A set-membership approach to consensus problems with bounded measurement errors

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***Abstract*— This paper analyzes two classes of consensus algorithms in presence of bounded measurement errors. The protocols taken into account adopt an updating rule based either on constant or vanishing weights. The bounded error assumption allows one to cast the consensus problem in a set-membership framework, and to study the team agreement in terms of the evolution of the feasible state set. It is shown that consensus cannot be guaranteed with respect to all possible noise realizations. Moreover, bounds on the asymptotic difference between the states of the agents are explicitly derived, in terms of the bounds on the measurement noise and the eigenvalues of the weight matrix.**

I. Introduction

In recent years, consensus algorithms have received increasing interest within the context of multi-agent systems. The ability of a team of interacting agents to reach an agreement on some quantity of interest is often a key issue for the solution of many problems in different application domains, like cooperative control of autonomous vehicles, information fusion, distributed sensor networks (see [1] and reference therein). Several solutions to the consensus problem have been proposed by now and nice theoretical results are available both in case of stationary and time-varying communication networks (e.g., see [2],[3],[4] and the survey [5])

Compared to the huge amount of papers analyzing how the topology of the communication graph affects the convergence properties of the consensus protocols, relatively few ones have addressed the behavior of consensus algorithms in presence of noisy measurements. In [6] classical consensus algorithms are shown to be input-to-state stable, thus implying that small disturbances do not completely disrupt the team agreement. A consensus protocol mimicking stochastic approximation algorithms with a decreasing step size has been proposed in [7]. The authors show that in case of noisy measurements, the adoption of vanishing weights guarantees the convergence in probability of the agents' states to the same value. A different description of the uncertainty is adopted in [8], where the measurement noise is only assumed to be bounded. A rule for selecting an estimate of the neighbors' state among all those compatible with the measurements and the noise bound is presented. The overall consensus protocol ensures the convergence of the states in a tube whose radius depends on the maximum amplitude of the measurement noise.

In this paper we analyze two classes of consensus algorithms in a set-theoretic framework. Under the assumption of unknown but bounded measurement errors, the feasible state set (i.e. the set of all states compatible with the bounds on the noise) is explicitly derived. This kind of sets naturally arise in the context of the set-membership estimation theory, which was originally developed for dynamic system identification and filtering problems, with the specific purpose of guaranteeing a worst case bound in the estimation of the model parameters or of the state vector [9],[10]. The evolution of the feasible state set is used to evaluate the asymptotic level of agreement of the team. Linear consensus protocols adopting both constant weights and vanishing weights are considered, in case of undirected and stationary communication graph. It is shown that for both types of protocols, asymptotic consensus cannot be guaranteed with respect to all possible noise realizations, and bounds on the asymptotic difference of the agents' states are explicitly derived, as a function of the bounds on the measurement errors and the eigenvalues of the weight matrix.

The paper is organized as follows. Section II presents an overview of the consensus protocols to be analyzed. In Section III the consensus problem is formulated in a set-membership framework, under the assumption of bounded measurement errors. The main contributions of the paper are presented in Section IV, where the asymptotic difference among the agents' states is related to the bounds on the measurement noise and to the weights used in the consensus protocol. Section V presents some numerical results illustrating the behavior of the algorithms for different noise realizations. Finally, in Section VI some conclusions are drawn and future directions of research are outlined.

II. Motivation and related work

Consider a system of n agents $\mathcal{V} = \{1, \ldots, n\}$ communicating among them according to an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes the vertex set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. An edge $(i, j),\ i \neq j$, belongs to $\mathcal{E}$ if and only agents i and j can communicate. Since $\mathcal{G}$ is undirected, if $(i, j) \in \mathcal{E}$ then also $(j, i) \in \mathcal{E}$. A path between two vertices $i, j \in \mathcal{V}$ is a sequence of edges $(l_k, l_{k+1}) \in \mathcal{E}$,

$k = 1, \ldots, s-1$ such that $l_1 = i$ and $l_s = j$. The graph $\mathcal{G}$ is connected if there exists a path between any two nodes $i, j \in \mathcal{V}$. The valence matrix D of $\mathcal{G}$ is the $n \times n$ diagonal matrix whose i-th entry on the diagonal corresponds to the degree of vertex i, i.e. to the number of edges incident on i. The adjacency matrix A of $\mathcal{G}$ is the $n \times n$ matrix whose ij-th entry is 1 if $(i,j) \in \mathcal{E}$, 0 otherwise. The Laplacian L of $\mathcal{G}$ is the $n \times n$ matrix $L = D - A$. Finally, we denote by $\mathcal{N}_i$ the set of neighbors of agent i, i.e. $\mathcal{N}_i = \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$.

Let $x_i(t) \in \mathbb{R}$ be the state of agent i at time $t \in \mathbb{N}$. At the same time instant, agent i is given a noisy information of the state of all its neighbors

$$y_j^{(i)}(t) = x_j(t) + v_j^{(i)}(t), \quad i = 1, \ldots, n, \quad j \in \mathcal{N}_i. \tag{1}$$

The quantities $y_j^{(i)}(t)$ can be thought of as measurements taken by agent i on the state of its neighbors, or as information sent to agent i through the communication network. In both cases, it is realistic to assume that each agent does not have access to the actual value of its neighbors' state. The term $v_j^{(i)}(t)$ models the uncertainty affecting the knowledge of the state of agent j, from agent i's viewpoint.

Each agent updates its state according to the equation

$$x_i(t+1) = x_i(t) + u_i(t), \quad i = 1, \ldots, n,$$

where $u_i(t)$ is the input of the i-th agent. If we denote by $Y^{(i)}(t) = \{y_j^{(i)}(t)\}_{j \in \mathcal{N}_i}$ all the information available to agent i at time t, the objective of a consensus algorithm is to find for each agent a control law $u_i(t) = f\left(x_i(t), Y^{(i)}(t)\right)$, $i = 1, \ldots, n$, ensuring the convergence of the agents' state to a common value, i.e. such that

$$\lim_{t \to \infty} |x_i(t) - x_j(t)| = 0, \quad i, j = 1, \ldots, n.$$

In the ideal case of noiseless information (i.e., when $v_j^{(i)}(t) = 0$, $\forall t$), a number of different solutions have been proposed, both for stationary and time-varying topology of the communication network, as well as for directed and undirected communication graphs [1],[5]. The vast majority of the proposed algorithms adopt a feedback control law $f(\cdot)$ which is a linear function of the agent states, the so called *linear consensus protocols*. When the topology of the communication graph is stationary, a linear consensus protocol takes on the form

$$x_i(t+1) = x_i(t) + \sum_{j \in \mathcal{N}_i} w_{ij}(x_j(t) - x_i(t)), \tag{2}$$

for $i = 1, \ldots, n$. If we stack the states of all agents into a single vector $x(t) = [x_1(t) \ \ldots \ x_n(t)]'$, equations (2) can be rewritten as

$$x(t+1) = (I + W)x(t) \tag{3}$$

where the entry (i,j) of the matrix W is given by

$$[W]_{ij} = \begin{cases} w_{ij} & \text{if } j \in \mathcal{N}_i \\ w_{ii} = -\sum_{j \in \mathcal{N}_i} w_{ij} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

One may think to the coefficient w_{ij} as a weight associated to the edge (i,j). In this case, the matrix W is symmetric (since the graph is undirected). Clearly, the communication graph determines the structure of W. It is well-known that if the graph is connected then there exist many possible choices of the weight matrix ensuring consensus, and if in addition W is symmetric the consensus value is simply the average of the initial agents' states (*average consensus* problem [11]). A typical example of such a matrix is

$$W = -\frac{1}{n} L, \tag{5}$$

where L denotes the Laplacian of the communication graph.

When the true state is not accessible, and noisy measurements like in (1) are used to replace the actual state value, the updating rule (2) becomes

$$x_i(t+1) = x_i(t) + \sum_{j \in \mathcal{N}_i} w_{ij}(x_j(t) - x_i(t)) + \sum_{j \in \mathcal{N}_i} w_{ij} v_j^{(i)}(t) \tag{6}$$

for $i = 1, \ldots, n$. Let

$$v(t) = [v^{(1)}(t), \ \ldots, \ v^{(n)}(t)]', \quad v^{(i)}(t) = \sum_{j \in \mathcal{N}_i} w_{ij} v_j^{(i)}(t), \tag{7}$$

then equations (6) can be rewritten in matrix form as

$$x(t+1) = (I + W)x(t) + v(t). \tag{8}$$

Due to the presence of the forcing term $v(t)$, consensus cannot be guaranteed anymore. However, using input-to-state stability arguments, it can be shown that the maximum difference between any two states remains bounded provided that the measurement noise $v_j^{(i)}(t)$ is bounded, and asymptotically vanishes if $v_j^{(i)}(t)$ tends to zero [6].

On the other hand, in order to make the effect of a persistent noise tend to zero, the measurements $y_j^{(i)}(t)$ should be weighted lesser and lesser over time. Let $a(t)$ be a positive function such that $\lim_{t \to \infty} a(t) = 0$. If the weights w_{ij} in (6) are replaced by $a(t) w_{ij}$, the updating rule becomes

$$x(t+1) = (I + a(t)W)x(t) + a(t)v(t). \tag{9}$$

In this case the agents' states evolve according to a time-varying linear system, fed by a vanishing input. If the measurement disturbances are modeled as independent stochastic variables, with zero mean and finite variance, then choosing $a(t) \sim \frac{1}{t^r}$, $0.5 < r \leq 1$, ensures that the state of all agents converges in probability to the same limit [7].

Driven by the aforementioned observations, the objective of this paper is twofold. Suppose W is selected so as to guarantee consensus in the noise-free case (see equation (3)). Moreover, assume that the measurement noise $v_j^{(i)}(t)$ is bounded. The first goal is to quantify the difference between the states as a function of the noise bound, in case constant weights are used (equation (8)). The second goal is to study the achievement of consensus when vanishing weights are adopted (equation (9)). Also in this case, should consensus not be guaranteed, the relation between state disagreement and noise bound will be pursued. The bounded error assumption naturally leads to cast these problems in a set-membership framework, as it will be shown in the next section.

III. Set-theoretic consensus

Let the state $x(t)$ be updated according to equation (8), where the input $v(t)$ is given by (7). Assume that the measurement noise is unknown-but-bounded (UBB), i.e.

$$|v_i^{(j)}(t)| \le \bar{\epsilon}, \quad i = 1, \dots, n, \quad j \in \mathcal{N}_i, \ \forall t, \tag{10}$$

where $\bar{\epsilon} > 0$ is a known quantity. The UBB assumption on the measurement noise immediately reflects on the possible values taken by the disturbance $v(t)$, i.e.

$$|v^{(i)}(t)| \le \epsilon_i, \tag{11}$$

where

$$\epsilon_i = \bar{\epsilon} \sum_{j \in \mathcal{N}_i} w_{ij} = \bar{\epsilon}|w_{ii}|. \tag{12}$$

For a given initial condition $x(0)$, it is possible to define the *feasible state set* (FSS) $\mathcal{X}_C(t)$ through the recursion

$$\begin{aligned} \mathcal{X}_C(0) &= \{x(0)\} \\ \mathcal{X}_C(t+1) &= (I + W)\mathcal{X}_C(t) + D_\epsilon \mathcal{B}_\infty \end{aligned} \tag{13}$$

where $\mathcal{B}_\infty$ denotes the unit ball in the ∞-norm, defined as $\|x\|_\infty = \max_i |x_i|$, and D_ϵ is the diagonal matrix whose i-th entry on the diagonal is equal to ϵ_i. Algebraic operators in (13) are to be intended as set operators, i.e. given the sets $\mathcal{S}, \mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{R}^n$, and a matrix $M \in \mathbb{R}^{n \times n}$

$$M\mathcal{S} = \{y \in \mathbb{R}^n : y = \mathcal{S}x, x \in \mathcal{S}\},$$

$$\mathcal{S}_1 + \mathcal{S}_2 = \{y \in \mathbb{R}^n : y = x_1 + x_2, x_1 \in \mathcal{S}_1, x_2 \in \mathcal{S}_2\}.$$

The set $D_\epsilon \mathcal{B}_\infty$ is a box in $\mathbb{R}^n$ and contains all the possible realizations of the disturbance $v(t)$ which satisfy the UBB assumption (10). Consequently, the set $\mathcal{X}_C(t)$ contains all the states at time t compatible with the initial condition $x(0)$ and the error bounds (10). By expanding the recursion (13), it can be checked that the feasible state set at time t can be written as

$$\mathcal{X}_C(t) = \{x \in \mathbb{R}^n : x = x_c(t) + T(t)\alpha, \ \alpha \in \mathbb{R}^{tn}, \ \|\alpha\|_\infty \le 1\} \tag{14}$$

where

$$x_c(t) = F^t x(0), \tag{15}$$
$$T(t) = [T_1 \ T_2 \ \dots \ T_t] \in \mathbb{R}^{n \times tn}, \tag{16}$$

and

$$F = I + W, \tag{17}$$
$$T_i = F^{t-i} D_\epsilon \in \mathbb{R}^{n \times n}, \quad i = 1, \dots, t. \tag{18}$$

The set $\mathcal{X}_C$ is a *parpolygon* in $\mathbb{R}^n$, with center x_c and edges parallel to the columns s_i of $T = [s_1 \ \dots \ s_{tn}]$. All the elements of $\mathcal{X}_C$ can thus be obtained by adding to x_c a linear combination of the segments identified by the columns s_i of T [10].

Similarly to the case of constant weights, if the state evolves according to equation (9), under the UBB assumption (10) the feasible state set $\mathcal{X}_V(t)$ at time t is given by

$$\begin{aligned} \mathcal{X}_V(0) &= \{x(0)\} \\ \mathcal{X}_V(t+1) &= (I + a(t)W)\mathcal{X}_V(t) + a(t)D_\epsilon \mathcal{B}_\infty \end{aligned} \tag{19}$$

Hence the feasible state set is still a parpolygon like (14), but with different center and edges

$$\mathcal{X}_V(t) = \{x \in \mathbb{R}^n : x = x_c(t) + T(t)\alpha, \ \alpha \in \mathbb{R}^{tn}, \ \|\alpha\|_\infty \le 1\} \tag{20}$$

where

$$x_c(t) = \Phi(t, 0)x(0), \tag{21}$$
$$T(t) = [T_1 \ T_2 \ \dots \ T_t] \in \mathbb{R}^{n \times tn}, \tag{22}$$

and

$$F(t) = I + a(t)W, \tag{23}$$
$$\Phi(t_2, t_1) = F(t_2 - 1)F(t_2 - 2) \dots F(t_1), \quad 0 \le t_1 < t_2, \tag{24}$$
$$\Phi(t, t) = I, \tag{25}$$
$$T_i = a(i-1)\Phi(t, i)D_\epsilon \in \mathbb{R}^{n \times n}, \quad i = 1, \dots, t. \tag{26}$$

Since the states of the agents at time t are constrained to belong to the parpolygon $\mathcal{X}_{(\cdot)}(t)$, the achievement of consensus can be established by studying the time evolution of $\mathcal{X}_{(\cdot)}(t)$. Specifically, consensus is reached if and only if all the segments defining $\mathcal{X}_{(\cdot)}(t)$ (the columns of matrix $T(t)$) eventually align with the vector $\mathbf{1} = [1 \ 1 \ \dots \ 1]' \in \mathbb{R}^n$, i.e. if and only if the parpolygon degenerates into a line. Moreover, should this not happen, a measure of the disagreement of the team is given by the maximum size of the projection of $\mathcal{X}_{(\cdot)}(t)$ on the subspace orthogonal to $\mathbf{1}$, denoted by $S^{\mathbf{1}^\perp}$. As a matter of fact, let $P^{\mathbf{1}^\perp} = I - \frac{\mathbf{1}\mathbf{1}'}{n}$ be the projector of a vector of $\mathbb{R}^n$ on $S^{\mathbf{1}^\perp}$. Then the projection of $\mathcal{X}_{(\cdot)}(t)$ on $S^{\mathbf{1}^\perp}$ is given by

$$\mathcal{X}_{(\cdot)}^{\mathbf{1}^\perp}(t) = \{x^{\mathbf{1}^\perp} \in \mathbb{R}^n : x^{\mathbf{1}^\perp} = P^{\mathbf{1}^\perp}x, \ x \in \mathcal{X}_{(\cdot)}(t)\}.$$

and

$$r_{(\cdot)}(t) = \max_{x \in \mathcal{X}_{(\cdot)}^{\mathbf{1}^\perp}(t)} \|x\|_2 \tag{27}$$

is the maximum deviation from consensus (in the 2-norm) at time t.

IV. Characterization of the feasible state set

Let the weight matrix W be defined as in (4), with $w_{ij} = w_{ji} > 0$, $\forall i, j$, and $\sum_{j \in \mathcal{N}_i} w_{ij} < 1$, $\forall i$. Assume that the communication graph is connected. From now on, let λ_i^X denote the i-th largest eigenvalue of a symmetric matrix X. We are now ready to present the main results of the paper.

A. Constant weights

Let us suppose that the state of the agents evolves according to the stationary dynamic model (8). By construction F (defined in (17)) is a symmetric, doubly stochastic matrix

$$F\mathbf{1} = \mathbf{1}, \quad \mathbf{1}'F = \mathbf{1}', \tag{28}$$

and $\lambda_i^F = 1 + \lambda_i^W, \quad i = 1, \ldots, n$. If the communication graph is connected, then 1 is the only eigenvalue of modulus one, with all other ones having modulus strictly smaller than one

$$-1 < \lambda_n^F \le \lambda_{n-1}^F \le \cdots < \lambda_1^F = 1, \tag{29}$$

and hence $-2 < \lambda_n^W \le \lambda_{n-1}^W \le \cdots < \lambda_1^W = 0$.

Proposition 1: The feasible state set $\mathcal{X}_C$ given by (14) is asymptotically unbounded along the direction identified by the vector $\mathbf{1}$.

Proof: Denote by $\mathcal{X}_C^{\mathbf{1}}(t)$ the orthogonal projection of $\mathcal{X}_C(t)$ on the subspace spanned by $\mathbf{1}$, i.e.

$$\mathcal{X}_C^{\mathbf{1}}(t) = \{x^{\mathbf{1}} \in \mathbb{R}^n : x^{\mathbf{1}} = \frac{\mathbf{1}\mathbf{1}'}{n}x, \ x \in \mathcal{X}_C(t)\}.$$

Let us consider the state $\hat{x}(t) \in \mathcal{X}_C(t)$ such that $\hat{x}(t) = x_c(t) + T(t)\hat{\alpha}$, with $\hat{\alpha} = [1\ 1\ \ldots\ 1]' \in \mathbb{R}^{tn}$. Then, from (14)-(17) and exploiting (28), its projection $\hat{x}^{\mathbf{1}}(t)$ on $\mathbf{1}$ is given by

$$\begin{aligned}\hat{x}^{\mathbf{1}}(t) &= \frac{\mathbf{1}\mathbf{1}'}{n}\hat{x}(t) = \frac{\mathbf{1}\mathbf{1}'}{n}(x_c(t) + T(t)\hat{\alpha}) \\ &= \frac{\mathbf{1}}{n}(\mathbf{1}'x(0) + [\mathbf{1}'D_\epsilon\ \mathbf{1}'D_\epsilon\ \ldots \mathbf{1}'D_\epsilon]\hat{\alpha}) \\ &= \frac{\mathbf{1}}{n}(\mathbf{1}'x(0) + t\sum_{i=1}^{n}\epsilon_i)\end{aligned}$$

By construction $\hat{x}^{\mathbf{1}}(t) \in \mathcal{X}_C^{\mathbf{1}}(t)$, and its norm tends to infinity

$$\lim_{t\to\infty} \left\|\hat{x}^{\mathbf{1}}(t)\right\|_2 = +\infty,$$

which concludes the proof. ■

Proposition 1 states that in case of constant weights, when consensus is achieved, the consensus value is not necessarily bounded. This means that even when the difference among the agents' state vanishes, the state of each agent can diverge.

Proposition 2: The set $\mathcal{X}_C^{\mathbf{1}^\perp}(t)$ is bounded for all t. Moreover, $r_C(t)$ in (27) satisfies

$$\max_{i=2,\ldots,n} \frac{\underline{\delta}_i}{1-|\lambda_i^F|} \le \lim_{t\to\infty} r_C(t) \le \frac{\overline{\delta}}{1-\lambda_M^F},$$

where $\underline{\delta}_i = \bar{\epsilon}\left\|\text{diag}(W)e_i\right\|_1$, $\overline{\delta} = \max_{v\in D_\epsilon\mathcal{B}_\infty}\left\|P^{\mathbf{1}^\perp}v\right\|_2$, $\lambda_M^F = \max\{\lambda_2^F, -\lambda_n^F\}$, $\text{diag}(W)$ is the diagonal matrix whose i-th diagonal entry is w_{ii}, and e_i is the unitary eigenvector of F associated to λ_i^F.

Proof: If $x(t) \in \mathcal{X}_C(t)$ then $x(t)$ satisfies (8), with the initial condition $x(0)$. Hence its projection evolves as

$$x^{\mathbf{1}^\perp}(t+1) = Fx^{\mathbf{1}^\perp}(t) + P^{\mathbf{1}^\perp}v(t), \tag{30}$$

with the initial condition $x^{\mathbf{1}^\perp}(0) = P^{\mathbf{1}^\perp}x(0)$. In (30) it has been exploited $P^{\mathbf{1}^\perp}F = FP^{\mathbf{1}^\perp} = F - \frac{\mathbf{1}\mathbf{1}'}{n}$, from (28). Since F is symmetric it admits an orthogonal decomposition $F = \sum_{i=1}^n \lambda_i^F e_i e_i'$ where e_i is the eigenvector related to the eigenvalue λ_i^F, and $e_i'e_j = 1$ if $i = j$, zero otherwise. Since $e_1 = \frac{1}{\sqrt{n}}\mathbf{1}$, the vectors $\{e_2,\ \ldots,\ e_n\}$ form an orthonormal basis of the subspace $S^{\mathbf{1}^\perp}$. Hence

$$\left\|Fx^{\mathbf{1}^\perp}(t)\right\|_2 \le \lambda_M^F\left\|x^{\mathbf{1}^\perp}(t)\right\|_2, \tag{31}$$

where $\lambda_M^F = \max\{\lambda_2^F, -\lambda_n^F\}$ denotes the modulus of the second largest (in modulus) eigenvalue of F. From the UBB assumption (10)-(12), the disturbance vector $v(t)$ belongs the box $D_\epsilon\mathcal{B}_\infty$; hence its projection $P^{\mathbf{1}^\perp}v(t)$ belongs to the projection $P^{\mathbf{1}^\perp}D_\epsilon\mathcal{B}_\infty$ and

$$\left\|P^{\mathbf{1}^\perp}v(t)\right\|_2 \le \overline{\delta}. \tag{32}$$

Let $z(t) \triangleq \left\|x^{\mathbf{1}^\perp}(t)\right\|_2$. From (30)-(32) it follows

$$z(t+1) \le \lambda_M^F z(t) + \overline{\delta}. \tag{33}$$

By (33), one gets

$$z(t) \le (\lambda_M^F)^t z(0) + \sum_{k=0}^{t-1}(\lambda_M^F)^{t-k-1}\overline{\delta} \triangleq \mu(t).$$

Since $0 \le \lambda_M^F < 1$ from (29), then

$$\lim_{t\to\infty}\mu(t) = \frac{\overline{\delta}}{1-\lambda_M^F}.$$

Hence, $\lim_{t\to\infty} r_C(t) \le \frac{\overline{\delta}}{1-\lambda_M^F}$.

In order to find an asymptotic lower bound for $r_C(t)$, notice that $z(t) \ge \max_{i=2,\ldots,n}|e_i'x^{\mathbf{1}^\perp}(t)|$. Let $\nu_i(t) \triangleq |e_i'x^{\mathbf{1}^\perp}(t)|$ and $\underline{\delta}_i \triangleq \max_{v\in D_\epsilon\mathcal{B}_\infty}|e_i'v|$, $i = 2, \ldots, n$. From the definition of $D_\epsilon\mathcal{B}_\infty$ and recalling (12), one has

$$\underline{\delta}_i = e_i'D_\epsilon\text{sgn}\ (e_i) = \bar{\epsilon}\left\|\text{diag}(W)e_i\right\|_1 \tag{34}$$

where sgn (e_i) has to be intended componentwise. Now, if we consider a noise realization such that

$$v_i(t) = \text{sgn}\ (e_i'x^{\mathbf{1}^\perp}(t))D_\epsilon\text{sgn}\ (e_i), \tag{35}$$

from (30) and (34) one gets

$$\begin{aligned}e_i'x^{\mathbf{1}^\perp}(t+1) &= \lambda_i^F e_i'x^{\mathbf{1}^\perp}(t) + e_i'v_i(t) \\ &= \lambda_i^F e_i'x^{\mathbf{1}^\perp}(t) + \text{sgn}\ (e_i'x^{\mathbf{1}^\perp}(t))\underline{\delta}_i.\end{aligned}$$

Notice that by construction $v_i(t)$ in (35) belongs to $D_\epsilon\mathcal{B}_\infty$. As a result of this noise realization, the dynamics of $\nu_i(t)$ becomes

$$\nu_i(t+1) = |\lambda_i^F|\nu_i(t) + \underline{\delta}_i,$$

and therefore $\lim_{t\to\infty}\nu_i(t) = \frac{\underline{\delta}_i}{1-|\lambda_i^F|}$. This concludes the proof. ■

Remark 1: Proposition 1 and 2 state that asymptotically the feasible state set $\mathcal{X}_C$ is contained in an infinite cylinder aligned with the vector $\mathbf{1}$, and whose radius r_C is bounded by $\max_{i=2,\ldots,n}\frac{\underline{\delta}_i}{1-|\lambda_i^F|} \le r_C \le \frac{\overline{\delta}}{1-\lambda_M^F}$. Notice that the upper bound is tight if the vector $\overline{\delta}e_M$ belongs to $P^{\mathbf{1}^\perp}D_\epsilon\mathcal{B}_\infty$, where e_M is the unitary eigenvector associated to the largest (in modulus) eigenvalue of F. This means that there should exist a feasible noise realization giving rise to a disturbance $\bar{v}$ whose projection on $\mathcal{S}^{\mathbf{1}^\perp}$ is $P^{\mathbf{1}^\perp}\bar{v} = \delta e_M$. In this case, if $x^{\mathbf{1}^\perp}(0) = x_0 e_M$ and $v(t) = \bar{v}$, $\forall t$, then $z(t) = \mu(t)$, $\forall t$. However, this condition is not satisfied in general, being dependent on the specific choice of W.

B. Vanishing weights

Let us suppose that the state of the agents evolves according to the time-varying dynamic model (9). Let $a(t)$ be a sequence such that

$$0 < a(t) < \frac{1}{\max_i |w_{ii}|}, \quad t \geq 0, \tag{36}$$

$$\lim_{t\to\infty} a(t) = 0, \tag{37}$$

$$\sum_{t=0}^{\infty} a(t) = +\infty. \tag{38}$$

Examples of such functions are $a(t) = 1/(t+\gamma)^r$, $\gamma > 1$, $0 < r \leq 1$. Condition (36) ensures that $F(t)$, defined in (23), is still a doubly stochastic matrix, satisfying (28), $\forall t$. Under the assumption of connected communication graph, the eigenvalues of $F(t)$ are

$$-1 < \lambda_n^{F(t)} \leq \lambda_{n-1}^{F(t)} \leq \cdots < \lambda_1^{F(t)} = 1, \quad \forall t \tag{39}$$

and $\lambda_i^{F(t)} = 1 + a(t)\lambda_i^W$. Condition (37) ensures that there exists a time instant $\bar{t}$, such that

$$\lambda_i^{F(t)} = 1 + a(t)\lambda_i^W > 0, \qquad \forall t > \bar{t}, \tag{40}$$

and hence

$$\lambda_M^{F(t)} = 1 + a(t)\lambda_2^W > 0, \qquad \forall t > \bar{t}, \tag{41}$$

where $\lambda_M^F(t)$ denotes the second largest (in modulus) eigenvalue of $F(t)$. For example, $\bar{t}$ can be the smallest time t such that $a(t) < 1/2$, $\forall t > \bar{t}$, and its existence is guaranteed by (37).

Proposition 3: Let $a(t)$ satisfy (36)-(38). Then, the feasible state set $\mathcal{X}_V$ given by (20) is asymptotically unbounded along the direction identified by the vector $\mathbf{1}$.

Proof: First note that by definition (24), and exploiting (28), it holds

$$\mathbf{1}'\Phi(t_2, t_1) = \mathbf{1}', \quad 0 \leq t_1 < t_2. \tag{42}$$

Denote by $\mathcal{X}_V^{\mathbf{1}}(t)$ the orthogonal projection of $\mathcal{X}_V(t)$ on the subspace spanned by $\mathbf{1}$

$$\mathcal{X}_V^{\mathbf{1}}(t) = \{x^{\mathbf{1}} \in \mathbb{R}^n : x^{\mathbf{1}} = \frac{\mathbf{1}\mathbf{1}'}{n}x, \ x \in \mathcal{X}_V(t)\}.$$

and consider the state $\hat{x}(t) \in \mathcal{X}_C(t)$ corresponding to $\hat{\alpha} = [1 \ 1 \ ... \ 1]' \in \mathbb{R}^{tn}$. Then, from (20)-(26) and exploiting (42), its projection $\hat{x}^{\mathbf{1}}(t)$ on $\mathbf{1}$ is given by

$$\hat{x}^{\mathbf{1}}(t) = \frac{\mathbf{1}}{n}(\mathbf{1}'x(0) + \sum_{j=1}^{n} \epsilon_j \sum_{i=0}^{t-1} a(i)).$$

By construction $\hat{x}^{\mathbf{1}}(t) \in \mathcal{X}_C^{\mathbf{1}}(t)$ and its norm tends to infinity by assumption (38), which concludes the proof. ■

The following proposition is the counterpart of Proposition 2 in case of vanishing weights.

Proposition 4: Let $a(t)$ satisfy (36)-(38). Then, the set $\mathcal{X}_V^{\mathbf{1}^\perp}(t)$ is bounded for all t. Moreover, $r_V(t)$ in (27) satisfies

$$\max_{i=2,\ldots,n} \frac{\underline{\delta}_i}{1-\lambda_i^F} \leq \lim_{t\to\infty} r_V(t) \leq \frac{\bar{\delta}}{1-\lambda_2^F},$$

where $\underline{\delta}_i = \bar{\epsilon}\,\|\text{diag}(W)e_i\|_1$, $\bar{\delta} = \max_{v \in D_\epsilon \mathcal{B}_\infty} \left\|P^{\mathbf{1}^\perp} v\right\|_2$, $\text{diag}(W)$ is the diagonal matrix whose i-th diagonal entry is w_{ii}, and e_i is the unitary eigenvector of F associated to λ_i^F.

Proof: If $x(t) \in \mathcal{X}_V(t)$ then, from (9), the dynamics of its projection is

$$x^{\mathbf{1}^\perp}(t+1) = F(t)x^{\mathbf{1}^\perp}(t) + a(t)P^{\mathbf{1}^\perp}v(t). \tag{43}$$

Like in the proof of Proposition 2, we can resort to an orthogonal decomposition of $F(t) = \sum_{i=1}^n \lambda_i^{F(t)} e_i e_i'$, where the eigenvectors e_i do not depend on t, which leads to the following upper bound for the 2-norm of $x^{\mathbf{1}^\perp}(t)$

$$z(t+1) \leq \lambda_M^{F(t)} z(t) + a(t)\bar{\delta}, \quad t \geq 0,$$

where $z(t) \triangleq \left\|x^{\mathbf{1}^\perp}(t)\right\|_2$. The main difference with respect to the case of constant weights is that now the maximum eigenvalue is time varying and the driving input on the r.h.s. tends to zero. From (41), the inequality above can be rewritten as

$$z(t+1) \leq (1 + a(t)\lambda_2^W) z(t) + a(t)\bar{\delta}, \quad t > \bar{t},$$

which makes the dependency of $\lambda_M^{F(t)}$ on $a(t)$ explicit. To get an upper bound on $z(t)$ let us consider

$$\begin{aligned} \mu(t+1) &= (1 + a(t)\lambda_2^W)\mu(t) + a(t)\bar{\delta}, \quad t > \bar{t} \\ \mu(\bar{t}) &= z(\bar{t}) \geq 0 \end{aligned} \tag{44}$$

Now, we will show that $\mu_e = -\frac{\bar{\delta}}{\lambda_2^W}$ is an asymptotically stable equilibrium point for system (44). To this end, let us consider the candidate Lyapunov function

$$V(\mu, t) = \left(\mu(t) + \frac{\bar{\delta}}{\lambda_2^W}\right)^2. \tag{45}$$

Computing (45) at time $t+1$, one gets

$$V(\mu, t+1) = (1 + a(t)\lambda_2^W)^2 V(\mu, t), \quad t > \bar{t}.$$

This allows one to compute the evolution of $V(\mu, t)$ along the solution of (44) as

$$V(\mu, t) = \left[\prod_{k=\bar{t}}^{t-1}(1 + a(k)\lambda_2^W)\right]^2 V(\mu, \bar{t}).$$

It is known that an infinite product of the form $\prod_{k=0}^{\infty}(1-a_k)$, with $0 \leq a_k < 1$, converges to a non zero value if and only if $\sum_{k=0}^{\infty} a_k < +\infty$, [12]. This implies that if $\sum_{k=0}^{\infty} a_k = +\infty$, then $\prod_{k=0}^{\infty}(1-a_k) = 0$. Hence, by assumption (38)

$$\lim_{t\to\infty} V(\mu, t) = 0,$$

for all $V(\mu, \bar{t})$. Since $V(\mu, t) = 0$ implies $\mu(t) = -\frac{\bar{\delta}}{\lambda_2^W}$, then we have

$$\lim_{t\to\infty} \mu(t) = -\frac{\bar{\delta}}{\lambda_2^W}.$$

Notice that convergence is ensured for any initial condition $\mu(\bar{t})$. Since by construction $z(t) \leq \mu(t)$, $t > \bar{t}$, and $\lambda_2^W = \lambda_2^F - 1$, we can conclude that $\lim_{t\to\infty} r_V(t) \leq \frac{\bar{\delta}}{1-\lambda_2^F}$.

Let us now turn the attention to the lower bound. As noted in the proof of Proposition 2,

$$z(t) \geq |e_i' x^{\mathbf{1}^\perp}(t)| \stackrel{\triangle}{=} \nu_i(t).$$

If we consider the noise realization $v_i(t)$, defined in (34)-(35), $\forall t > \bar{t}$, from (43) the dynamics of $\nu_i(t)$ becomes

$$\nu_i(t+1) = (1 + a(t)\lambda_i^W)\nu_i(t) + a(t)\underline{\delta}_i, \quad t > \bar{t}.$$

Notice that for $t > \bar{t}$, $1 + a(t)\lambda_i^W > 0$ from (40). Hence, by applying the same arguments used to prove the convergence of $\mu(t)$, one gets $\lim_{t\to\infty} \nu_i(t) = -\frac{\underline{\delta}_i}{\lambda_i^W} = \frac{\underline{\delta}_i}{1-\lambda_i^F}$. This concludes the proof. ■

Remark 2: Proposition 4 shows that also in case of vanishing weights, the feasible state set does not shrink to the line spanned by $\mathbf{1}$. This means that consensus cannot be guaranteed, with respect to all possible noise realizations satisfying the UBB assumption (10). Similarly to what happens for constant weights, the feasible state set $\mathcal{X}_V$ is contained in an infinite cylinder aligned with the vector $\mathbf{1}$, and whose radius r_V is bounded by $\max_{i=2,\ldots,n} \frac{\underline{\delta}_i}{1-\lambda_i^F} \leq r_V \leq \frac{\overline{\delta}}{1-\lambda_2^F}$. Again, the upper bound is tight if the disturbance $\overline{\delta} e_2$ belongs to $P^{\mathbf{1}^\perp} D_\epsilon \mathcal{B}_\infty$.

C. Discussion

For the classes of consensus algorithms considered, it turns out that if the updating scheme ensures the achievement of consensus in the noise-free scenario, then the feasible state set is asymptotically contained in an infinite cylinder aligned with $\mathbf{1}$, and whose radius can be bounded by functions of the eigenvalues of the weight matrix and of the maximum amplitude of the measurement errors. It is worth remarking that the radius does not depend on the initial state of the agents. This means that the maximum deviation from consensus, with respect to all possible noise realizations, is independent of the initial disagreement of the team.

Both the upper and the lower bounds found in case of vanishing weights are smaller than or equal to their counterpart in case of constant weights. This means that an algorithm like (9) could provide some improvement also in a worst-case scenario, at least in principle. However, whenever matrix W is such that $\lambda_M^F = \lambda_2^F$ (which is often the case, see [11]) the upper bound is the same for both classes. Moreover, it depends only on the second largest eigenvalue of matrix F. It is known that such an eigenvalue determines the convergence rate to the consensus value in absence of measurement noise (the smaller meaning the faster [2],[11]). Hence, a faster mixing network guarantees also a smaller worst-case asymptotic difference among the agents' states.

It is interesting to observe that the upper bound in case of vanishing weights does not depend on the rate of convergence of the weighting sequence $a(t)$, as long as it satisfies (38). Note that if $a(t)$ is selected such that $\sum_{t=0}^{\infty} a(t) < +\infty$, then $\prod_{t=\bar{t}}^{\infty}(1 + a(t)\lambda_2^W) > 0$, and the limiting values of the bounding sequences $\mu(t)$, $\nu_i(t)$ depend on the initial conditions (see the proof of Proposition 4). As a side effect, even if $v(t) = 0$, $\forall t \geq 0$, i.e. in the noise-free scenario, consensus cannot be reached. Nonetheless, an $a(t)$ such that its summation converge, would have the advantage of bounding the feasible state set also along the direction $\mathbf{1}$ (see the proof of Proposition 3). Hence, in a worst-case analysis it could be of interest to choose a faster vanishing $a(t)$, in order to trade off boundedness of the agents' states and maximum asymptotic disagreement of the team.

Finally, it is worth remarking that the main reason why protocols like (9) do not guarantee consensus in a set-membership framework (differently from what happens when measurement noise is modeled in a stochastic setting [7]) is that the noise is only assumed to be bounded, and biased noise realizations are allowed as well.

V. Numerical results

In this section we report some simulation results, involving a team of six agents, whose communication graph is depicted in Figure 1. The weight matrix W is selected as in (5). With this choice, $\lambda_M^F = \lambda_2^F = 0.80$ and $\lambda_i^F > 0$, $i = 1, \ldots, n$. The measurement noise is assumed to be bounded as in (10), with $\bar{\epsilon} = 0.1$. The bounds on $r_C(t)$ and $r_V(t)$ provided by Propositions 2 and 4 coincide and are given by

$$\underline{r} = \max_{i=2,\ldots,n} \frac{\underline{\delta}_i}{1-\lambda_i^F} = \frac{\underline{\delta}_2}{1-\lambda_2^F} = 0.45$$

$$\overline{r} = \frac{\overline{\delta}}{1-\lambda_M^F} = 0.62$$

In the updating rule with vanishing weights, the sequence $a(t) = \frac{1}{(t+5)^{0.6}}$ is used.

Figures 2-3 summarize the results of 100 simulation runs, starting from initial conditions $x(0)$ randomly generated and normalized such that $\left\| x^{\mathbf{1}^\perp}(0) \right\|_2 = 1$. The average and the maximum of $\left\| x^{\mathbf{1}^\perp}(t) \right\|_2$ are depicted at each time instant, for both choices of the updating rule. The asymptotic upper bound $\overline{r}$ (dash-dotted line) is also shown. In a first set of experiments, the measurement noise $v_j^{(i)}(t)$ is uniformly distributed in the interval $[-\bar{\epsilon},\ \bar{\epsilon}]$. When constant weights are used, the average of $\left\| x^{\mathbf{1}^\perp}(t) \right\|_2$ settles around 0.05, while the maximum values oscillates around 0.1 (Figure 2). In case of vanishing weights, the same quantities at the final simulation time are one order of magnitude smaller. A second set of simulations, where the measurement noise $v_j^{(i)}(t)$ is simulated by a white process taking the values $\pm\bar{\epsilon}$ with equal probability, has been carried out. The behavior of $\left\| x^{\mathbf{1}^\perp}(t) \right\|_2$ is basically the same as before, just larger values are observed due to the different noise characteristics (Figure 3). In both scenarios, the theoretical upper bound on the asymptotic value of $r(t)$ turns out to be quite conservative. However, it is worth recalling that such a bound must hold for all possible noise realizations satisfying (10). Even though the worst-case noise $v_j^{(i)}(t)$ is not easy to determine, some unfavorable realizations can be figured out. In Figure 4, $\left\| x^{\mathbf{1}^\perp}(t) \right\|_2$ is shown when $v_j^{(i)}(t)$ is generated such that $v(t) = v_2(t)$ as defined in equation (35). In this case, the actual value of

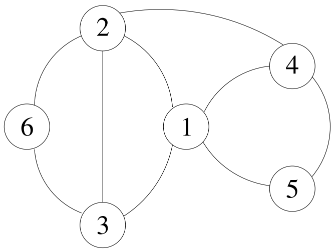

Fig. 1. Communication graph used in the simulations.

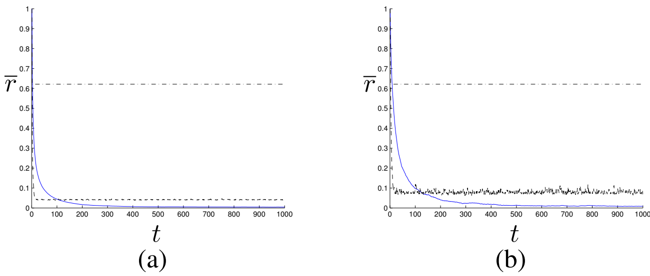

Fig. 2. Average value (a) and maximum (b) of $\left\|x^{\mathbf{1}^\perp}(t)\right\|_2$, over 100 simulation runs, in case of constant weights (dashed line) and vanishing weights (solid line). Measurement noise is uniformly distributed.

$\left\|x^{\mathbf{1}^\perp}(t)\right\|_2$ is eventually larger than $\underline{r}$ (lower dash-dotted line in Figure 4), and the final value is the same for both choices of the weights.

VI. Conclusions and Future Work

In this paper, the asymptotic properties of two classes of linear consensus algorithms have been analyzed, in presence of bounded measurement errors. The consensus protocols taken into account differ for the way the weighting matrix is chosen, being either constant over time or vanishing as time increases. Under the assumption of bounded errors the consensus problem has been formulated in a set-theoretic framework. By studying the evolution of the feasible state set, a worst-case analysis on the asymptotic disagreement of the team has been performed.

It has been shown that for both kinds of algorithms, consensus cannot be guaranteed with respect to all possible noise realizations, but the difference among the agents' states is asymptotically bounded. Both upper and lower bounds have been derived, as a function of the bounds on the measurement noise and of the eigenvalues of the weight matrix.

There is a number of issues related to set-membership consensus which are going to be addressed in future work. One is the characterization of the noise realization giving rise to the maximum disagreement of the team, in order to achieve possible tighter bounds. Closely related to this topic is the synthesis of the weight matrix minimizing the worst-case asymptotic deviation from consensus. The possibility to extend these results to the case of time-varying topology of the communication network is also under investigation.

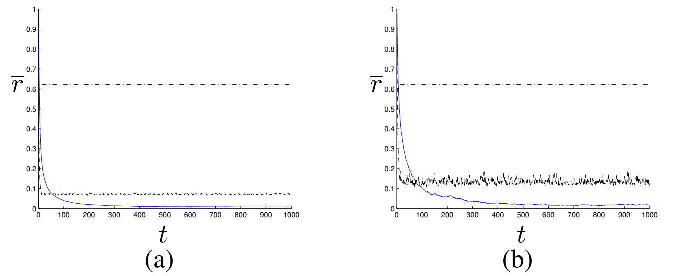

Fig. 3. Average value (a) and maximum (b) of $\left\|x^{\mathbf{1}^\perp}(t)\right\|_2$, over 100 simulation runs, in case of constant weights (dashed line) and vanishing weights (solid line). Measurements noise is $\pm\epsilon$ with equal probability.

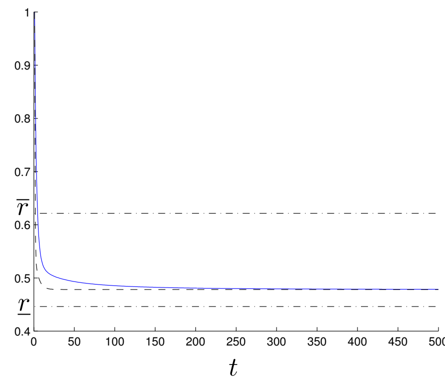

Fig. 4. Value of $\left\|x^{\mathbf{1}^\perp}(t)\right\|_2$ for noise realization (35), in case of constant weights (dashed line) and vanishing weights (solid line). Dash-dotted lines are the upper and lower bounds provided by Propositions 2 and 4.

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