Periodic Dynamics in Queuing Networks

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Abstract

This paper deals with state-dependent open Markovian (or exponential) queuing networks, for which arrival and service rates, as well as routing probabilities, may depend on the queue lengths. For a network of this kind, following Mandelbaum and Pats, we provide a formal definition of its associated fluid model, and we focus on the relationships which may occur between the network stochastic dynamics and the deterministic dynamics of its corresponding fluid model, particularly focusing on queuing networks whose fluid models have global periodic attractors.

Key words: periodic motion, state dependent queuing networks

1 Introduction

The study of complex systems pervades all of science, from cell biology to ecology, from computer science to meteorology. A paradigm of a complex system is a network where complexity may come from different sources: topological structure, network evolution, connection and node diversity, and/or dynamical evolution. The macroscopic behavior of a network is influenced by both the dynamical rules governing the nodes and the flow occurring along the links. Much research on complex networks has been focused so far on their topological structure; however, most networks offer support for various dynamical

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processes and in this paper we propose to study some aspects of periodic dynamics associated with *queuing networks* (Q-networks).

Queuing theory has its roots in the early twentieth century, in the work of the Danish engineer A. K. Erlang, who studied traffic loads in telephone systems. In the parlance of queuing theory, a queuing system is a place where *customers* arrive (according to an arrival process) to receive service from a service facility (i.e., a *server*). Several different real-world problems, including industrial production lines, telephony systems, computer networks, and motorized vehicular traffic, to mention only a few, have been successfully modeled using queuing theory. In these models the generic terms 'customers' and 'servers' may take various forms in different areas of application. In the case of a data switching network, 'customers' are packets arriving at the switching node and 'servers' are transmission channels. In a CPU job scheduling problem, 'customers' are computer processes (jobs or transactions) and 'servers' are various computer resources, such as CPU, I/O devices. Arazi, Ben-Jacob, and Yechiali [1] have suggested queuing theory as a possible way of understanding the joint action of genes, proteins and RNA molecules (interwoven in intricate interdependencies commonly known as genetic networks). In this approach, customers can represent molecules of some sort, waiting for chemical reactions.

In Q-networks one considers networks of queues forming in front of servers [2]. In these systems, the dynamics of customers moving around the network is ruled by random events, according to several types of random processes. In this paper we consider *state-dependent open Markovian (or exponential) Q-networks*, for which arrival and service rates, as well as routing probabilities, may depend on queue lengths. The stationary analysis of state-dependent networks started with the seminal work of Jackson [3]; for a summary of progress over the years see [4]. The transient evolution of such networks has been studied, among others, by Mandelbaum and Pats [5]. Non stationary Markov processes in which parameters like arrival and service rates, routing topologies for the network, and the number of servers at a given node are all functions of time, as well as the current state of the system, are discussed in [6]. Mandelbaum, Massey and Reiman [6] employ the theory of strong approximations to obtain functional strong laws of large numbers and functional central limit theorems for these networks, which gives a tractable set of network fluid and diffusion approximations. Further references on time dependent queues can be found in [7,8] and references therein.

Queuing theory often focuses on the steady-state analysis of stable networks, for which, as time tends to infinity, the system stabilizes, and its stochastic behavior becomes essentially independent of the initial conditions: the distribution function describing the system's state approaches an invariant (or stationary) distribution – a distribution that does not change over time. Much recent work on Q-networks has focused on fluid models for stability analysis,

diffusion approximations for performance analysis, and the relationship between the two methods. Fluid models are first-order approximations to the original Q-networks, and typically in the fluid approximation of a Q-network one adopts an abstract deterministic description of the average network dynamics through a set of ordinary differential equations, thus neglecting the effect of stochastic fluctuations on the network dynamics.

In this paper we consider Q-networks with associated fluid models that have global periodic attractors. We stress that *for state-dependent Q-networks, a fluid model may have a periodic attractor*. As already mentioned, queuing theory often focuses on the analysis of steady-state behavior of the network. To approximate the steady-state behavior of the network, it is first necessary to derive conditions under which a network admits a stationary distribution. For state-dependent Q-networks where the fluid model has a periodic attractor, we are not aware of a general mathematically rigorous proof for the existence and uniqueness of a stationary measure for the corresponding queuing network, but rather only on numerical evidence, as is also presented in this paper. The motivation for this research is twofold. First, we think that not only periodic behavior, but also different types of complex dynamics may be found in Q-networks, and these deserve to be studied since *the fluid model (deterministic dynamical system) can display a wide range of dynamics, including steady states, limit cycles and chaos*. Second, we think that fluid (and diffusion) models of queuing networks provide a promising approach for scalable and accurate performance analysis of real-world networks; at least this appears to be the case, e.g., for large TCP/IP networks [9–14].

Previous works – There are several papers which discuss oscillations and chaos in Q-networks. Chase et al. [15] analyze multiclass, deterministic flow models operating at full capacity. They examine both a switched-arrival and a switched-server model. For the former model, operating under a threshold-type policy (switch the incoming flow to a queue whenever its level drops below a threshold), they show that chaos can occur by analyzing the process embedded at the switching times as a discrete-time dynamical system. Whitt [16] studies a deterministic multiclass network of queues and shows that the evolution of queue lengths at the various nodes exhibits behavior that is sensitive to initial conditions - characteristic of chaotic systems. The authors of [17] are interested in stability and chaos of the dynamic input-pricing mechanism rather than the queueing process itself. In [18] the authors study a fluid model of a digital switching system when the system's workload is caused by interactions between the system and incident traffic, and show that there are several steady-state modes of operation, including chaotic.

Our work – In this paper we adopt the theory developed in [5] for the transient evolution of Q-networks to study systems whose deterministic fluid models have globally stable limit cycles (periodic attractors). In such cases, the max-

imum Lyapunov exponent over the stable cycle is equal to zero, in contrast to stable fixed points where the maximum Lyapunov exponent is negative. This fact has consequences on the behavior of Q-networks and the variable which corresponds to the zero Lyapunov exponent, the phase variable, is neutrally stable and hence the phase diffusion is the dominant effect in such networks. Following Mandelbaum and Pats [5], for a given Q-network we provide a formal definition of its associated fluid model, and we focus on the relationships which may occur between the network stochastic dynamics and the deterministic dynamics of its corresponding fluid model. The paper is organized as follows. Section 2 introduces necessary terminology and definitions. Following [5], Section 3 summarizes fluid models for open Markovian Q-networks, for which arrival and service rates, as well as routing probabilities, may depend on the state. In Sections 4 and 5 we discuss two examples of simple two-node Q-networks, whose fluid models have globally stable limit cycles, discussing what kind of information about the dynamics of such networks might be obtained by analyzing their deterministic fluid model. Finally, in Section 6, we close the paper with conclusions.

2 Preliminaries

A single class, single station queuing system consists of a queuing buffer and one or more identical servers. A server can only serve one customer at a time and hence it is either in a "busy" or in an "idle" state. If all the servers are busy upon the arrival of a customer, the newly arriving customer is buffered, assuming that buffer space is available, and waits for its turn. When the customer currently in service departs, one of the waiting customers is selected for service according to a *service discipline* (e.g., the FIFO discipline: *first in, first out*). The arrivals of customers and their services are considered to be the outcomes of stochastic processes, often assumed mutually independent. The *average customer interarrival time* is denoted by T_λ and its reciprocal is called the *arrival rate* $\lambda = 1/T_\lambda$. Similarly T_μ denotes the *average service time* and its reciprocal is called the *service rate* $\mu = 1/T_\mu$. Elementary queuing system models are described using the Kendall's notation: "$A/B/m$ - service discipline", where A identifies the type of stochastic process assumed for the arrival process, B relates to the service process, and m is the number of servers in the facility. Among those symbols used for A, B the letter M is used for denoting an exponential distribution (Markovian - memoryless property).

A network in which each node is a single-station queuing system is referred to as a *Q-network*. A Q-network is said to be *open* if customers enter from outside the network, circulate among the service facilities and eventually leave the network. Even if the customer flows are granular and stochastic, average customer flow rates can be sensibly introduced, similarly to the case of a

single-station queuing system described above. Accordingly, we refer to the following notation: $Q_i(t)$ denotes the total number of customers at node i at time t (i.e., the number of customers in service plus the number of customers waiting in the buffer); the exogenous arrival rate and the service rate for node i are denoted by λ_i and μ_i, respectively. For a n node network, λ_i and μ_i, $1 \leq i \leq n$, can be functions of both the time t and the network *internal state* $\mathbf{Q}(t) \in \mathbb{R}^n$, that is $\lambda_i(t) = f_i(\mathbf{Q}, t), \mu_i(t) = g_i(\mathbf{Q}, t)$, for some functions $f_i, g_i : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the set of nonnegative real numbers.

Queuing networks are rather difficult to explore, and in order to enable some sort of analysis to be conducted, one must usually make some assumptions on the stochastic characterization of the network. In this paper we consider n-node single class open Q-networks, in which each node is an $M_\xi/M_\xi/1-FIFO$ single class station (in queuing theory notation, the subscript ξ is often used to remark the state dependency). It is worth noting that without loss of generality this model can be used for treating the case in which each node has $m > 1$ identical servers, assuming for each node an equivalent single station with a *load-dependent* service rate [1]. A customer that has been served by node i can randomly either leave the system or instead he can join any queue depending on *transition probabilities* P_{ij}, $1 \leq i, j \leq n$, each one characterizing the customer flow from node i to node j. We will call the square matrix

$$\mathbf{P} = \begin{pmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & \ddots & \vdots \\ P_{n1} & \cdots & P_{nn} \end{pmatrix} \tag{1}$$

the *transition probability matrix* of the Q-network, where $0 \leq P_{ij} \leq 1$ and $\sum_{j=1}^n P_{ij} \leq 1$. In the general case the transition matrix $\mathbf{P}$ can be both dependent on the network state $\mathbf{Q}(t)$ and the time t, and the probability that a customer exits the network after being served at node i agrees with the quantity $S_i = 1 - \sum_{j=1}^n P_{ij}$.

Let us recall the well known *traffic equations* for an open network of $M/M/1-FIFO$ queues with constant rates. The traffic equation for node i deals with the *nominal aggregate customer arrival rate* γ_i, that is the nominal cumulative flow rate of customers which join the queue at node i whether coming from outside or from inside the network. Thus the traffic equation for node i is

$$\gamma_i = \lambda_i + \sum_{j=1}^{n} \rho_j \, \mu_j \, P_{ji}, \tag{2}$$

[1] For example, let us consider a node with $m > 1$ identical servers with service rate μ_0. The equivalent one-server station has the load-dependent service rate μ given by $\mu(Q(t)) = Q(t)\,\mu_0$, if $Q(t) \leq m$, or $\mu(Q(t)) = m\,\mu_0$, otherwise.

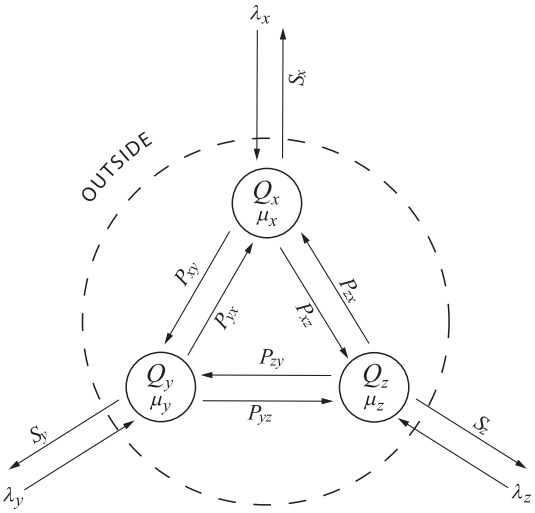


Fig. 1. A three-node Q-network: each circle represents a service station. The service rates are written inside the circles; the arrival rates are at the begin of the arrows; routing probabilities are beside arrows.

where $\rho_j = \gamma_j/\mu_j$, if $\gamma_j < \mu_j$, or $\rho_j = 1$, otherwise. The term ρ_j represents the nominal proportion of time that node j is serving a customer, and $\rho_j\,\mu_j\,P_{ji}$ is nominal effective rate of flow of customers from node j to node i.

In 1957 and 1963, J. R. Jackson published some fundamental results about open Q-networks considering a special case in which the exogenous arrival rates λ_i, the service rates μ_i, as well the transition matrix $\mathbf{P}$ are assumed constant [3]. In such case Jackson could express analytically the stationary distribution of the Q-network. For more complicated scenarios, especially when the analysis of the *dynamic evolution* of state-dependent Q-networks is considered, a different approach must be used.

3 Fluid models of state dependent Q-networks

The theoretical setup that we propose largely follows Mandelbaum and Pats [5]. We consider an n-node open Q-network in which each node is like a $M_\xi/M_\xi/1\ -FIFO$ single station with one server and in which both arrival rates $\boldsymbol{\lambda} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ and service rates $\boldsymbol{\mu} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ as well as the transition matrix $\mathbf{P} : \mathbb{R}^n_+ \to \mathbb{R}^{n\times n}_+$ are locally Lipschitz continuous functions of the state. In this case, by denoting with $\mathbb{N}$ the set of natural numbers, the dynamic evolution of the Q-network is the realization of a $\mathbb{N}^n$-valued stochastic process $\mathbf{Q} = \{\mathbf{Q}(t), t \geq 0\}$ that satisfies a relation of the form

$$\mathbf{Q}(t) = \mathbf{Q}(0) + \mathbf{A}(t) + \mathbf{F}(t) - \mathbf{D}(t), \qquad t \geq 0, \tag{3}$$

where the column vector $\mathbf{Q}(0)$ is the initial state of the Q-network, $\mathbf{A}(\cdot)$ is an n-dimensional counting process related to the arrivals from the outside of the network (exogenous arrivals), $\mathbf{F}(\cdot)$ is an n-dimensional counting process related to the flows inside the network (endogenous arrivals), and $\mathbf{D}(\cdot)$ is an n-dimensional counting process related to the service completions (departures). For the i-th component in (3) we assume

$$A_i(t) = N_i^+ \left(\int_0^t \lambda_i \left(\mathbf{Q}(s) \right) ds \right), \tag{4}$$

$$D_i(t) = N_i^- \left(\int_0^t \mu_i \left(\mathbf{Q}(s) \right) 1_{(0,\infty)}(Q_i(s)) ds \right), \tag{5}$$

$$F_i(t) = \sum_{j=1}^{n} \int_0^t 1_{\pi_{ji}(s-)} \left(U_j(D_j(s)) \, dD_j(s), \right. \tag{6}$$

where the notation have the following meaning. For a generic function $f : \mathbb{R}_+ \to \mathbb{R}_+$ the writing $f(s-)$ denotes the left limit of $f(\cdot)$ at s, assuming $f(0-) = f(0)$. For $i = 1, \ldots, n$, the quantities $N_i^+(\cdot), N_i^-(\cdot)$ are mutually independent Poisson processes (with rate 1). Given any set B, the indicator function $1_B : B \to \{0, 1\}$ is defined as $1_B(x) = 1$ if $x \in B$; 0 otherwise.

In (5) the set $(0, \infty)$ is the set of strictly positive reals and the indicator function $1_{(0,\infty)}(\cdot)$ is used since no customer departure from a station can occur while this station is empty. For $j = 1, \ldots, n$, the sequences $\{U_j(l)\}_{l \in \mathbb{N}}$ are assumed to be sequences of i.i.d. random variables uniformly distributed on $[0, 1)$, and for $i, j = 1, \ldots, n$ and $s \geq 0$, the set $\pi_{ji}(s-)$ is equal to

$$\pi_{ji}(s-) = \left\{ x \in [0,1) : \sum_{k=1}^{i-1} P_{jk}(\mathbf{Q}(s-)) \leq x < \sum_{k=1}^{i} P_{jk}(\mathbf{Q}(s-)) \right\}, \tag{7}$$

and for $j, k = 1, \ldots, n$, the terms P_{jk} are elements of the transition matrix function $\mathbf{P}$. In this expression, the left limit is used since, when a customer is served at time t_0, she/he is instantaneously routed according to the transition matrix $\mathbf{P}(\mathbf{Q}(t_0-))$ which refers to the network state at $\mathbf{Q}(t_0-)$. Indeed, the transition matrix $\mathbf{P}(\mathbf{Q}(t_0))$ is defined only when the customer served at t_0 has been definitely routed somewhere. The quantities introduced above have the following interpretations: for $t \geq 0$, $A_i(t)$ and $F_i(t)$ represent the cumulative number of exogenous and endogenous arrivals respectively to station i during $[0, t]$; when a customer at time t is served at node j, the quantity $D_j(t)$ increases by 1 from $D_j(t-)$. If the random variable $U_j(D_j(t)) \in \pi_{jk}(t-)$ then the customer departing form node j is routed to node k. Note that for each $t \geq 0$, $\bigcup_{i=1}^n \pi_{ji}(t) \subseteq [0, 1)$. It is worth emphasizing that the processes N_i^+ and N_i^- and the sequences $\{U_i(l)\}_{l \in \mathbb{N}}$ are assumed mutually independent. Moreover we assume that the spectral radius $r(\mathbf{P}(\cdot))$ of the transition matrix $\mathbf{P}$ is such that $\sup\{r(\mathbf{P}(\mathbf{v})), \mathbf{v} \in \mathbb{R}_+^n\} < 1$, and that the two vector-valued functions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ satisfy a linear growth constraint, i.e., there exists a positive constant Δ such that $\max\{\|\boldsymbol{\lambda}(\mathbf{v})\|, \|\boldsymbol{\mu}(\mathbf{v})\|\} \leq \Delta(1 + \|\mathbf{v}\|)$, for all $\mathbf{v} \in \mathbb{R}_+^n$.

As discussed in [5], since the Q-network process described above is a Markov jump process on $\mathbb{N}^n$, this latter assumption represents a sufficient condition for the non-explosion of the network's state $\mathbf{Q}$, i.e., $P\{\|\mathbf{Q}(t)\| < \infty, t \geq 0\} = 1$. Almost surely (in probabilistic terms), the paths of the stochastic process $\{\mathbf{Q}(t), t \geq 0\}$ consist of n piecewise constant right-continuous non-negative integer-valued functions that have finite left limit at each time. Moreover, for $i = 1, \ldots, n$, the stochastic processes $\{A_i(t), t \geq 0\}, \{D_i(t), t \geq 0\}$ can be referred to as *time-changed Poisson processes*. The expressions (4)-(6) can be rearranged in a form that is more amenable to analysis. Towards this end, we first introduce the following.

Definition 1 *Let $N = \{N(t), t \geq 0\}$ be a standard Poisson process (rate 1). The stochastic process $\hat{N}(t) = N(t) - E\{N(t)\} = N(t) - t$, $t \geq 0$, is said to be the* centering *of $N(\cdot)$ by its mean rate function.*

The previous definition can be extended to the case of time-changed Poisson processes. Therefore, we introduce the compensation of each of the stochastic processes (4)-(6) by a quantity equal to the integration over $[0, t]$ of the *effective rates* associated with each process. Accordingly, for $i = 1, \ldots, n$, we define

$$\hat{A}_i(t) = A_i(t) - \alpha_i(t), \qquad \hat{D}_i(t) = D_i(t) - \delta_i(t), \qquad \hat{F}_i(t) = F_i(t) - \phi_i(t),$$

where

$$\alpha_i(t) = \int_0^t \lambda_i\left(\mathbf{Q}(s)\right) ds,$$

$$\delta_i(t) = \int_0^t \mu_i\left(\mathbf{Q}(s)\right) 1_{(0,\infty)}\left(Q_i(s)\right) ds,$$

$$\phi_i(t) = \sum_{j=1}^{n} \int_0^t P_{ji}\left(\mathbf{Q}(s-)\right) \mu_j\left(\mathbf{Q}(s)\right) 1_{(0,\infty)}\left(Q_j(s)\right) ds.$$

Denoting the initial condition $\mathbf{Q}(0)$ as $\mathbf{Q}_0$, eq. (3) can be rewritten as

$$\mathbf{Q}(t) = \mathbf{Q}_0 + \left[\boldsymbol{\Theta}(t) + \mathbf{R}(t)\right] + \left[\hat{\mathbf{A}}(t) + \hat{\mathbf{F}}(t) - \hat{\mathbf{D}}(t)\right], \tag{8}$$

where, by denoting by $\mathbf{I}$ the identity matrix and by $\mathbf{P}'$ the transpose of $\mathbf{P}$, for $t \geq 0$ we have

$$\boldsymbol{\Theta}(t) = \int_0^t \boldsymbol{\lambda}\left(\mathbf{Q}(s)\right) + \left[\mathbf{P}'\left(\mathbf{Q}(s-)\right) - \mathbf{I}\right] \boldsymbol{\mu}\left(\mathbf{Q}(s)\right) ds, \tag{9}$$

$$\mathbf{R}(t) = \int_0^t \left[\mathbf{I} - \mathbf{P}'\left(\mathbf{Q}(s-)\right)\right] d\mathbf{Y}(s), \tag{10}$$

$$\mathbf{Y}(t) = \int_0^t \mathbf{I}_{\{0\}}(\mathbf{Q}(s)) \boldsymbol{\mu}\left(\mathbf{Q}(s)\right) ds, \tag{11}$$

where for any set $A \subset \mathbb{R}$ and for any $\mathbf{v} \in \mathbb{R}^n$ we define the diagonal matrix

$$\mathbf{I}_A(\mathbf{v}) = \begin{pmatrix} 1_A(v_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1_A(v_n) \end{pmatrix}. \tag{12}$$

In expression (8), the term $\boldsymbol{\Theta}$ represents the integration over $[0, t]$ of the *potential* aggregate net customer flow rates, while the term $\mathbf{R}$ captures part (or the total) of this flow depending on the queues becoming empty. Accordingly, the quantity $\boldsymbol{\Theta}(t) + \mathbf{R}(t)$ can be referred to as the integration over $[0, t]$ of the *nominal* aggregate net customer flow rate and the sum $\hat{\mathbf{A}} + \hat{\mathbf{F}} - \hat{\mathbf{D}}$, that in [5] is proved to define a martingale, incorporates the effects of discrete random fluctuations around these values over the same period. In general, the only way to use stochastic equations (3) or (8) directly in a useful manner is to resort to numerical approaches (i.e., computer simulations). Nevertheless, in some circumstances simpler equations than (8) can be considered as an approximation. We discuss one such approximation below.

3.1 Fluid Approximations

Fluid approximations are formal deterministic approximations to Q-networks obtained as scaling limits, where arrival and service rates are speed up, while attenuating the stochastic fluctuations by the same quantity. As a result, starting from a Q-network, a deterministic dynamical system (in the following, the *fluid model*) can be defined which captures for each state of the network the nominal trend of the stochastic dynamics. Under certain circumstances, the solutions of the fluid model can be considered – *within limited time intervals* – as an approximation of what is expected to be the stochastic evolution of the Q-network. From a theoretical point of view, fluid models for queuing networks are obtained as limits of sequences of stochastic processes $\{\mathbf{q}^k(t), t \geq 0\}$, determined as discussed in the following. In the notation, the superscript k indicates that the corresponding quantity is related to the k-th network of such a sequence. Accordingly, let consider a Q-network with initial condition $\mathbf{Q}_0$ and whose stochastic evolution is described by a random process $\mathbf{Q}$ satisfying relationship (3) and arising assumptions. We first define a sequence of Q-networks $\{\mathbf{Q}^k(t), t \geq 0\}$, satisfying relationship (3) and arising assumptions, such that $\mathbf{Q}_0^k = k\mathbf{Q}_0$, and

$$\boldsymbol{\lambda}^k(\mathbf{Q}^k) = k\boldsymbol{\lambda}(\mathbf{Q}^k/k), \quad \boldsymbol{\mu}^k(\mathbf{Q}^k) = k\boldsymbol{\mu}(\mathbf{Q}^k/k), \quad \mathbf{P}^k(\mathbf{Q}^k) = \mathbf{P}(\mathbf{Q}^k/k). \tag{13}$$

The sequence of stochastic processes $\{\mathbf{q}^k(t), t \geq 0\}$ is then defined as $\{\mathbf{q}^k\} =$

$\{\mathbf{Q}^k/k\}$. Accordingly, we have $\mathbf{q}_0^k = k\mathbf{Q}_0/k = \mathbf{Q}_0$ and for each $k \in \mathbb{N}$ we have

$$\mathbf{q}^k(t) = \mathbf{Q}_0 + \mathbf{A}^k(t) + \mathbf{F}^k(t) - \mathbf{D}^k(t), \qquad t \geq 0, \tag{14}$$

where, for the i-th component in (14) we have

$$A_i^k(t) = \frac{1}{k} N_i^+ \left(k \int_0^t \lambda_i \left(\mathbf{q}^k(s) \right) ds \right), \tag{15}$$

$$D_i^k(t) = \frac{1}{k} N_i^- \left(k \int_0^t \mu_i \left(\mathbf{q}^k(s) \right) 1_{(0,\infty)}(q_i^k(s)) ds \right), \tag{16}$$

$$F_i^k(t) = \sum_{j=1}^{n} \int_0^t 1_{\pi_{ji}^k(s-)} \left(U_j(kD_j^k(s)) \right) dD_j^k(s), \tag{17}$$

where

$$\pi_{ji}^k(s-) = \left\{ x \in [0,1) : \sum_{h=1}^{i-1} P_{jh}^k(\mathbf{Q}^k(s-)) \leq x < \sum_{h=1}^{i} P_{jh}^k(\mathbf{Q}^k(s-)) \right\}. \tag{18}$$

By construction, in the Markovian stochastic process described by $\mathbf{q}^k$ an arrival event in node i is recorded by increasing the length of the i^{th} corresponding *rational-valued* 'queue' by the quantity $1/k$, and the same quantity is subtracted when a departure occurs. With respect to the original Q-network $\mathbf{Q}$, the process $\mathbf{q}^k$ has arrival and service rates speed up k times to compensate for the k-times reduced effects of the stochastic perturbations. By applying a centering operation to each of the stochastic processes (15)-(17), we can write

$$\hat{\mathbf{a}}^k(t) = \mathbf{A}^k(t) - \boldsymbol{\alpha}^k(t), \qquad \hat{\mathbf{d}}^k(t) = \mathbf{D}^k(t) - \boldsymbol{\delta}^k(t), \qquad \hat{\mathbf{f}}^k(t) = \mathbf{F}^k(t) - \boldsymbol{\phi}^k(t).$$

where

$$\alpha_i^k(t) = k \int_0^t \lambda_i \left(\mathbf{q}^k(s) \right) ds$$

$$\delta_i^k(t) = k \int_0^t \mu_i \left(\mathbf{q}^k(s) \right) 1_{(0,\infty)} \left(q_i^k(s) \right) ds$$

$$\phi_i^k(t) = k \sum_{j=1}^{n} \int_0^t P_{ji} \left(\mathbf{q}^k(s-) \right) \mu_j \left(\mathbf{q}^k(s) \right) 1_{(0,\infty)} \left(q_j^k(s) \right) ds.$$

Accordingly, for the stochastic process $\mathbf{q}^k$ we can write

$$\mathbf{q}^k(t) = \mathbf{Q}_0 + \left[\boldsymbol{\theta}^k(t) + \mathbf{r}^k(t) \right] + \left[\hat{\mathbf{a}}^k(t) + \hat{\mathbf{f}}^k(t) - \hat{\mathbf{d}}^k(t) \right] \tag{19}$$

where

$$\boldsymbol{\theta}^k(t) = \int_0^t \left\{ \boldsymbol{\lambda} \left(\mathbf{q}^k(s) \right) + \left[\mathbf{P}^{k\prime} \left(\mathbf{q}^k(s-) \right) - \mathbf{I} \right] \boldsymbol{\mu} \left(\mathbf{q}^k(s) \right) \right\} ds, \tag{20}$$

$$\mathbf{r}^k(t) = \int_0^t \left[\mathbf{I} - \mathbf{P}^{k\prime} \left(\mathbf{q}^k(s-) \right) \right] d\mathbf{y}^k(s), \tag{21}$$

$$\mathbf{y}^k(t) = \int_0^t \mathbf{I}_{\{0\}}(\mathbf{q}^k(s))\boldsymbol{\mu}\left(\mathbf{q}^k(s)\right) ds. \tag{22}$$

For the dynamical behavior of the asymptotic process $\mathbf{q}^k$ when $k \to \infty$ the following theorem holds (this is a special case of Theorem 4.6 in [5]).

Theorem 1 (Fluid Limit Theorem) *The sequence of stochastic processes* $\{\mathbf{q}^k(t), t \geq 0\}$ *for* $k \to \infty$ *converges in probability uniformly over compacts in* $[0, \infty)$ *to the unique deterministic nonnegative absolutely continuous function* $\mathbf{q} : \mathbb{R}_+ \to \mathbb{R}_+^n$ *which is the solution of*

$$\mathbf{q}(t) = \mathbf{Q}_0 + \int_0^t \left[\boldsymbol{\lambda}\left(\mathbf{q}(s)\right) + \left[\mathbf{P}'\left(\mathbf{q}(s)\right) - \mathbf{I}\right]\boldsymbol{\mu}\left(\mathbf{q}(s)\right)\right] ds + \mathbf{r}(t), \qquad t \geq 0, \tag{23}$$

where the term $\mathbf{r}(t)$ *takes into account the cumulative effect of the system's state-dependent reflection on the boundaries*

$$\mathbf{r}(t) = \int_0^t \left[\mathbf{I} - \mathbf{P}'\left(\mathbf{q}(s)\right)\right] d\mathbf{y}(s), \quad t \geq 0. \tag{24}$$

The function $\mathbf{y} : \mathbb{R}_+ \to \mathbb{R}_+^n$ *is continuous, with nondecreasing components, such that* $\mathbf{y}(0) = \mathbf{0}$ *and*

$$\int_0^\infty \mathbf{I}_{(0,\infty)}(\mathbf{q}(t)) d\mathbf{y}(t) = \mathbf{0}. \tag{25}$$

The unique solution $\mathbf{q}$ of (23) is referred to as the *fluid limit* of the sequence of processes $\{\mathbf{q}^k(t), t \geq 0\}$ derived from the original Q-network $\{\mathbf{Q}(t), t \geq 0\}$ with initial condition $\mathbf{Q}_0$. By condition (25), since $\mathbf{y}$ is non-negative and due to the continuity of $\mathbf{q}$, it turns out that if the initial condition $\mathbf{Q}_0$ have all its components greater than 0, then there exists an interval $[0, t_s]$ for which $\mathbf{r}(t) = \mathbf{0}\ \forall t \in [0, t_s]$. Under this hypothesis, the fluid limit $\mathbf{q}$ over $[0, t_s]$ agrees with the unique solution of the system of ordinary differential equations (ODEs)

$$\frac{d\mathbf{q}}{dt} = \boldsymbol{\lambda}\left(\mathbf{q}\right) + \left[\mathbf{P}'\left(\mathbf{q}\right) - \mathbf{I}\right]\boldsymbol{\mu}\left(\mathbf{q}\right) \equiv \mathbf{h}(\mathbf{q}), \tag{26}$$

assuming $\mathbf{q}(0) = \mathbf{Q}_0$. The ith component in (26) has the simple form

$$\frac{dq_i}{dt} = \lambda_i\left(\mathbf{q}\right) + \sum_{j=1}^{n} P_{ji}(\mathbf{q})\mu_j(\mathbf{q}) - \mu_i(\mathbf{q}). \tag{27}$$

When taking into account the boundaries, instead of (26), the following differential equation with discontinuous right-hand side must be considered [5]

$$\frac{d\mathbf{q}}{dt} = \mathbf{h}(\mathbf{q}) + [\mathbf{I} - \mathbf{P}'(\mathbf{q})]\tilde{\mathbf{m}}(\mathbf{q}) \equiv \mathbf{H}(\mathbf{q}), \tag{28}$$

where the function $\mathbf{h}$ is defined in (26) and the function $\tilde{\mathbf{m}}$ is unambiguously determined by the state-dependent reflection, as described in the following.

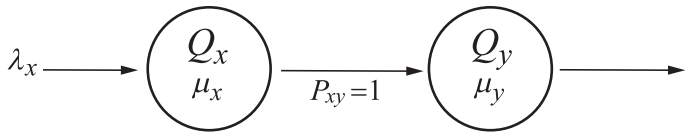


Fig. 2. A two-station tandem Q-network.

Denote by $\mathcal{I}(\mathbf{q}) \subseteq \{1, 2, \ldots, n\}$ the subset of indexes i such that $h_i(\mathbf{q}) < 0$ and such that the corresponding state's component q_i is zero, i.e., we have $i \in \mathcal{I}(\mathbf{q}) \Leftrightarrow q_i = 0, h_i(\mathbf{q}) < 0$ for $i = 1, 2, \ldots, n$. Accordingly, the components of $\tilde{\mathbf{m}}(\mathbf{q})$ can be obtained solving the linear system

$$\begin{cases} h_i(\mathbf{q}) + \tilde{m}_i(\mathbf{q}) - \sum_{j \in \mathcal{I}(\mathbf{q})} P_{ji}(\mathbf{q}) \tilde{m}_j(\mathbf{q}) = 0, & \text{if } i \in \mathcal{I}(\mathbf{q}), \\ \tilde{m}_i(\mathbf{q}) = 0, & \text{otherwise.} \end{cases} \tag{29}$$

In the following, expression (28) will be referred to as the *fluid model equation* for the Q-network with queue length process $\mathbf{Q}$.

3.2 Example

Consider the simple two-station $M_\xi/M_\xi/1-FIFO$ tandem Q-network of Fig. 2 with parameters

$$\begin{aligned} \lambda_x &= 200, & \lambda_y &= 0, \\ \mu_x &= 2Q_x, & \mu_y &= \begin{cases} \frac{Q_x Q_y}{10}, \text{ if } Q_x Q_y < 10^6 \\ 10^5, \quad \text{otherwise,} \end{cases} \end{aligned} \tag{30}$$

and $\mathbf{P} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. A sequence of processes $\{\mathbf{q}^k\}$, can be built using

$$\begin{aligned} \lambda_x^k &= 200 \cdot k, & \lambda_y^k &= 0, \\ \mu_x^k &= 2q_x^k \cdot k, & \mu_y^k &= \begin{cases} \frac{k q_x^k q_y^k}{10}, & \text{if } q_x^k q_y^k < 10^6 \\ k \cdot 10^5, \text{ otherwise,} \end{cases} \end{aligned} \tag{31}$$

and $\mathbf{P}^k = \mathbf{P}$. On the basis of Theorem 1 and subsequent comments, the fluid limit for this sequence of stochastic processes is the solution of (28), where

$$\begin{cases} h_x(\mathbf{q}) = 200 - 2q_x \\ h_y(\mathbf{q}) = 2q_x - \dfrac{q_x q_y}{10} \end{cases} \qquad \text{if } q_x q_y < 10^6 \tag{32}$$

$$\begin{cases} h_x(\mathbf{q}) = 200 - 2q_x \\ h_y(\mathbf{q}) = 2q_x - 10^5 \end{cases} \qquad \text{otherwise.} \tag{33}$$

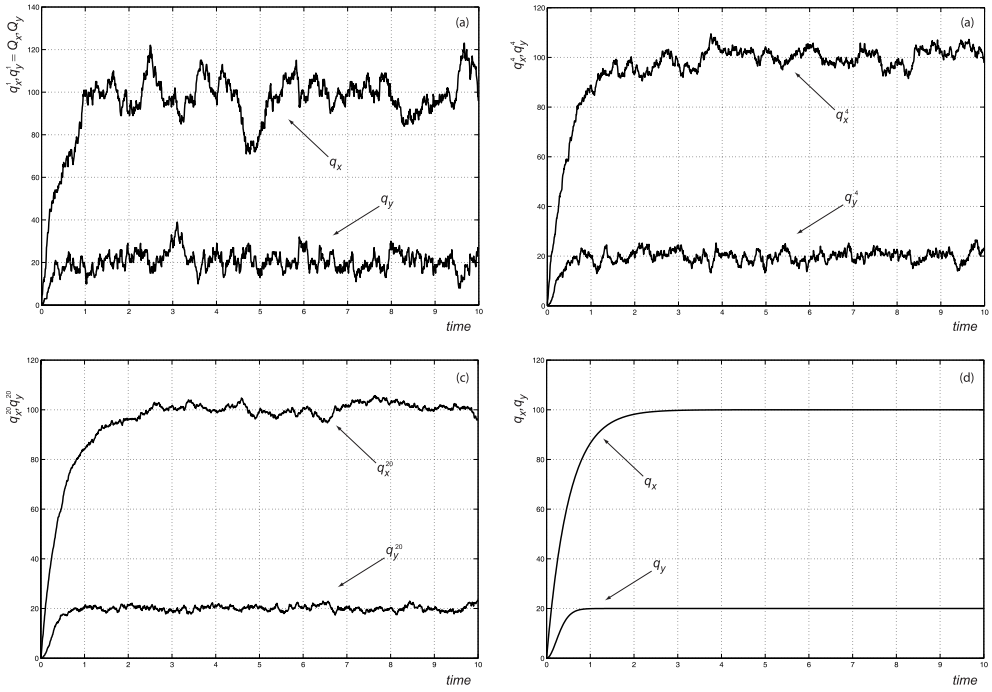


Fig. 3. From top left to bottom right: three realization of the stochastic process $\mathbf{q}^k$ respectively for $k = 1, 4$ and 20 and the ideal trajectory of the fluid limit ($k \to \infty$).

It is easy to verify that for $(q_x, q_y) \in \mathbb{R}^2_+$, if $q_x = 0$ then $h_x \geq 0$, and if $q_y = 0$, then $h_y \geq 0$. Moreover, the trajectories of the system $\frac{d\mathbf{q}}{dt} = \mathbf{h}(\mathbf{q})$ have always nonnegative components, when initial conditions are in $\mathbb{R}^2_+$. Therefore, we have $\tilde{\mathbf{m}} = 0$. Moreover, by a simple inspection of (32)-(33), it is easy to check that the dynamical system has a fixed point at $(q_x, q_y) = (100, 20)$, which is a globally stable attractor. It is an interesting issue to investigate with numerical simulations how, for a given generic $k > 0$, the stochastic dynamics of the process $\mathbf{q}^k$ is related to the deterministic behavior of dynamical system $\frac{d\mathbf{q}}{dt} = \mathbf{h}(\mathbf{q})$ (note that $\mathbf{q}^1 = \mathbf{Q}$). As it appears clear in Fig. 3, when k is increased, the stochastic fluctuations around what appears to be the main trajectory of the dynamics are progressively destroyed.

4 Fluid Models as Deterministic Approximations

The fluid model of a state-dependent Markovian Q-network, described in Section 3.1, is a deterministic dynamical system whose functional form is related to that one of both the network topology and its parameters. In a generic state dependent Q-network, *since no strong constraints are imposed on the resulting functional form of the deterministic fluid model*, the resulting dynamical system (28) can display a wide range of dynamics, including steady states, periodic behavior and chaos. The similarity between the dynamics plotted in Fig. 4, suggests two fundamental questions: what general information can be

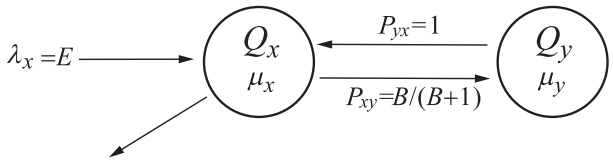


Fig. 4. Two-station Q-network with parametric primitives (34)-(35).

provided about the stochastic dynamics of a Q-network by analysis of its fluid model? In this paper we focus this question on Q-networks whose fluid models have global periodic attractors, and toward this end we introduce the special family of *amplitude-scale Q-networks* $\mathbf{Q}^E = \{\mathbf{Q}^E(t), t \geq 0\}$ for which the arrival rates, the service rates and the transition probability matrix have the form $\boldsymbol{\lambda}(\mathbf{v}) = E\boldsymbol{\lambda}_0\left(\frac{\mathbf{v}}{E}\right)$, $\boldsymbol{\mu}(\mathbf{v}) = E\boldsymbol{\mu}_0\left(\frac{\mathbf{v}}{E}\right)$, $\mathbf{P}(\mathbf{v}) = \mathbf{P}_0\left(\frac{\mathbf{v}}{E}\right)$, where $\mathbf{v} \in \mathbb{R}^n_+$, E is a positive number and $\boldsymbol{\lambda}_0, \boldsymbol{\mu}_0 : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ and $\mathbf{P}_0 : \mathbb{R}^n_+ \to \mathbb{R}^{n\times n}_+$ are Lipschitz continuous functions. In such a Q-network, by changing parameter E we change *the way the network reacts to its workload*. This aspect is quite interesting, since these systems allow for analyzing the relationships between the stochastic dynamics and the fluid-deterministic dynamics, framing this issue into a question of scalability and dependence on the parameter E. Moreover, as it is made clearer in the following, when increasing the scaling factor in an amplitude-scale network the overall dynamics appears to become progressively insensitive to the stochastic variations of customer flows with respect to their expected rates. This result in some aspects is equivalent to the progressive noise suppression introduced in the sequence of stochastic process $\{\mathbf{q}^k\}$.

4.1 An example with no reflection on the boundaries

We introduce a simple example, which does not have a definite relevance for some real existing Q-network, but that turns out to be particularly instructive thanks to its behavior. Referring to Fig. 4, let us consider the family of amplitude-scale state-dependent two-station $M_\xi/M_\xi/1 - FIFO$ Q-networks with the following parameters:

$$\lambda_x = E, \qquad \lambda_y = 0, \qquad \mu_x = Q_x^E(B+1), \tag{34}$$

$$\mu_y \begin{cases} \frac{{Q_x^E}^2 Q_y^E}{E^2}, & \text{if } \frac{{Q_x^E}^2 Q_y^E}{E^2} < 100E, \\ 100E, & \text{otherwise,} \end{cases} \qquad \mathbf{P} = \begin{pmatrix} 0 & \frac{B}{B+1} \\ 1 & 0 \end{pmatrix}, \tag{35}$$

where B is a positive real parameter and E is the scaling factor of the sequence. For each E, the stochastic process $\mathbf{Q}^E$, describing above network with parameters (34)-(35), satisfies all assumptions discussed in Section 3.1, so we can use results provided by Theorem 1 and refer to the system (28) as the

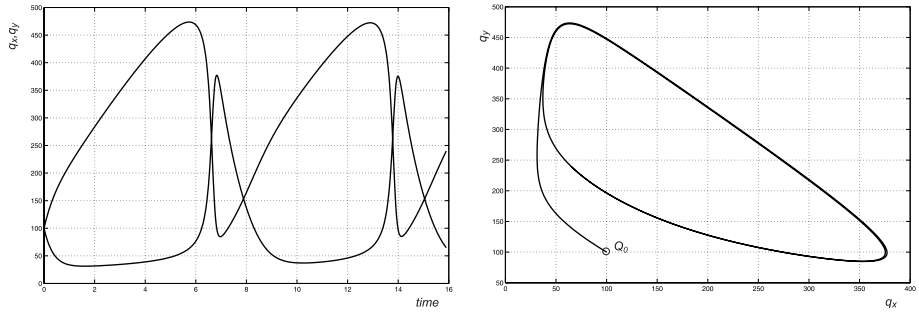


Fig. 5. Periodic attractor for the fluid model (36)-(37): $E = 100, B = 3$.

fluid model of $\mathbf{Q}^E$. The fluid model has the form

$$\frac{d\mathbf{q}}{dt} = \mathbf{h}^E(\mathbf{q}) + [\mathbf{I} - \mathbf{P}']\tilde{\mathbf{m}}^E(\mathbf{q}) \equiv \mathbf{H}^E(\mathbf{q}),$$

where the function $\mathbf{h}^E = (h_x^E, h_y^E)$ is given by

$$\begin{cases} h_x^E(\mathbf{q}) = & E + \dfrac{q_x^2 q_y}{E^2} - q_x(B+1) \\ h_y^E(\mathbf{q}) = & Bq_x - \dfrac{q_x^2 q_y}{E^2} \end{cases} \qquad \text{if } \frac{q_x^2 q_y}{E^2} < 100E \tag{36}$$

and

$$\begin{cases} h_x^E(\mathbf{q}) = & 101E - q_x(B+1) \\ h_y^E(\mathbf{q}) = & Bq_x - 100E \end{cases} \qquad \text{otherwise.} \tag{37}$$

From the expression (36) of $\mathbf{h}^E$ it is immediate to verify that for all $E > 0$ when $(q_x, q_y) \in \mathbb{R}^2_+$ if $q_x = 0$ then $h_x^E > 0$ and if $q_y = 0$ then $h_y^E > 0$, with the result that the function $\tilde{\mathbf{m}}$ is always zero. This is a consequence of the fact that the autonomous dynamical system $\frac{d\mathbf{q}}{dt} = \mathbf{h}^E(\mathbf{q})$ has solutions with always-non negative components when the initial condition belongs to $\mathbb{R}^2_+$. If the ODE systems (36) and (37) are analyzed separately as independent systems, it turns out that the former system agrees with a scaled version of the Brusselator dynamical model, well known in chemistry, while the latter system has solution eventually attracted toward $(\frac{101E}{B+1}, -\infty)$ if $B < 100$ or toward $(\frac{101E}{B+1}, +\infty)$ if $B > 100$. If $B = 100$ this system eventually stabilizes on a fixed point that depends on the initial condition. The independent Brusselator system (36) has an unique fixed point at $(q_x, q_y) = (E, BE)$ which is stable for $0 < B \leq 2$ and unstable for $B > 2$. At $B = 2$ the fixed point undergoes a Hopf bifurcation, and any solution of the system is eventually periodic for $B > 2$. A deeper analysis shows that for $2 < B < 100$ the dynamical system defined by the two joint expressions (36)-(37) has the same periodic limit cycle of the independent Brusselator system (36) for all initial conditions belonging to $\mathbb{R}^2_+$ (Fig. 5). Several Q-networks with primitives (34)-(35) have been simulated setting $B = 3$ and varying E in the range $15 \div 500$ (Fig. 6). As it can be seen by inspection of these plots, even though in all cases the fluid model trajectories

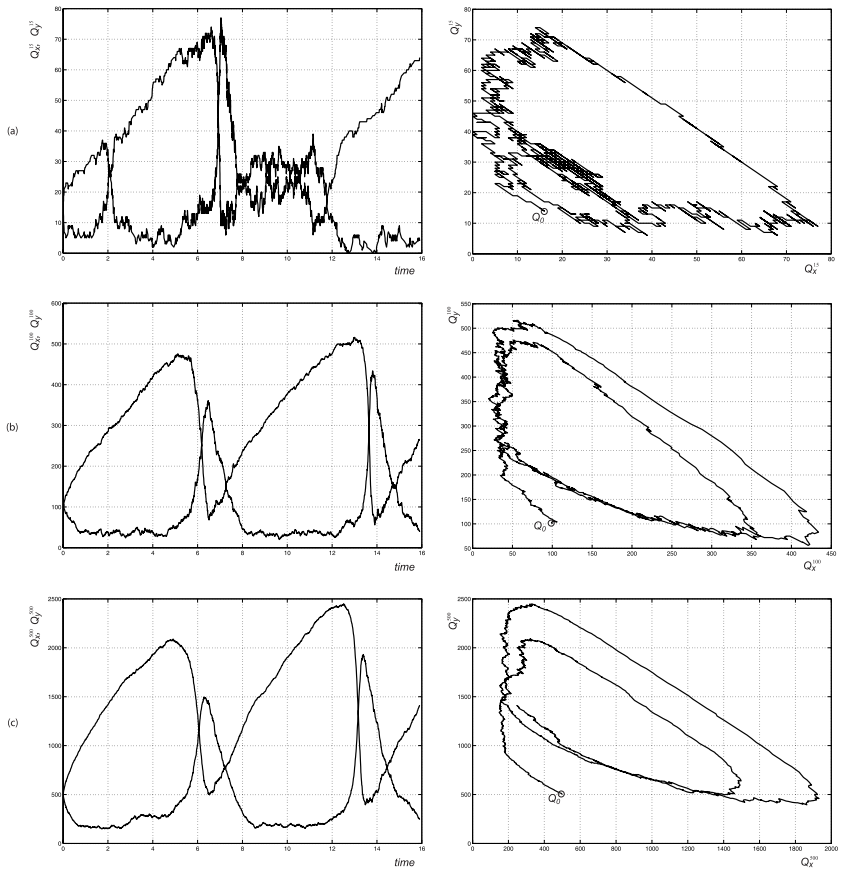


Fig. 6. Three simulations of the $M/M/1-FIFO$ Q-network example based on parameters (34)-(35) with $B = 3$ and $E = 15$ (a), $E = 100$ (b), $E = 500$ (c). The initial conditions were set to (E, E).

are similar to that one depicted in Fig. 5 (just scaled in amplitude, according to parameter E), the similarity between the deterministic dynamics and the stochastic dynamics increases as the scaling factor E is increased. If the fluid model has a global attractor, as in the case of the example described in this section, this must affect the dynamics of the Q-network, as well, identifying a region of the state space for which the probability of finding the network's state in the distant future is greater than somewhere else. This fact can be easily verified by computer simulations (Fig. 7).

Finally, we discuss the autocorrelation function for the Q-network, which in general, if it exists, is defined as $r_{ii}^E(\tau) \equiv \lim_{t\to\infty} \frac{1}{t}\int_0^t Q_i^E(u+\tau)Q_i^E(u)du$, where i is either x or y. For a Q-network in an oscillatory regime, one might expect the autocorrelation function behaves as $r_{ii}^E(\tau) = a_0 + a_1 e^{-\gamma\tau}\cos(\omega\tau + \phi) + \ldots$, where $a_0, a_1, \ldots$, ω and γ depend on E, $\omega = 2\pi/T$ is the period and $1/\gamma$ is so-called correlation time of the limit cycle. Numerically we find that the correlation time of a limit cycle, when the Q-network has an overall oscillatory dynamical behavior, diverges with the parameter E as E goes to infinity. Figure 8 shows the autocorrelation function for Q_y^E-component of

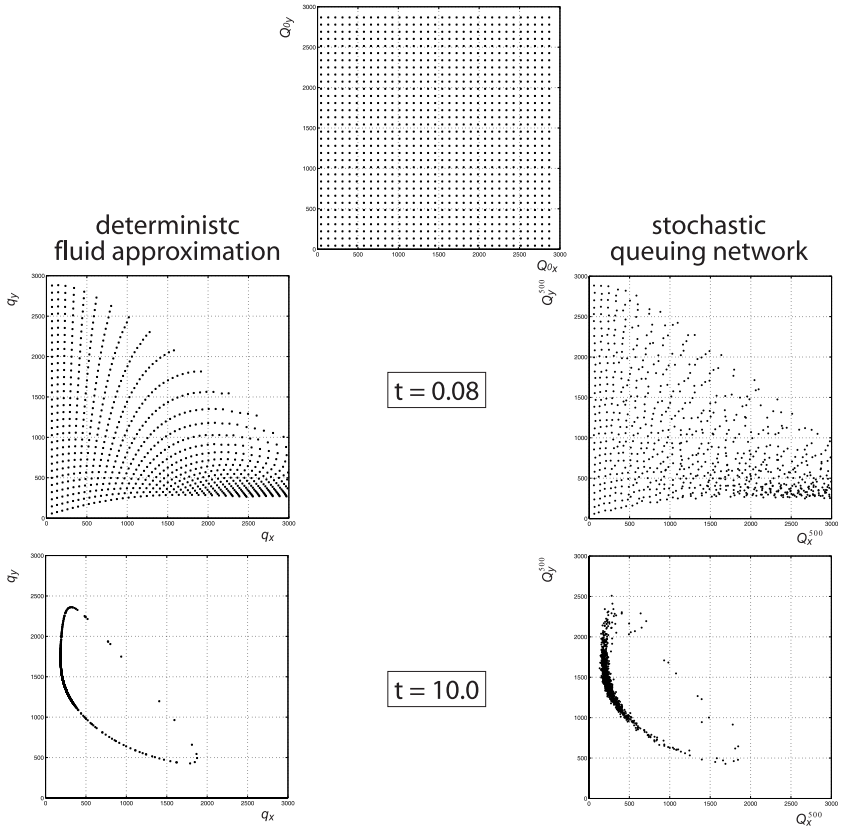


Fig. 7. Effects of the attractor on 1000 realizations originated from 1000 uniformly distributed initial conditions, for both the Brusselator stochastic Q-network (34)-(35) (*right side*) and its related fluid model (*left side*) ($B = 3, E = 500$).

the Brusselator Q-network, for several different values of E. We observe that the autocorrelation function is strongly damped for a small values of E such as $E = 30$. From this analysis one can also provide an estimation for the minimum value of E required for the network to act as an oscillator and, hence, to produce correlated oscillations.

4.2 An example with reflection on the boundaries

This example, introduced in [5], differs from the previous case due to a non zero function $\tilde{\mathbf{m}}$ in the associated fluid model (28). Referring again to Fig. 2, let us consider the sequence of two-station $M_\xi/M_\xi/1-FIFO$ tandem Q-network with parameters

$$\lambda_x(\mathbf{Q}^E) = \begin{cases} 6E, & \text{if } 0 \le Q_x^E, Q_y^E \le E \\ 6E + 20(Q_x^E - E), & \text{if } Q_x^E > E \\ 6E + \frac{5}{E}(Q_x^E - E)(Q_y^E - E), & \text{otherwise,} \end{cases} \tag{38}$$

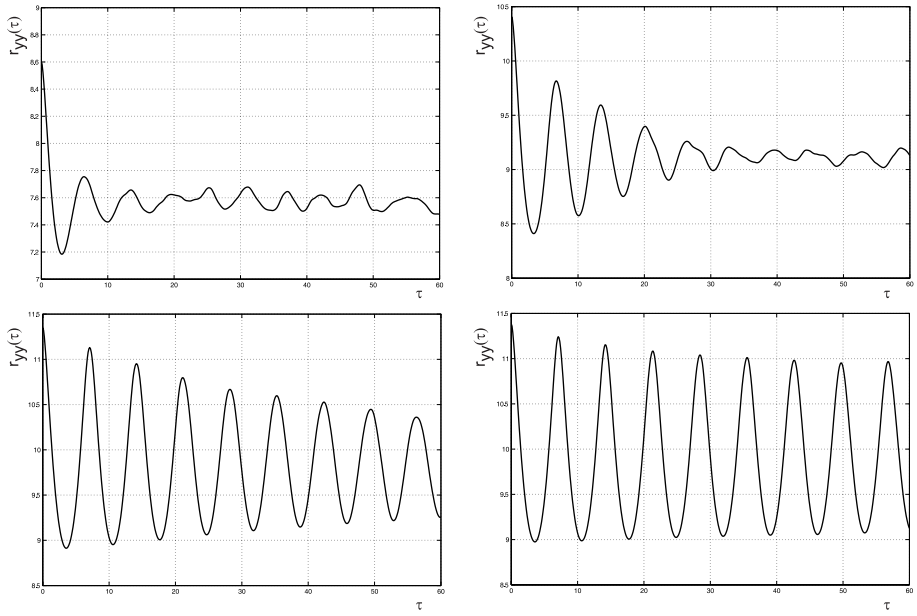


Fig. 8. Autocorrelation function for the Q_y^E-component of a realization of the Brusselator Q-network (34)-(35) for $B = 3$ and different values of E (from top left to bottom right): $E = 30, E = 100, E = 500, E = 1000$.

$$\mu_x(\mathbf{Q}^E) = \begin{cases} 3E, & \text{if } 0 \le Q_x^E, Q_y^E \le E \\ 3E + 20(Q_x^E - E)^+ + 4(Q_y^E - E), & \text{if } Q_y^E > E \\ 3E + 20(Q_x^E - E), & \text{otherwise,} \end{cases} \tag{39}$$

$$\lambda_y = 0, \qquad \mu_y = 5E, \qquad \mathbf{P} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \tag{40}$$

where E is the scaling factor and, for any real number a, we have $a^+ = \max\{0, a\}$. For each E the stochastic process $\mathbf{Q}^E$, describing the Q-network with parameters (38)-(40), satisfies all assumptions discussed in Section 3.1, so we can use results provided by Theorem 1 and refer to the system (28) as the fluid model of $\mathbf{Q}^E$. The fluid model has the form

$$\frac{d\mathbf{q}}{dt} = \mathbf{h}^E(\mathbf{q}) + [\mathbf{I} - \mathbf{P}']\tilde{\mathbf{m}}^E(\mathbf{q}) \equiv \mathbf{H}^E(\mathbf{q}),$$

where the function $\mathbf{h}^E = (h_x^E, h_y^E)$ is given by

$$h_x^E(\mathbf{q}) = \begin{cases} 3E, & \text{if } 0 \le q_y \le E \\ 3E - 4(q_y - E), & \text{if } q_x, q_y > E \\ 3E + \frac{5}{E}(q_x - E)(q_y - E) - 4(q_y - E), & \text{otherwise,} \end{cases} \tag{41}$$

$$h_y^E(\mathbf{q}) = \begin{cases} -2E, & \text{if } 0 \le q_x, q_y \le E \\ -2E + 20(q_x - E)^+ + 4(q_y - E), & \text{if } q_y > E \\ -2E + 20(q_x - E), & \text{otherwise,} \end{cases} \tag{42}$$

The analysis of (41)-(42) reveals that there are points on the boundary ($q_x = 0$) in which h_x^E is negative for all $E > 0$, and that there are some points on the boundary ($q_y = 0$) in which h_y^E is negative for all $E > 0$. Indeed we have

$$h_x^E(0, q_y) = \begin{cases} 3E, & \text{if } 0 \le q_y \le E \\ 12E - 9q_y, & \text{otherwise,} \end{cases} \tag{43}$$

$$h_y^E(q_x, 0) = \begin{cases} -2E, & \text{if } 0 \le q_x \le E \\ 20q_x - 22E, & \text{otherwise,} \end{cases} \tag{44}$$

from which it results $h_x^E(0, q_y) < 0$ if $q_y > 4E/3$ and $h_y^E(q_x, 0) < 0$ if $0 \le q_x < 11E/10$. Accordingly, solving system (29), function $\tilde{\mathbf{m}}^E$ has components

$$\tilde{m}_x^E(\mathbf{q}) = \begin{cases} -h_x(\mathbf{q}) = 9q_y - 12E, & \text{if } q_x = 0, q_y > \frac{4E}{3} \\ 0, & \text{otherwise,} \end{cases} \tag{45}$$

$$\tilde{m}_y^E(\mathbf{q}) = \begin{cases} -h_y(\mathbf{q}) = 22E - 20q_x, & \text{if } 0 \le q_x \le \frac{11E}{10}, q_y = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{46}$$

The dynamical system (28) with $\mathbf{h}^E$ and $\tilde{\mathbf{m}}^E$ defined as in (41)-(42) and (45)-(46) has the global periodic attractor plotted in solid line in Fig. 9.b for $E = 500$. For this system any trajectory hits the boundary $q_x = 0$ in $q_y > 4E/3$ entering the periodic cycle after a transient evolution flattened on the q_y axis. In particular, the limit cycle leaves the boundary $q_x = 0$ as soon as h_x^E becomes positive, i.e., for $q_y < 4E/3$. We remark that for this fluid model any trajectory enters the limit cycle in a finite time, in contrast to the previous example.

5 Periodic Attractors in Q-networks: Long-Run Behavior

In this section we analyze the Q-network dynamics from a probabilistic point of view. Let write the fluid model (28) as $\frac{d\mathbf{q}}{dt} = \mathbf{H}(\mathbf{q})$, and let assume that for each initial condition it has one unique solution, eventually attracted toward a globally attractive limit cycle with support $\varphi \subset \mathbb{R}^n$. Accordingly, the period T of such limit cycle is related to the velocity field $v(\mathbf{q}) = \|\mathbf{H}(\mathbf{q})\|$ by $T = \int_\varphi \frac{dl}{v(\mathbf{q})}$. This latter integral always exists, since our hypothesis implies $v(\mathbf{q}) > 0$

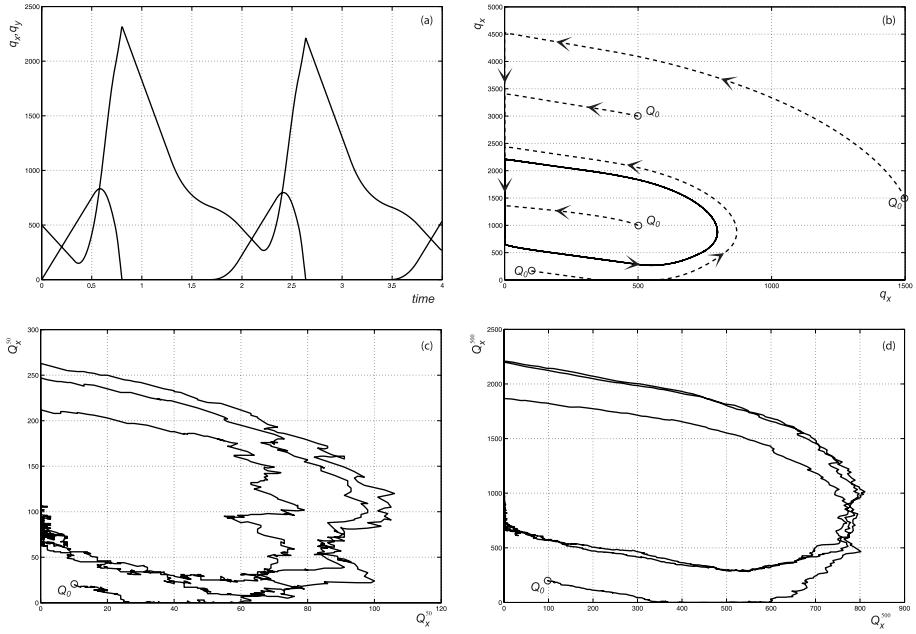


Fig. 9. *Subplot (a)*: dynamical evolution of (q_x, q_y) for the fluid trajectory over the limit cycle ($E = 500$); *subplot (b)*: the global attractiveness of the limit cycle is here shown for several initial conditions ($E = 500$). *Subplot (c)* and *(d)*: stochastic dynamics of two Q-networks with primitives (38)-(40) for $E = 50$ and $E = 500$.

everywhere. By denoting with $\mathcal{B}(\mathbb{R}^n)$ the Borel σ-algebra of subsets of $\mathbb{R}^n$, we define the function $\Pi : \mathcal{B}(\mathbb{R}^n) \to [0, 1]$ as

$$\Pi(A) = \int_{A \cap \varphi} \frac{dl}{T \cdot v(\mathbf{q})} \tag{47}$$

and we state that Π is a probability measure over $\mathbb{R}^n$. Indeed, it results $\Pi(\varnothing) = 0$ and for a given sequence $\{A_n\}$ of disjoint subsets of $\mathbb{R}^n$ belonging to $\mathcal{B}(\mathbb{R}^n)$ we have $\Pi(\cup_n A_n) = \sum_n \Pi(A_n)$. Moreover, from the definition of T, $\Pi(\mathbb{R}^n) = \frac{1}{T} \int_\varphi \frac{dl}{v(\mathbf{q})} = 1$. The following fundamental property holds

Theorem 2 *The probability measure* Π *defined in (47) is the unique invariant probability measure over* $\mathbb{R}^n$ *with respect to the dynamical system* $\frac{d\mathbf{q}}{dt} = \mathbf{H}(\mathbf{q})$.

The proof of Theorem 2 is nontrivial, but it is here omitted since it can be traced back into the well known and more general ergodic theory of differentiable dynamical systems developed by Sinai, Ruelle and Bowen [19–21].

Once assuming the state of a dynamical system as a random variable, invariant measures describe stationary state distributions. This aspect is well known in ergodic theory applied to, e.g., chaotic systems [22]. In our case, we are considering eventually periodic systems for which the stationary distribution related to the probability measure (47) is impulsive over the zero-measure support of the limit cycle: since φ is attractive, for this kind of systems any

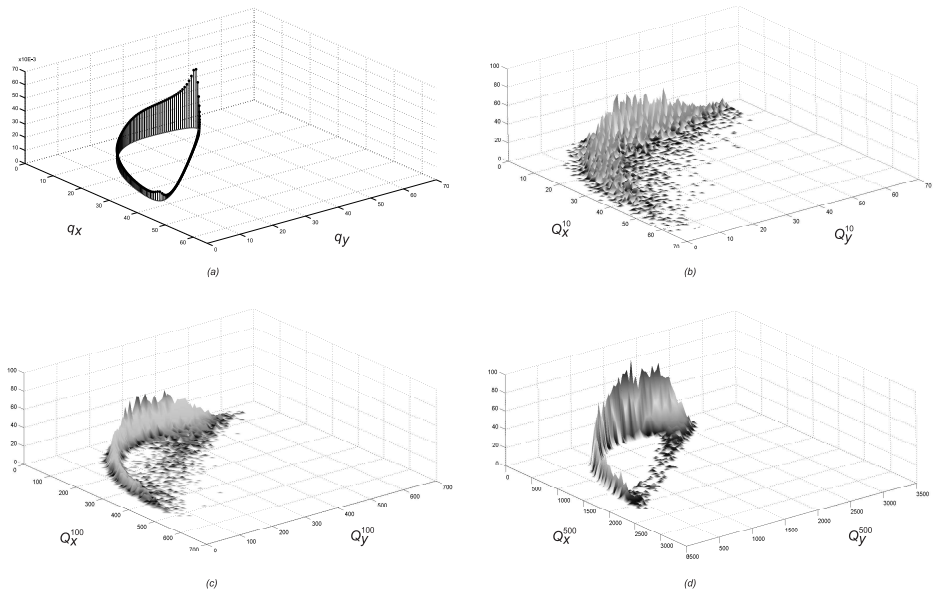


Fig. 10. *Subplot (a)* - The unique invariant impulsive distribution over the limit cycle support φ computed for the deterministic fluid model of the Brusselator Q-network (34)-(35) with $B = 3, E = 10$. *Subplot (b), (c) and (d)* - Distributions of 10000 realizations evaluated at $t = 10^4$s of the stochastic process $\mathbf{Q}^E(\cdot)$ for $E = 10, 100$ and 500 ($B = 3$). All the realizations start from the same initial condition $\mathbf{Q}_0^E = (E, E)$.

stationary distribution must be equal to 0 for points not belonging to φ. From (47), the unique stationary distribution is described over φ by a curvilinear probability density function (pdf) $\psi : \varphi \to \mathbb{R}_+$ defined as $\psi(\mathbf{q}) = \frac{1}{Tv(\mathbf{q})}$. The probabilistic interpretation of the curvilinear pdf ψ is the following: when assuming the fluid system state as a random variable distributed according to ψ, the probability for the state $\mathbf{q}$ to belong to a given measurable portion (subset) of φ remains unchanged under the dynamics. It is worth noting the simple property that is captured by ψ: the slower a portion of the limit cycle is covered, the higher is the probability to find the system's state inside it.

We now provide numerical evidence that when the fluid model of a state-dependent n-node $M_\xi/M_\xi/1 - FIFO$ open Q-network $\mathbf{Q}$ has a globally stable periodic attractor, then a stationary invariant distribution most likely exists. These results are in agreement with other previous works on the general analysis of stochastic dynamical systems (see, e.g., [23]). The numerical results reported in this section shows that, regardless of the initial condition, the state distribution of a periodic Q-network approaches in the distant future a stationary invariant distribution which is strongly related to the unique impulsive stationary distribution of the related fluid model. Referring to the two previous amplitude-scale network examples, this issue appears clear in Fig. 10 and 11. The numerical analysis of the stochastic evolution showed that, regardless of the initial distribution choice, for the considered periodic Q-networks crater-shaped stationary distributions are approached, being the ridge of the

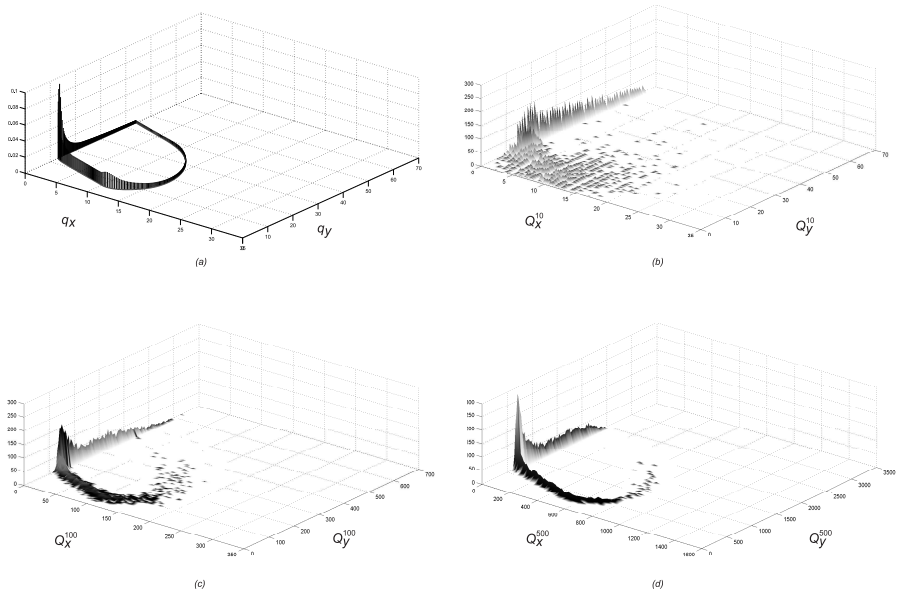


Fig. 11. *Subplot (a)* - The unique invariant impulsive distribution over the limit cycle support φ computed for the deterministic fluid model of the Q-network example (38)-(40) with $E = 10$. *Subplot (b), (c) and (d)* - Distributions of 10000 realizations evaluated at $t = 10^4$s of the stochastic process $\mathbf{Q}^E(\cdot)$ for $E = 10, 100$ and 500. All the realizations start from the same initial condition $\mathbf{Q}_0^E = (E, E)$.

craters positioned over the supports φ of the fluid limit cycles.

As previously discussed, in an amplitude-scale Q-network the increase of the scaling factor E involves a relative suppression of the granular noise perturbations. This fact influences the state distribution dynamics and, as confirmed by simulations, the greater is E and the more concentrated around φ is the approached stationary distribution, with a ridge outline proportional to the curvilinear pdf ψ. This can be explained observing that the Q-network evolves with a strong influence of the underlying fluid deterministic velocity field. Since in our case the fluid system has a globally attractive limit cycle, over the periodic trajectory the deterministic state variable which corresponds to the zero Lyapunov exponent, the phase variable, is neutrally stable and hence, when E is sufficiently large, the dominant effect of the stochastic perturbations is a phase diffusion, and around φ a distribution proportional to the natural invariant curvilinear pdf ψ is consequently approached. It is interesting noting that the simple analysis of the deterministic fluid model can provide useful information for *predicting* the stochastic evolution of a Q-network, especially when its fluid model has a globally attractive limit cycle. The last two examples show that the overall stochastic dynamics can result to be strongly influenced by the deterministic velocity field even if the queues become empty (i.e., even if some nodes in the Q-network has a very small workload). This is in contrast to the intuitive idea that fluid approximations can be sensibly taken into account only for Q-networks with heavily overloaded servers.

6 Conclusions

In this paper we have studied periodic dynamics in state-dependent open Markovian Q-networks. We have adopted the theory developed in [5] for the transient evolution of Q-networks to study networks for which the fluid limit has a globally stable limit cycle (periodic attractor). Two examples of simple two-node amplitude-scale queueing networks, whose fluid approximations have globally stable limit cycles, have shown that the simple analysis of the deterministic fluid model can provide useful information for predicting the stochastic evolution of a Q-network. In particular, the stochastic dynamics has been analyzed by taking into account the overall dynamical characterization of the fluid deterministic model, even considering the reflection on the boundaries. For periodic Q-networks, the correlation between different stochastic realizations has been discussed and numerical results have been reported for the analysis of the network long-run stochastic dynamical behavior.

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