathbf{p}_1(0)$, $\mathbf{p}_2(0)$,…, $\mathbf{p}_n(0)$ are not collinear, then, by Lemma 2, $\mathbf{p}_1(t)$, $\mathbf{p}_2(t)$,…, $\mathbf{p}_n(t)$ are not collinear for every $t > 0$, and $\mathsf{A}(\mathrm{co}\{\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t)\}) \neq 0$ for every finite time t. Therefore, inclusion (14) implies $\mathsf{A}\left(\mathrm{co}\{\mathbf{p}_1(t_2), \ldots, \mathbf{p}_n(t_2)\}\right) < \mathsf{A}\left(\mathrm{co}\{\mathbf{p}_1(t_1), \ldots, \mathbf{p}_n(t_1)\}\right)$, for all $t_2 > t_1$, from which part *i)* of the statement follows.

Let us now prove part *ii)*. The perimeter of the polygon with vertices $\mathbf{p}_1(t)$, $\mathbf{p}_2(t), \ldots$, $\mathbf{p}_n(t)$ is given by,

$$\mathsf{P}(t) = \sum_{i=1}^{n} \|\mathbf{p}_i(t) - \mathbf{p}_{i+1}(t)\|. \tag{15}$$

The time derivative of (15) along the trajectories of system (5) is,

$$\begin{aligned}\dot{\mathsf{P}} &= \sum_{i=1}^{n} k_{i+1}\,\frac{(\mathbf{p}_i - \mathbf{p}_{i+1})^{\rm T}(\mathbf{p}_{i+1} - \mathbf{p}_{i+2})}{\|\mathbf{p}_i - \mathbf{p}_{i+1}\|} - k_i\,\|\mathbf{p}_i - \mathbf{p}_{i+1}\| \\ &= \sum_{i=1}^{n} \frac{k_{i+1}}{\|\mathbf{p}_i - \mathbf{p}_{i+1}\|}\left[(\mathbf{p}_i - \mathbf{p}_{i+1})^{\rm T}(\mathbf{p}_{i+1} - \mathbf{p}_{i+2})\right. \\ &\qquad \left. - \|\mathbf{p}_i - \mathbf{p}_{i+1}\| \cdot \|\mathbf{p}_{i+1} - \mathbf{p}_{i+2}\|\right],\end{aligned}$$

which is always nonpositive by the definition of scalar product. Note that $\dot{\mathsf{P}}(t) = 0$ only if all the robots are aligned at time $t > 0$, but this can never happen by our hypothesis of non-collinear agents at time $t = 0$ and by Lemma 2. Hence, $\mathsf{P}(t)$ is monotonically decreasing. ■

Note that for $n = 3$, the area $\mathsf{A}(t)$ of the triangle with vertices $\mathbf{p}_1(t)$, $\mathbf{p}_2(t)$ and $\mathbf{p}_3(t)$, decreases to zero according to the simple formula $\mathsf{A}(t) = e^{-(k_1+k_2+k_3)t}\,\mathsf{A}(0)$.

In order to make our cyclic pursuit model more realistic, let us now introduce suitable *connectivity constraints* among the robots.

***Definition** 6 (Connectivity constraints):* Let us suppose that robot R_{i+1} has a communication set modeled as a disk of finite radius r. R_{i+1} is said to be *connected* with robot R_i at time t (to which it transmits the position information $\mathbf{p}_{i+1}(t)$ necessary for the computation of the control input $\mathbf{u}_i$(t)), if R_i is within its communication disk at time t, i.e.,

$$\|\mathbf{p}_i(t) - \mathbf{p}_{i+1}(t)\| \leq r. \tag{16}$$

◇

If the robots have limited communication capabilities, then there will exist trajectories in which the connectivity of the robots is preserved and "critical trajectories" leading to a connectivity loss. Note that the fulfillment of condition (16) for all $t \geq 0$ and for all i, depends upon three factors: the *control gains* k_i of the robots, the *initial positions* $\mathbf{p}_i(0)$, and the *communication radius* r.

In the next theorem, the main result of this section, we adopt a geometric viewpoint and show the role played by these factors on the connectivity of the robotic network.

Theorem 2 (Connectivity maintenance condition): Consider system (3) with control input (4), and let $k_i > 0$ for all i. Then, for every connected initial condition (i.e., for every $\mathbf{p}_i(0)$ such that $\|\mathbf{p}_i(0) - \mathbf{p}_{i+1}(0)\| \le r$, $i \in \{1, 2, \dots, n\}$), the robots are connected for all $t > 0$, if and only if

$$k_1 = k_2 = \ldots = k_n. \tag{17}$$

Proof: Let us first prove the *necessity* of condition (17). Consider the first pair of robots R_1 and R_2, and suppose that they are in a critical connectivity configuration, i.e., $\|\mathbf{p}_1 - \mathbf{p}_2\| = r$. We want to study under which conditions on the gains k_1 and k_2, the following inequality holds:

$$\frac{d}{dt}\Big(\frac{1}{2}\|\mathbf{p}_1(t) - \mathbf{p}_2(t)\|^2\Big) = k_2(\mathbf{p}_1(t) - \mathbf{p}_2(t))^{\mathrm{T}}(\mathbf{p}_2(t) - \mathbf{p}_3(t)) - k_1\,\|\mathbf{p}_1(t) - \mathbf{p}_2(t)\|^2 \;\le\; 0, \tag{18}$$

that is $(\mathbf{p}_1 - \mathbf{p}_2)^{\mathrm{T}}(\mathbf{p}_2 - \mathbf{p}_3) \le \frac{k_1}{k_2}\, r^2$ being $\|\mathbf{p}_1 - \mathbf{p}_2\|^2 = r^2$. By simple geometric arguments (see Fig. 2), we note that the solution of the following problem,

$$\max_{\mathbf{p}_1,\,\mathbf{p}_2,\,\mathbf{p}_3 \,\in\, \mathbb{R}^2} \;(\mathbf{p}_1 - \mathbf{p}_2)^{\mathrm{T}}(\mathbf{p}_2 - \mathbf{p}_3)$$
$$\text{s.t.}\;\; \|\mathbf{p}_1 - \mathbf{p}_2\| = r,\;\; \|\mathbf{p}_2 - \mathbf{p}_3\| \le r,$$

is exactly equal to r^2. Therefore, we obtain the following condition on the gains k_1 and k_2:

$$k_1 \ge k_2. \tag{19}$$

By repeating the same procedure as above for the other $n-1$ pairs of robots (namely, R_2 and R_3, R_3 and R_4, $\dots$, R_n and R_1), we end up with the following additional constraints on the controller gains:

$$k_2 \ge k_3,\; \ldots, k_{n-1} \ge k_n,\; k_n \ge k_1. \tag{20}$$

Putting (19) and (20) together, condition (17) is found.

For the *sufficiency*, let us suppose that $k_1 = k_2 = \ldots = k_n = k$. Consider again the first pair of robots R_1 and R_2. Condition (18) becomes in this case:

$$\frac{d}{dt}\Big(\frac{1}{2}\|\mathbf{p}_1(t) - \mathbf{p}_2(t)\|^2\Big) = k\big[(\mathbf{p}_1(t) - \mathbf{p}_2(t))^{\mathrm{T}}(\mathbf{p}_2(t) - \mathbf{p}_3(t)) - \|\mathbf{p}_1(t) - \mathbf{p}_2(t)\|^2\big]. \tag{21}$$

If we now suppose that R_1 and R_2 are in a critical connectivity configuration, i.e., $\|\mathbf{p}_1 - \mathbf{p}_2\| = r$ (and that $\|\mathbf{p}_2 - \mathbf{p}_3\| \le r$), then it turns out that (21) is nonpositive. By repeating the same reasoning for the other pairs of robots, we deduce that the agents are connected for all $t > 0$. ■

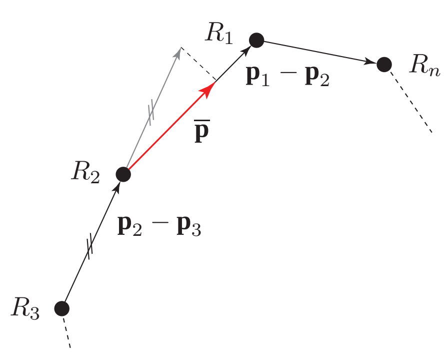

Fig. 2. The projection of vector $\mathbf{p}_2 - \mathbf{p}_3$ onto $\mathbf{p}_1 - \mathbf{p}_2$ is the vector $\overline{\mathbf{p}} = \dfrac{(\mathbf{p}_1 - \mathbf{p}_2)^{\mathrm{T}}(\mathbf{p}_2 - \mathbf{p}_3)}{r^2}\,(\mathbf{p}_1 - \mathbf{p}_2)$.

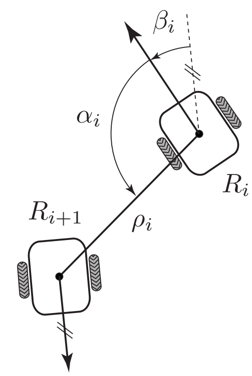

Fig. 3. Parameters of the nonlinear cyclic pursuit model.

Remark 1: It is worth emphasizing that condition (17) of Th. 2 is necessary and sufficient for connectivity maintenance, *for every* connected initial condition. Note that if the control gains are *not* identical, there will always exist a connected initial condition and a finite time instant at which at least one of the agents loses the connectivity. ◇

IV. Nonlinear Cyclic Pursuit

In this section we extend the scenario studied in Sect. III, to one in which each robot is an unicycle,

$$\begin{aligned}
\dot{x}_i(t) &= v_i(t)\cos(\theta_i(t)),\\
\dot{y}_i(t) &= v_i(t)\sin(\theta_i(t)), \quad i \in \{1, 2, \dots, n\},\\
\dot{\theta}_i(t) &= \omega_i(t),
\end{aligned}$$

where $[x_i(t),\, y_i(t)]^{\mathrm{T}} \in \mathbb{R}^2$ denotes the position of robot i at time t, $\theta_i(t) \in \mathbb{R}$ its heading (note that $\theta_i(t)$ is allowed to take values in $\mathbb{R}$ in order to avoid discontinuities in the angular control we will design later in this section), and $[v_i(t),\, \omega_i(t)]^{\mathrm{T}}$ its forward and angular velocities. In order to write a relative model of robots' dynamics, following [9, Sect. IV], we introduce the three new variables ρ_i, α_i and β_i, where $\rho_i > 0$ denotes the distance between robot R_i and robot R_{i+1}, α_i is the difference between the i-th robot's heading and the heading that would take it directly toward its leading neighbor, and angle β_i is defined as in Fig. 3. After some algebraic manipulation, the following equations in the new variables ρ_i, α_i and β_i are obtained [9, Sect. IVA]:

$$\begin{aligned}
\dot{\rho}_i &= -v_i\cos\alpha_i - v_{i+1}\cos(\alpha_i + \beta_i),\\
\dot{\alpha}_i &= \frac{1}{\rho_i}\big[v_i\sin\alpha_i + v_{i+1}\sin(\alpha_i + \beta_i)\big] - \omega_i,\\
\dot{\beta}_i &= \omega_i - \omega_{i+1}.
\end{aligned} \tag{22}$$

In analogy with (4) and similarly to [11, Sect. 2], let us choose the following controls for robot i: $v_i = 1$ and $\omega_i = k_i\,\alpha_i$, $k_i > 0$. Substituting these controls into (22), yields a system of n cyclically interconnected subsystems of the form:

$$\begin{aligned}
\dot{\rho}_i &= -\cos\alpha_i - \cos(\alpha_i + \beta_i),\\
\dot{\alpha}_i &= \frac{1}{\rho_i}\left[\,\sin\alpha_i + \sin(\alpha_i + \beta_i)\,\right] - k_i\,\alpha_i,\\
\dot{\beta}_i &= k_i\,\alpha_i - k_{i+1}\,\alpha_{i+1}.
\end{aligned}$$

Let $\boldsymbol{\xi}_i \triangleq [\rho_i,\, \alpha_i,\, \beta_i]^{\mathrm{T}}$ and $\boldsymbol{\xi} \triangleq [\boldsymbol{\xi}_1^{\mathrm{T}}, \boldsymbol{\xi}_2^{\mathrm{T}}, \ldots, \boldsymbol{\xi}_n^{\mathrm{T}}]^{\mathrm{T}}$, then the overall system will be of the form:

$$\dot{\boldsymbol{\xi}} = \mathbf{f}(\boldsymbol{\xi}). \tag{23}$$

The equilibria $\bar{\boldsymbol{\xi}}$ of (23) for $k_i = k$, $i \in \{1, 2, \ldots, n\}$, have been studied in [9, Th. 5]. As in Sect. III, in order to make our nonlinear cyclic pursuit model more realistic, we introduce suitable *connectivity constraints* among the robots. In the notation of this section, we will say that robot R_{i+1} is connected with robot R_i at time t if $\rho_i(t) \le r$ (cf. Def. 6). The following theorem sheds some light into the role played by the control gains, initial pose and communication radius of the unicycles, on the connectivity of the multi-agent system. The proof is based on Th. 1.

***Theorem** 3:* Consider the system (23). For every choice of the positive control gains $k_1, k_2, \ldots, k_n$ and for every communication radius r, there always exist a connected initial condition and a finite time t^*, such that $\rho_i(t^*) > r$ for some $i \in \{1, 2, \ldots, n\}$.

Proof: Let us define the following polyhedral set $\mathcal{S}$:

$$\mathcal{S} \triangleq \big\{[\rho_i,\, \alpha_i,\, \beta_i]^{\mathrm{T}} \,|\, 0 < r_m \le \rho_i \le r,\, |\alpha_i| \le \Gamma_\alpha, \\ |\beta_i| \le \Gamma_\beta,\; i \in \{1, 2, \ldots, n\}\big\}, \tag{24}$$

where r_m, Γ_α, Γ_β are positive constants. Γ_α, Γ_β are bounds introduced to render $\mathcal{S}$ compact. Note that if we find at least one point $\boldsymbol{\xi}^* \in \partial\mathcal{S}$ such that $\mathbf{f}(\boldsymbol{\xi}^*) \notin \mathcal{T}_{\mathcal{S}}(\boldsymbol{\xi}^*)$, then the thesis is proved. The polyhedral set (24) admits the following description in terms of hyperplanes,

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{bmatrix} \boldsymbol{\xi} \le \begin{bmatrix} r \\ -r_m \\ \Gamma_\alpha \\ \Gamma_\alpha \\ \Gamma_\beta \\ \Gamma_\beta \\ \vdots \\ r \\ -r_m \\ \Gamma_\alpha \\ \Gamma_\alpha \\ \Gamma_\beta \\ \Gamma_\beta \end{bmatrix}.$$

Let us check condition (2) on vertex $\mathbf{v} = [\, r, -\Gamma_\alpha, \Gamma_\beta, r, \Gamma_\alpha, \Gamma_\beta, \ldots, r, \Gamma_\alpha, \Gamma_\beta \,]^{\mathrm{T}}$ of $\mathcal{S}$. It is easy to verify that,

$$\mathcal{T}_{\mathcal{S}}(\mathbf{v}) = \left\{\mathbf{z} \in \mathbb{R}^{3n} \,\middle|\, \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{z} \le \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}, \tag{25}$$

and

$$\mathbf{f}(\mathbf{v}) = \begin{bmatrix} -\cos\Gamma_\alpha - \cos(-\Gamma_\alpha + \Gamma_\beta) \\ \frac{1}{r}\left[-\sin\Gamma_\alpha + \sin(-\Gamma_\alpha + \Gamma_\beta)\right] + k_1\,\Gamma_\alpha \\ -\Gamma_\alpha(k_1 + k_2) \\ \vdots \\ -\cos\Gamma_\alpha - \cos(\Gamma_\alpha + \Gamma_\beta) \\ \frac{1}{r}\left[\sin\Gamma_\alpha + \sin(\Gamma_\alpha + \Gamma_\beta)\right] - k_n\,\Gamma_\alpha \\ \Gamma_\alpha(k_n + k_1) \end{bmatrix}. \tag{26}$$

Combining (26) with (25) we obtain $3n$ inequalities, the last of which is $k_1 \le -k_n$, that is never satisfied being all the control gains positive by hypothesis. ∎

Note that differently from the linear case, in nonlinear cyclic pursuit there does *not* exist a choice of the control gains that guarantees the connectivity among the robots at all times, *for every* connected initial condition.

V. Simulation Results

To illustrate Th. 2, simulation experiments have been performed with a team of $n = 4$ robots. Fig. 4 shows $\max_{t \in [0,10]} \|\mathbf{p}_i(t) - \mathbf{p}_{i+1}(t)\|$, $i \in \{1, 2, 3, 4\}$, (black, red, blue, green dots), for 20 random generated connected initial conditions in a 6×10 m box, when $k_1 = \ldots = k_4 = 1$. The communication radius is $r = 5$ m, (dashed line). As it is evident from the figure, all the dots lie below the dashed line, thus confirming the sufficiency of condition (17), (i.e., if the control gains are identical, the connectivity is preserved at all times, *for every* connected initial condition).

Fig. 5(a) shows the trajectory of four robots moving from a connected initial condition with control gains $k_1 = 1$, $k_2 = 2$ $k_3 = 3$, $k_4 = 4$. Fig. 5(d) reports the time history of $\|\mathbf{p}_i(t) - \mathbf{p}_{i+1}(t)\|$, $i \in \{1, 2, 3, 4\}$: the communication radius is $r = 3$ m (dashed line). From Fig. 5(d), we see the connectivity is preserved at all times. The robots eventually rendezvous at the point $(0.22, 3.08)$. Fig. 5(b) shows the trajectory of four robots moving from the same connected initial condition as in Fig 5(a), but with control gains $k_1 = 0.1$, $k_2 = 18$, $k_3 = 0.5$, $k_4 = 14$, (as before, $r = 3$ m). From Fig. 5(e), we observe that this time robots R_2, R_1 and R_4, R_3 lose the connectivity respectively 5 ms and 8 ms after the beginning of the simulation (cf. Remark 1). Note that to improve the readability of Figs. 5(b), 5(e), we did not stop the simulation when the connectivity between the robots is lost: in fact, we assumed that the agents always receive the position information from their leading neighbors.

In the simulation results shown in Figs. 5(c), 5(f), four unicycles start moving with the same heading $\theta_i(0) = \pi/2$ rad, $i \in \{1, 2, 3, 4\}$, from the vertices of a square of side 4 m centered at the origin. The communication radius is $r = 5$ m, hence all the robots are initially connected. Fig. 5(c) shows the trajectory of the four unicycles and Fig. 5(f) the time history of $\rho_i(t)$, $i \in \{1, 2, 3, 4\}$, when $k_1 = \ldots = k_4 = 0.75$. Note that differently from the linear cyclic pursuit, in this case, even though all the control gains

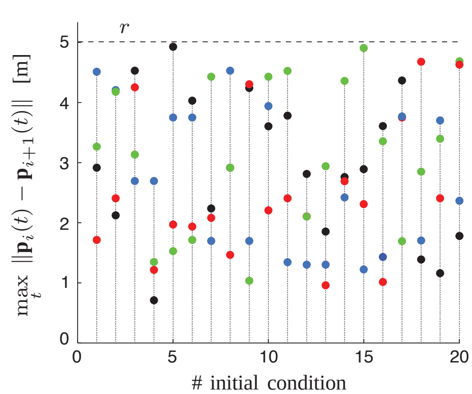

Fig. 4. Maxima of $\|\mathbf{p}_i(t) - \mathbf{p}_{i+1}(t)\|$, $t \in [0, 10]$, $i \in \{1, 2, 3, 4\}$ (black, red, blue, green dots), for 20 random generated connected initial conditions: $k_1 = \ldots = k_4 = 1$ and $r = 5$ m (dashed line).

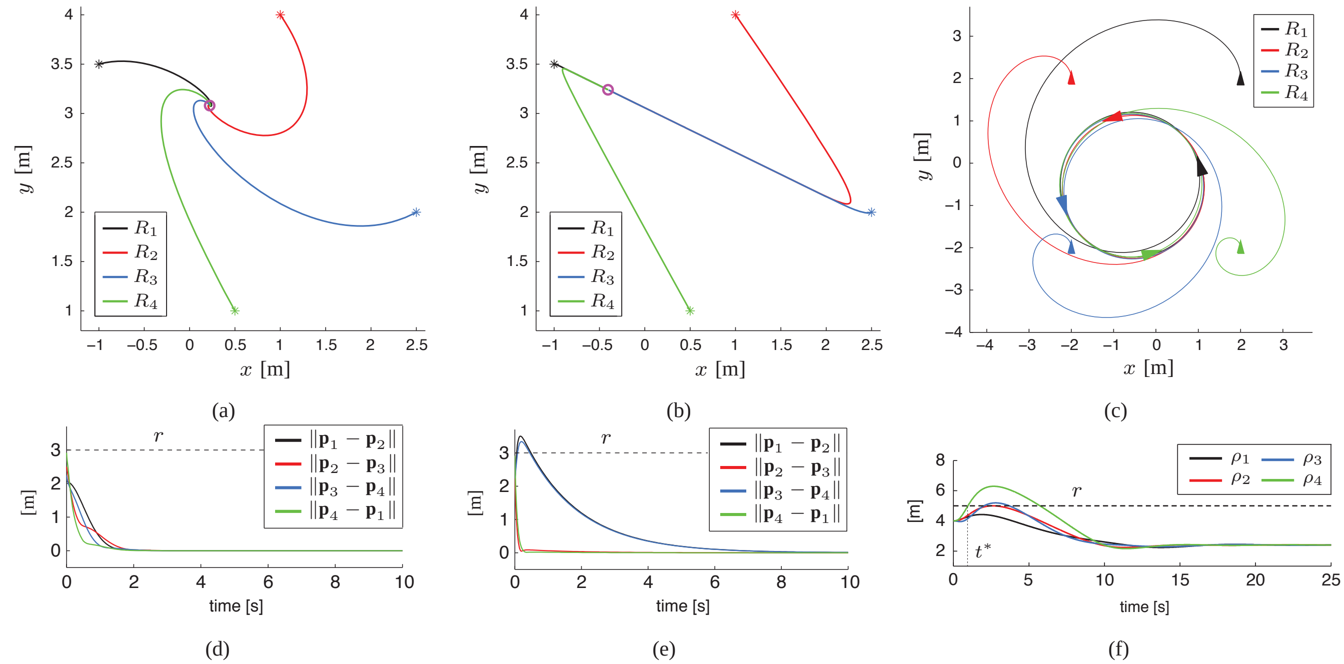

Fig. 5. *First column:* Trajectory of the four robots for $k_1 = 1$, $k_2 = 2$ $k_3 = 3$, $k_4 = 4$ and time history of $\|\mathbf{p}_i(t) - \mathbf{p}_{i+1}(t)\|$, $i \in \{1, 2, 3, 4\}$; *Second column:* Trajectory of the four robots for $k_1 = 0.1$, $k_2 = 18$, $k_3 = 0.5$, $k_4 = 14$ and time history of $\|\mathbf{p}_i(t) - \mathbf{p}_{i+1}(t)\|$, $i \in \{1, 2, 3, 4\}$; *Third column:* Trajectory of the four unicycles for $k_1 = \ldots = k_4 = 0.75$ and time history of $\rho_i(t)$, $i \in \{1, 2, 3, 4\}$.

are identical, robots R_1 and R_4 lose the connectivity at time $t^* = 0.96$s (see Fig. 5(f) and cf. Th. 3). As above, in order to make Figs. 5(c), 5(f) clearly readable, we did not interrupt the simulation when the connectivity between robots R_1 and R_4 is lost. Thus, the unicycles eventually reach the equilibrium configuration $\bar{\boldsymbol{\xi}} = [2.4,\ \pi/4,\ \pi/2,\ \ldots,\ 2.4,\ \pi/4,\ \pi/2]^{\mathrm{T}}$ (see Fig. 5(c) and cf. [9, Sect. V]).

VI. Conclusions and future work

In this paper we have studied the connectivity maintenance problem in linear and nonlinear cyclic pursuit, when different control gains are assigned to the agents. In the single-integrator case we have shown that for every connected initial condition, the connectivity among the robots is preserved at all times if and only if the control gains are identical (Theorem 2). In the nonlinear scenario, Theorem 3 states that for every choice of the positive control gains and for every communication radius, there always exists a connected initial condition from which the robots lose the connectivity.

Future research aims at extending our geometric analysis of connectivity to robotic networks of general topology, and at designing distributed control strategies for connection recovery and connection loss prevention. We also plan to study the effect of communication errors/delays on network's connectivity. In nonlinear cyclic pursuit, two research directions are currently under investigation. The first one is to determine the subset of connected initial conditions that guarantee the connectivity of the robots at all times, for an assigned set of control gains. Vice versa, the second is to identify all the control gains that guarantee the connectivity of the unicycles at all times, for a given subset of connected initial conditions.

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