Invariant Measures of Tunable Chaotic Sources: Robustness Analysis and Efficient Estimation

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⋆ RESEARCH MANUSCRIPT ⋆

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Abstract

In this paper a theoretical approach for studying the robustness of the chaotic statistics of piecewise affine maps with respect to parameter perturbations is discussed. The approach is oriented toward the study of the effects that the non-idealities derived from the circuit implementation of these chaotic systems have on their dynamics. The ergodic behavior of these systems is discussed in detail adopting the approach developed by Boyarsky and Góra, with particular reference to the family of Sawtooth maps, and the robustness of their invariant measures is studied. Although the paper is particularly focused on this specific family of maps, the proposed approach can be generalized to other piecewise affine maps considered in the literature for ICT applications. Moreover, in the paper an efficient method for estimating the unique invariant density for stochastically stable piecewise affine maps is proposed. The method is an alternative to Monte-Carlo methods and to other methods based on the discretization of the Frobenius-Perron operator.

1 Introduction

In the past decades both Ergodic Theory and Information Theory have gained increasing attention in the Engineering community, thanks to the high potential use of their theoretical framework in Information and Communications Technology (ICT) applications. Within the explored research topics, discrete time chaotic dynamical systems have been widely investigated for the generation of sequences with predetermined statistical properties [1–4]. Such kind of dynamical systems can be suitably used, e.g., for designing truly random number generators [4, 5], for issuing spreading sequences in DS-CDMA telecommunication systems [6], or for enhancing the electromagnetic compatibility of specific

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electronic devices [7]. In these applications, chaotic systems are used for the design of symbolic stochastic sources with symbols chosen from a properly defined finite alphabet, and the knowledge of the system and of its invariant measure allows for determining all the statistical features of the symbolic source [2, 3]. Several types of discrete time chaotic dynamical systems have been proposed for the implementation of stochastic sources with *tunable statistical characteristics*: in such cases, depending on the application, by changing the system parameters one or more statistical features of the source can be modulated at will (e.g., the autocorrelation, or any higher order expectations) [2–4, 8].

Among the chaotic systems taken into account for these ICT applications, *piecewise affine maps* (also called piecewise linear maps) are good candidates for being implemented by electronic circuits, since their analytical expressions typically involve simple algebraic operations [4, 9–11]. Nevertheless, the statistical characteristics of these dynamical systems turn out to be extremely sensible to parameter changes, and this aspect represents a critical issue when the tolerances of any implementation process are taken into account. This fact has been deeply investigated, e.g., in [4].

The aim of this paper is twofold:

1) to present a theoretical approach for studying the robustness of the chaotic statistics of piecewise affine maps with respect to parameter perturbations;

2) to propose an efficient method for estimating the invariant measure for stochastically stable piecewise affine maps.

The first topic has been studied by several authors adopting different points of view (see, e.g., [12–14] and references cited therein). Our approach is oriented toward the theoretical analysis of the effects that the non-idealities derived from the circuit implementation of these chaotic systems have on their dynamics. Referring to the theory developed by Boyarsky and Góra [13], we discuss in detail the ergodic behavior of piecewise affine expanding transformations, with particular reference to the family of the Sawtooth maps considered in [4] for truly random number generation.

Ergodic Theory classifies dynamical systems depending on the *complexity degree* of their dynamics.[2] Referring to the approach of Boyarsky and Góra we identify a region within the parameter space in which the Sawtooth map preserves its exactness property. Moreover, a particular effort was devoted to frame the family of Sawtooth maps within the Ergodic Theory developed for expanding transformations and to prove, for the considered special case, that all the conditions necessary for adopting and using the known theoretical results about the stability of invariant measures are satisfied. Although this paper is focused on the Sawtooth map special case, we stress that the proposed approach can be generalized for studying the effects that the implementation non-idealities have on the dynamics of other piecewise affine maps proposed in the literature for ICT applications.

As mentioned before, in this paper we also propose and discuss an efficient method for estimating the invariant measure of stochastically stable piecewise affine maps. The proposed approach is simple, and it is an alternative to the Monte-Carlo methods and to other methods based on the discretization of the Frobenius-Perron operator [2, 15–21]. The estimation accuracy of this method is

[2] E.g., with an increasing level of complexity Ergodic Theory defines ergodic, weakly mixing, strong mixing and exact transformations.

discussed in detail and, as an example, the approximation of the invariant measure for the Sawtooth map, considering different non-trivial parameter values, is calculated.

The presentation of original theorems, propositions and results is enclosed with proofs and detailed discussions.

Paper organization: the paper is organized as follows. In Section 2 we define both the terminology and the notation that are used in the rest of the paper. Section 3 frames the study of piecewise expanding maps within the general Ergodic Theory, following the approach of Boyarsky and Góra [13]. In Section 4 we present the family of Sawtooth maps and we discuss the robustness of its exactness property with respect to parameter variations. In Sections 5 and 6 we discuss and apply the general theoretical tools necessary for studying the robustness of invariant measures for the family of Sawtooth maps. In Sections 7 and 8 we face the problem of computing invariant measures for piecewise affine maps with non-trivial parameter values, and we propose an efficient estimation method alternative to other well known approaches. The paper is closed with Conclusions and References.

2 Basic Setup and Notation

In this paper we adopt the following notation and terminology. Let I be an interval. We denote with $\mathcal{B}_I$ the usual Borel σ-algebra of subsets of I, and with $\lambda : \mathcal{B}_I \to \mathbb{R}^+$ the Lebesgue measure. The triplet $(I, \mathcal{B}_I, \lambda)$ is the Lebesgue space measure over I. We say that a property holds almost everywhere (a.e.) on I if the subset $F \subset I$ on which the property fails has zero measure, i.e., $\exists H \in \mathcal{B}_I$ such that $F \subseteq H$ and $\lambda(H) = 0$. With reference to the Lebesgue integration theory, the notation $L^p(I)$ denotes the set of functions $f : I \to \mathbb{R}$ such that $\int_I |f(x)|^p dx < \infty$, with $0 < p \in \mathbb{N}$, whereas $L^\infty(I)$ is the set of a.e. bounded measurable functions. We recall that $L^p(I)$ and $L^\infty(I)$ can be made Banach spaces with reference to the norms $\|f\|_p = (\int_I |f(x)|^p dx)^{\frac{1}{p}}$ and $\|f\|_\infty = \inf\{M \in \mathbb{R}^+ \text{ such that } \{x \in I : |f(x)| > M\} \text{ has zero measure}\}$, respectively. We furthermore recall that if H is a linear operator from the Banach space $(X, \|\cdot\|)$ into itself, then

$$\|H\| = \sup \left\{ \frac{\|Hf\|}{\|f\|}, f \in X, \|f\| \neq 0 \right\}. \tag{1}$$

We define $\mathcal{P}_I$ as the set of all finite partitions of intervals[3] of I, and if $Q \in \mathcal{P}_I$, we denote the endpoints of Q with $\mathcal{E}_Q = \{p_0, \ldots, p_q\}$, with $p_i < p_j$ if $i < j$.

Definition 1 *If there exists a positive number M such that $\forall Q \in \mathcal{P}_I$*

$$V_I(f, Q) = \sum_{k=1}^{\#Q} |f(p_k) - f(p_{k-1})| \leq M, \tag{2}$$

then f is said to be of bounded variation *on I and the quantity $V_I(f) = \sup_{\mathcal{P}_I}\{V_I(f, Q)\}$ is the* total variation *of f on I.*

[3] If $P \in \mathcal{P}_I$ then $P = \{I_i, i = 1, \ldots, p, \text{with } p > 0\}$, with $I_i \cap I_j = \varnothing$ for $i \neq j$ and $\cup_{i=1}^{p} I_i = I$.

We can now define the following subset of $L^1(I)$

$$BV(I) = \{f \in L^1(I) : \\ \exists g \in L^1(I) \text{ such that } V_I(g) < \infty \text{ and } g = f \text{ a.e.}\}. \quad (3)$$

Note that $BV(I)$ also contains functions with infinite variation that are a.e. equal to a bounded variation function. In such case we define the *infimum total variation* of $f \in BV(I)$ as $\tilde{V}_I(f) = \inf\{V_I(g) : g \in BV(I), g = f \text{ a.e.}\}$. The set $BV(I)$ is a vector space of functions, which can be made a Banach space with the norm $\|f\|_{BV} = \|f\|_1 + \tilde{V}_I(f)$, and it can be proved that that $BV(I)$ is dense in $L^1(I)$ [13]. Since in this paper we deal with probability density functions (pdfs), we introduce the subset $D_1(I) = \{f \in L^1(I) : \|f\|_1 = 1\}$ and the subset $D_{BV}(I) = BV(I) \cap D_1(I)$.

In this paper we consider one dimension dynamical systems $x_{n+1} = \tau(x_n)$, where $\tau : I \to I$ is a nonsingular transformation, i.e., if $A \in \mathcal{B}_I \Rightarrow \tau^{-1}(A) \in \mathcal{B}_I$, and if for any $A \in \mathcal{B}_I$ such that $\lambda(A) = 0$ we have that $\lambda(\tau^{-1}(A)) = 0$. In particular, by denoting with $\tau|_A$ the restriction of τ to the subset $A \subseteq I$, we introduce the special class of maps that follows [13].

Definition 2 *We denote with $\mathcal{T}_{exp}(I)$ the class of piecewise expanding transformations $\tau : I \to I$ that satisfy*

1. *there exists a partition $Q \in \mathcal{P}_I$, with $Q = \{I_1, \ldots, I_q\}$ and $\mathcal{E}_Q = \{p_0, \ldots, p_q\}$ the endpoints of Q, such that $\tau|_{I_i}$ is continuous and derivable on I_i, and $|\tau'(x)| \geq 1 + \varepsilon$, $\varepsilon > 0$ for any $1 \leq i \leq q$ and $p_{i-1} < x < p_i$.*
2. *the functions $g_1(x) = \frac{1}{|\tau'_-(x)|}$ and $g_2(x) = \frac{1}{|\tau'_+(x)|}$ are functions of bounded variation on I, where*

$$\tau'_-(x) = \lim_{y \to x^-} \tau'(y), \qquad \tau'_+(x) = \lim_{y \to x^+} \tau'(y).$$

In the following we consider partitions Q satisfying the point 1 of the previous definition such that it has the smallest necessary number of elements.

We conclude this section defining a non-empty set $A \subseteq I$ as *τ-invariant* if and only if $\tau(A) = A$. It is worth noting that $\tau(A) = A$ does not imply that $\tau^{-1}(A) = A$, whereas the converse is true. Moreover, $x \in A \Rightarrow \tau^n(x) \in A, \forall n \in \mathbb{N}$.

3 General Framework: Invariant Measures

Invariant measures play a key role in the analysis of the stochastic aspects of chaotic dynamics, since they provide necessary information for retrieving any order statistics of the chaotic process [2, 22].

Definition 3 *Let $(I, \mathcal{B}_I, \mu)$ be a measure space. The measure $\mu : \mathcal{B}_I \to \mathbb{R}^+$ is* invariant with respect to $\tau : I \to I$ *if and only if for any $A \in \mathcal{B}_I$ we have that $\mu(A) = \mu(\tau^{-1}(A))$. We say that τ preserves μ.*

REMARK: in the following we consider measures absolutely continuous (a.c.) with respect to the Lebesgue measure, i.e., defined as

$$\mu(A) = \int_A f(x)dx, \qquad A \in \mathcal{B}_I, \tag{4}$$

where $f \in L^1(I)$, with $f \geq 0$, is called the density of μ. We abbreviate the previous expression using the compact notation $\mu = f \cdot \lambda$, and if $f \in D_1(I)$ then μ is a probabilistic measure. Not all nonsingular transformations admit an a.c. invariant measure [23], whereas the existence of a.c. invariant measures with density of bounded variation is assured for maps belonging to $\mathcal{T}_{exp}(I)$ [13].

3.1 The Frobenius-Perron operator

Let $\tau : I \to I$ be a non-singular transformation. We define the Frobenius-Perron operator $\Theta_\tau : L^1(I) \to L^1(I)$ as follows: for any $f \in L^1(I)$, the function $\Theta_\tau f$ is the unique (up to λ-a.e. equivalence) element in $L^1(I)$ such that

$$\int_A \Theta_\tau f(x)dx = \int_{\tau^{-1}(A)} f(x)dx, \qquad A \in \mathcal{B}_\mathbb{R}. \tag{5}$$

The validity of the above definition derives from the Radon-Nikodym theorem [13]. The mechanism ruled by the Frobenius-Perron operator is intuitively summarized by the following remark: assuming the initial condition x_0 of the dynamical system as a random variable distributed over I according to $f_0 \in D_1(I)$, the pdf $f_1 = \Theta_\tau f_0$ describes the distribution of the system state $x_1 = \tau(x_0)$. Accordingly, the operator Θ_τ rules the evolution of the initial pdf f_0 induced by the deterministic transformation τ. The link between invariant measures and densities is hereafter provided: a pdf f^* is said *invariant* with respect to τ iff $\Theta_\tau f^* = f^*$ a.e., and it can be proved that in such case the measure $\mu^* = f^* \cdot \lambda$ is τ-invariant [13].

The Frobenius-Perron operator has some important properties, that will be used in the following. In particular, $\Theta_\tau : L^1(I) \to L^1(I)$ is a linear operator such that $\|\Theta_\tau f\|_1 = \|f\|_1$ and if $f \geq 0$ then $\Theta_\tau f \geq 0$. Moreover, if Θ_{τ^n} is the Frobenius-Perron operator corresponding to $\tau^n = \tau \circ \overset{n}{\cdots} \circ \tau$, then $\Theta_{\tau^n} = \Theta_\tau \circ \overset{n}{\cdots} \circ \Theta_\tau$. We are interested in considering the Frobenius-Perron operator within the set $BV(I)$, and to this aim we report the following theorem, proved in [13].

Theorem 1 *Let $\tau \in \mathcal{T}_{exp}(I)$. If $f \in BV(I)$ then $\Theta_\tau f \in BV(I)$ and there exists two constant δ_1, δ_2 such that $\forall f \in BV(I)$*

$$\tilde{V}_I(\Theta_\tau f) \leq \delta_1 \tilde{V}_I(f) + \frac{\delta_1}{\delta_2} \|f\|_1 , \tag{6}$$

where, referring to the notation introduced in Def. 2 and defining

$$\gamma = \max_{1 \leq i \leq q} \{V_{I_i}(g_1)\} = \max_{1 \leq i \leq q} \{V_{I_i}(g_2)\}\}$$

we have $\delta_1 = \frac{2}{\alpha} + \gamma$, $\delta_2 = \min_{1 \leq i \leq q} \lambda(I_i)$, with $1 < \alpha \leq \inf_{x \in I} |\tau'(x)|$.

The above theorem states the important property that if $\tau \in \mathcal{T}_{exp}(I)$ then $\Theta_\tau(BV(I)) \subseteq BV(I)$, and thanks to the fact that $\|\Theta_\tau f\|_1 = \|f\|_1$ we have that $\Theta_\tau(D_{BV}(I)) \subseteq D_{BV}(I)$. An useful corollary can now be proved.

Corollary 1 *Let $\tau \in \mathcal{T}_{exp}(I)$ and let us consider the Banach space $(BV(I), \|\cdot\|_{BV})$. It results $\|\Theta_\tau\|_{BV} < \infty$.*

Proof. We have $\|\Theta_\tau f\|_{BV} = \|\Theta_\tau f\|_1 + \tilde{V}_I(\Theta_\tau f) = \|f\|_1 + \tilde{V}_I(\Theta_\tau f)$, and from Theorem 1, $\|\Theta_\tau f\|_{BV} \leq (\frac{\delta_1}{\delta_2} + 1)\|f\|_1 + \delta_1 \tilde{V}_I(f)$. Accordingly, $\forall f \in BV(I)$ such that $\|f\|_{BV} \neq 0$ we have

$$\frac{\|\Theta_\tau f\|_{BV}}{\|f\|_{BV}} \leq \frac{(\frac{\delta_1}{\delta_2} + 1)\|f\|_1 + \delta_1 \tilde{V}_I(f)}{\|f\|_1 + \tilde{V}_I(f)} \leq \frac{\delta_1}{\delta_2} + \delta_1 + 1, \tag{7}$$

and $\|\Theta_\tau\|_{BV} \leq \frac{\delta_1}{\delta_2} + \delta_1 + 1 < \infty$. □

3.2 Support of measures

Let us consider a measure space $(I, \mathcal{B}_I, \mu)$, and let us assume on the interval I the topology induced by the euclidean distance. The *support* $\mathcal{S}(\mu)$ of the measure μ is the smallest closed set of full measure, i.e.,

$$\mathcal{S}(\mu) = X \setminus \bigcup_{\substack{\mu(A)=0 \\ A \text{ is open}}} A. \tag{8}$$

If there is a set $A \in \mathcal{B}_I$ such that $\mu(E) = \mu(A \cap E)$ for every $E \in \mathcal{B}_I$, then μ is said to be *concentrated* on A. This is equivalent to requiring that $\mu(E) = 0$ whenever $E \cap A = \varnothing$. Finally, let μ_1 and μ_2 be measures on I. If there exists a pair of disjoint sets A and B such that μ_1 is concentrated on A and μ_2 is concentrated on B, then μ_1 and μ_2 are said to be *mutually singular*, written $\mu_1 \perp \mu_2$.

We can now provide some useful results about measures $\mu = f \cdot \lambda$ with density $f \in BV(I)$. In particular, if $g \in BV(I)$ then there exists a lower semicontinuous function[4] $f \in BV(I)$ equal to g a.e. [13]. We call the set

$$\Omega_f = \{x : f(x) > 0\} \tag{9}$$

the *support of the density* f, which is an open set if f is lower semicontinuous. Accordingly, for any $g \in BV(I)$ there exists an open set A such that $\lambda(A \Delta \Omega_g) = 0$, with the set A equal to the support of a lower semicontinuous density f of bounded variation a.e. equal to g.[5]

Given an a.c. measure $\mu = f \cdot \lambda$, the link between the support $\mathcal{S}(\mu)$ and its equivalent lower semicontinuos density $f \in BV(I)$ is given by the following

Proposition 1 $\Omega_f \subseteq \mathcal{S}(\mu)$.

Proof. We remark that $\Omega_f \subseteq I$ is an open set with positive measure. Directly from the definition of $\mathcal{S}(\mu)$ we have that $\Omega_f \subseteq \mathcal{S}(\mu)$. □

From the definition of Ω_f it results that μ is concentrated on Ω_f.

[4] A function $f : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous iff $f(y) \leq \liminf_{x \to y} f(x)$ for any $y \in \mathbb{R}$.

[5] The symbol Δ denotes the symmetric difference between sets, that is $A \Delta B = (A \cup B) \setminus (A \cap B)$.

3.3 Ergodicity and Exactness

Ergodic maps have a relevant importance in ICT applications since they allow for generating ergodic stochastic processes [24]. We report in this subsection some fundamental definitions and results that will be used for characterizing the family of Sawtooth maps. Let us consider a probability space $(I, \mathcal{B}_I, \mu)$, and a measure-preserving transformation $\tau : I \to I$.

Definition 4 *The map τ is called* ergodic *with respect to μ if and only if $\forall A \in \mathcal{B}_I$ such that $A = \tau^{-1}(A)$ we have $\mu(A) = 0$ or $\mu(A) = 1$. In such case μ is called an ergodic measure.*

In general a transformation can admit more than one probabilistic ergodic measure. About this point we recall the following [13]

Theorem 2 *If μ_1 and μ_2 are two different a.c. probabilistic ergodic measures for a transformation τ, then $\mu_1 \perp \mu_2$.*

Exact maps are an important subset of ergodic transformations since they exhibit the strongest chaotic behavior.

Definition 5 *A map τ such that $A \in \mathcal{B}_I \Rightarrow \tau(A) \in \mathcal{B}_I$ is said* exact *iff for each $A \in \mathcal{B}_I, \mu(A) > 0$,*

$$\lim_{n\to\infty} \mu(T^n(A)) = 1. \tag{10}$$

Proving the exactness of a given map directly from the definition is not trivial, since (10) must hold for any $A \in \mathcal{B}_I, \mu(A) > 0$. The importance of exact maps derives from its *stochastic stability*, discussed in the following [13, 14]

Theorem 3 *Let $\tau \in \mathcal{T}_{exp}(I)$. If τ is exact then it has the strong mixing property, i.e., it admits one unique invariant a.c. measure $\mu^* = f^* \cdot \lambda$, with $f^* \in D_{BV}(I)$, such that for all $g \in D_{BV}(I)$*

$$\lim_{n\to\infty} \|\Theta_\tau^n g - f^*\|_{BV} = 0. \tag{11}$$

In particular, depending on τ, a positive constant K and a constant $0 < r < 1$ exist such that for any p with $1 \le p \le \infty$ and for any $n \in \mathbb{N}$

$$\|\Theta_\tau^n g - f^*\|_p < K r^n \|g\|_{BV}. \tag{12}$$

3.4 Quasi-Compactness of the Frobenius-Perron Operator

We conclude this introductory theoretical Section with its main theorem, proved in [13]. Any invariant density $f \in BV(I)$ for a given map $\tau \in \mathcal{T}_{exp}(I)$ can be viewed as an eigenfunction associated with the eigenvalue 1 of the Frobenius-Perron linear operator $\Theta_\tau : BV(I) \to BV(I)$. More in detail, the dimension of the eigenspace $\{f \in BV(I) : \Theta_\tau f = f \text{ a.e.}\}$ agrees with the number of different invariant probability densities. The study of the complete set of eigenfunctions for the Frobenius-Perron operator is founded on an important result in

functional analysis proved by Ionescu-Tulcea and Marinescu: the spectral decomposition theorem for quasi-compact linear operators [13, 25]. For this kind of operators, among other properties, the number of eigenvalues with modulus 1 is finite, and the related eigenspaces have finite dimension as well. Moreover, the supremum of the moduli of the remaining eigenvalues is strictly lower than one.[6]

In the following, for a given subset $A \subset \mathbb{R}$, we denote with $1_A : \mathbb{R} \to \{0, 1\}$ the characteristic function of A, i.e., if $x \in A$ then $1_A(x) = 1$ otherwise $1_A(x) = 0$.

Theorem 4 *Let I be an interval and let $\tau \in \mathcal{T}_{exp}(I)$. Then the Frobenius-Perron Θ_τ is quasi-compact on the space $BV(I)$. In particular,*

1. *τ has a finite number n of ergodic a.c. probability measures $\mu_1, \ldots, \mu_n$ with densities $f_1, \ldots, f_n$ belonging to $D_{BV}(I)$;*
2. *for each $1 \leq i \leq n$ there exists a finite collection of $k(i)$ disjoint sets $\{A_{i,1}, \ldots, A_{i,k(i)}\}$ such that*

$$\mathcal{S}(\mu_i) = \bigcup_{j=1}^{k(i)} A_{i,j}, \tag{13}$$

such that for $\phi_{i,j} = f_i 1_{A_{i,j}}$, for $j = 1 \ldots, k(i)$ we have

$$\begin{cases} \phi_{i,1} = \Theta_\tau \phi_{i,k(i)} \\ \phi_{i,2} = \Theta_\tau \phi_{i,1} \\ \quad \vdots \\ \phi_{i,k(i)} = \Theta_\tau \phi_{i,k(i)-1}. \end{cases} \tag{14}$$

and such that for $j = 1 \ldots, k(i)$ the transformation $\tau^{k(i)} : A_{i,j} \to A_{i,j}$ is exact with respect to the a.c. measure $\mu_{i,j} = f_i \cdot \lambda$.

4 The Family of Sawtooth Maps

In this Section we introduce the family of Sawtooth maps, and on the basis of previous results we discuss some relevant properties of these systems. Let us consider the dynamical system $x_{n+1} = S(x_n)$, where

$$S(x) = \begin{cases} Bx + 1 & \text{if } x < P, \\ Bx - 1 & \text{if } x \geq P, \end{cases} \tag{15}$$

with $B \in \mathbb{R}^+$, and $P \in \mathbb{R}$. In the literature, this dynamical system has been investigated by several authors (see, e.g., [4, 26, 27]): in [26] it was proposed for studying the nonlinear dynamics of a class of analog-to-digital converters; in [27] it was proposed for the design of a chaos-based random bit generator;

[6] An introductory overview about the spectrum of the Frobenius-Perron operator for mixing maps is discussed n [2].

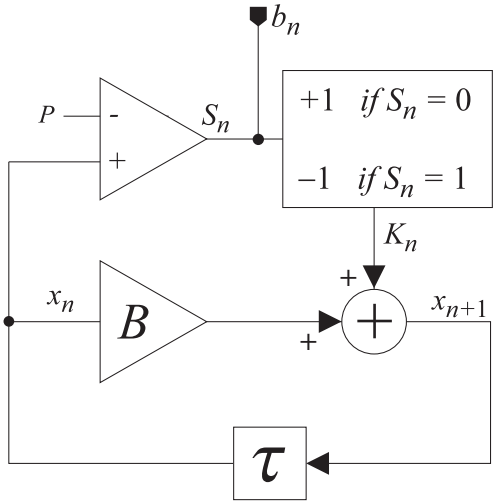


Figure 1: Simple block diagram for the implementation of the map (15).

in [4] the authors studied from a dynamical point of view the effects of the system parameters dispersion due to implementation tolerances.

This map can be implemented on the basis of the simple block diagram depicted in Fig. 1. In detail, the authors in [4] proved that, when implementing an electronic circuit on the basis of the block diagram depicted in Fig. 1, the non-idealities introduced by the implementation process tolerances define a different map that is equivalent[7] to a piecewise affine transformation $T : \mathbb{R} \to \mathbb{R}$ of the form

$$T(x) = \begin{cases} B(x-\alpha)+1 & \text{if } x < \alpha, \\ B(x-\alpha)-1 & \text{if } x \geq \alpha, \end{cases} \tag{16}$$

where $B \in \mathbb{R}^+$, and $\alpha \in \mathbb{R}$. Furthermore, by varying the parameters B and P in (15) all the possible equivalent systems in the family (16) can be obtained. In the following, we refer to (16) as the family of Sawtooth maps (Fig. 2).

For $B > \frac{2}{1+|\alpha|}$ any initial condition triggers sequences eventually attracted toward $\pm\infty$, whereas for $0 < B \leq \frac{2}{1+|\alpha|}$ the dynamical behavior depends on the specific values of parameters B and α (Fig. 3).

REMARK: in the following we focus on the *parameter space*

$$|\alpha| < \frac{\sqrt{2}-1}{1+\sqrt{2}}, \qquad \sqrt{2} < B \leq \frac{2}{1+|\alpha|}. \tag{17}$$

By defining the interval $\Lambda = [-1, 1) \subset \mathbb{R}$, it is easy to check that Λ is both T-invariant and attractive for any trajectory triggered by initial conditions chosen between the two fixed points $\pi_1 = \frac{B\alpha-1}{B-1}$ and $\pi_2 = \frac{B\alpha+1}{B-1}$. Finally, any other initial condition x_0 such that $x_0 > \pi_2$ or $x_0 < \pi_1$ triggers trajectories attracted

[7] Equivalent up to a topological conjugacy set by a linear homeomorphism [4]. The connection between (16) and the dynamics of (15) may loose its validity if for the implementation of the map (15) a block diagram different from that one depicted in Fig. 1 is adopted [4].

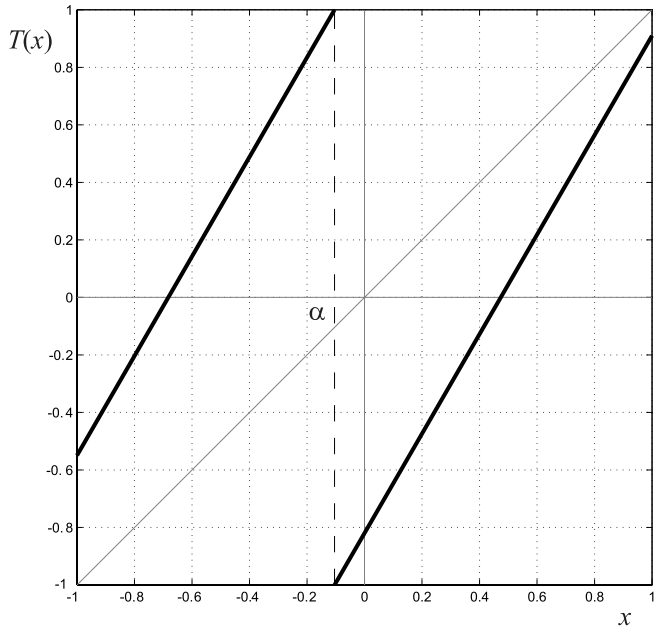


Figure 2: The Sawtooth map for $B = 1.7298$ and $\alpha = -0.1043$.

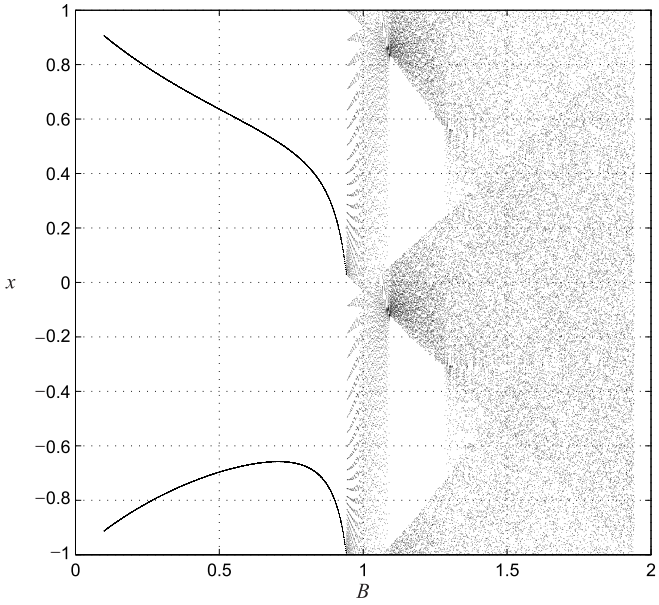


Figure 3: Bifurcation diagram for the Sawtooth map (16) with $\alpha = 0.03$ and B ranging in $0.1 \div 2/1.03$.

toward $\pm\infty$. Referring to the parameter space (17), the maximum allowed value for B is 2, obtainable setting $\alpha = 0$.

We note that if we define $I_1 = [-1, \alpha)$ and $I_2 = [\alpha, 1)$ the restrictions $T|_{I_1} = T_1 : I_1 \to \Lambda$ and $T|_{I_2} = T_2 : I_2 \to \Lambda$ are invertible. Accordingly, for any $A \in \mathcal{B}_{\mathbb{R}}$ we have $T^{-1}(A) = T_1^{-1}(A) \cup T_2^{-1}(A)$, where $T_1^{-1}(A) \cap T_2^{-1}(A) = \varnothing$ and

$$T_1^{-1}(x) = \frac{x-1}{B} + \alpha, \qquad T_2^{-1}(x) = \frac{x+1}{B} + \alpha. \tag{18}$$

The Sawtooth map (16) belongs to $\mathcal{T}_{exp}(\Lambda)$. Indeed, we have the partition $\{I_1, I_2\} \in \mathcal{P}_\Lambda$ such that T_{I_i} is continuous and derivable, for $i = 1, 2$, and $|T'(x)| = B = 1 + \epsilon > 1$, for all $p_{i-1} < x < p_i$. Moreover the functions $g_1(x) = g_2(x) = \frac{1}{B}$ have a total variation equal to 0. □

4.1 Invariant Special Sets

We have a particular interest in the following type of sets.

Definition 6 *A* multi-interval *(m-interval) J is a subset of $\mathbb{R}$ which is the union of a collection of intervals (even with infinite Lebesgue measure). An invariant m-interval J is* minimal *if it does not have any invariant subset $J' \subset J$ which is a m-interval.*

Single intervals represent a special class of m-intervals. It is easy to prove that unions of m-intervals are m-intervals, whereas on the contrary, the non-empty intersections of m-intervals may not be a m-interval. For example, $(-1, 0] \cap [0, 1) = \{0\}$, which is a singleton. Moreover, we remark that if a m-interval J is invariant for a generic transformation $\tau : I \to I$, then J is invariant for any transformation $\tau^n : I \to I$, $\forall n \in \mathbb{N}, n > 0$. On the other hand, a m-interval J which is invariant for τ^n, $n > 1$, may not be invariant for $\tau^m, m < n$. Indeed, let us consider the map $\tau : [0, 1) \to [0, 1)$ defined as $\tau(x) = (0.5 + x) \bmod 1$, where $a \bmod 1$ returns the fractional part of a. In this case the set $[0, 1)$ is invariant for both τ and τ^2, whereas the set $[0, 5)$ is invariant for τ^2 and not for τ. Note that this result shows that the minimality of an invariant m-interval may depend on the considered iteration τ^n of τ.

In the following, we prove some fundamental properties about invariant minimal m-intervals for the Sawtooth map. In the next subsections we refer to these results for proving the exactness for this family of maps.

Lemma 1 *Let J be a m-interval such that if $x \in J$ there exists $\epsilon > 0$ such that $[x, x + \epsilon) \subset J$. Then $T(J)$ is a m-interval satisfying the same property.*

Proof. J can be expressed as a countable union of intervals of the form $[a, b)$. If $\alpha \in (a, b)$ then $T([a, b)) = [T(a), 1) \cup [-1, T(b))$. If $\alpha \notin (a, b)$ then $T([a, b)) = [T(a), T(b))$. Accordingly $T(J)$ can be expressed again as a countable union of intervals of the form $[a, b)$. □

It is worth noting that an open m-interval satisfies the above property, and therefore if J is an open interval then $T^n(J)$ is an m-interval for all $n \in \mathbb{N}$.

Lemma 2 *Let $\delta \in \mathbb{R}$ with $0 < |\delta| < 1 - |\alpha|$, and let us consider the open interval $J = (\alpha - \frac{|\delta| - \delta}{2}, \alpha + \frac{|\delta| + \delta}{2})$. If $\alpha \in T(J)$ then $T^4(J) = \Lambda$.*

Proof. Let us assume $\delta > 0$ and $J = (\alpha, \alpha + \delta)$. We have $(-1, \alpha) \subset T(J)$ and $(T(-1), 1) \subset T^2(J)$. Recalling (17) we have $T(-1) = -B - B\alpha + 1 < -\sqrt{2} + \sqrt{2}\frac{\sqrt{2}-1}{\sqrt{2}+1} + 1 = \frac{\sqrt{2}-1}{\sqrt{2}+1} < \alpha$, and therefore $[\alpha, 1) \subset T^2(J)$. Let us now assume $\alpha \geq 0$, and let us focus on $A = (T(-1), 1) \subset T^2(J)$. In particular A can be written as the union of two disjoint intervals $K_1 = (T(-1), \alpha)$ and $K_2 = [\alpha, 1)$. We have $\lambda(K_1) = \alpha + B + B\alpha - 1$ and $\lambda(K_2) = 1 - \alpha$. The two intervals $T(K_1) = (T^2(-1), 1)$ and $T(K_2) = [-1, T(1))$ overlap since $\lambda(T(K_1)) + \lambda(T(K_1)) = B\alpha + B^2(1 + \alpha) - B + B - B\alpha = B^2(1 + \alpha)$ which is greater than 2 if $B > \frac{\sqrt{2}}{1+\alpha}$. Since $\alpha \geq 0$ this latter inequality is verified for all $B > \sqrt{2}$, and therefore $T^3(J) \subseteq T(K_1) \cup T(K_2) = \Lambda = T^4(J)$. If $\alpha < 0$ we can consider the evolution of $[\alpha, 1) \subset T^2(J)$. In particular $[-1, T(1)) \subset T^3(J)$ and recalling (17) we have $T(1) = B + B\alpha - 1 > \alpha$. Accordingly, we can focus on the set $A = [-1, T(1)) \subset T^3(J)$, which can be written as the union of $K_1 = [-1, \alpha)$ and $K_2 = [\alpha, T(1))$. Since $\lambda(K_1) = 1 - \alpha$ and $\lambda(K_2) = \alpha + B + B\alpha - 1$ then the two intervals $T(K_1) = (T(-1), 1)$ and $T(K_2) = [-1, T^2(1))$ overlap. Indeed we obtain that $T^2(1) > T(-1)$ if $B > \frac{\sqrt{2}}{1-\alpha}$, that is verified for all $B > \sqrt{2}$ since $\alpha < 0$. Therefore $T^4(J) \subseteq T(K_1) \cup T(K_2) = \Lambda$. A specular reasoning holds for $\delta < 0$. □

We can use the above Lemma for proving the following

Theorem 5 *Let $T : \Lambda \to \Lambda$ be the Sawtooth map with parameters (17). Then Λ is a minimal invariant m-interval for $T^n : \Lambda \to \Lambda$, $\forall n \in \mathbb{N}, n > 0$.*

Proof. Let us assume $J \subseteq \Lambda$ an invariant m-interval. If $I \subseteq J$ is an arbitrary interval then $T^k(I) \subseteq J$, for all $k \in \mathbb{N}$. We prove that there exists $n_0 \in \mathbb{N}$ such that $T^n(I) = \Lambda$ for all $n \geq n_0$ and therefore $J = \Lambda$ is the unique invariant m-interval for all iterates τ^n of τ.

Let $I_0 \subset \Lambda$ be an open interval such that $I_0 = (a_0, b_0)$ and let us define $\delta_0 = b_0 - a_0$. Let $k_0 \geq 0$ be the smallest integer such that $\alpha \in T^{k_0}(I_0)$. This integer exists since the Sawtooth map is linear and expanding on $\Lambda \setminus \{\alpha\}$, and until $\alpha \notin T^k(I_0) \subseteq \Lambda$ the intervals of the sequence $\{T^k(I_0)\}$ have measures $\lambda(T^k(I_0)) = B^k\delta_0$ which increases exponentially (recall that $B > \sqrt{2}$). Accordingly, we can split the interval $T^{k_0}(I_0) = (a_1, b_1)$ by the point α and we focus on the open interval $I_1 = (\alpha - \frac{|\delta_1| - \delta_1}{2}, \alpha + \frac{|\delta_1| + \delta_1}{2})$, where $\delta_1 = a_1 - \alpha$, if $\alpha - a_1 > b_1 - \alpha$, or $\delta_1 = b_1 - \alpha$ otherwise. It results that $|\delta_1| \geq B^{k_0}\frac{\delta_0}{2}$. Following the same reasoning, let $k_1 > 0$ be the smallest integer such that $\alpha \in T^{k_1}(I_1)$. If $k_1 = 1$ then from Lemma 2 we have that $T^{k_0+5}(I_0) = \Lambda$ and the proof is over. If otherwise $k_1 \geq 2$ then we split the interval $T^{k_1}(I_1) = (a_2, b_2)$ by the point α and we focus on the open interval $I_2 = (\alpha - \frac{|\delta_2| - \delta_2}{2}, \alpha + \frac{|\delta_2| + \delta_2}{2})$, where $\delta_2 = a_2 - \alpha$, if $\alpha - a_2 > b_2 - \alpha$, otherwise $\delta_2 = b_2 - \alpha$. It results that $|\delta_2| \geq B^{k_2}\frac{|\delta_1|}{2} > |\delta_1|$. Proceeding iteratively we can build a sequence of open intervals $\{I_n\}$ such that $I_n \subseteq J$ and such that $\lambda(I_{n+1}) = |\delta_{n+1}| \geq B^{k_n}\frac{|\delta_n|}{2} \geq B^{k_0}\frac{\delta_0}{2}\prod_{p=1}^{n}\frac{B^{k_p}}{2}$ with $k_p \geq 1$. Since until $k_p > 1$ for $p = 1, 2, \ldots$ the intervals of the sequence have increasing exponential measures, there must be a step n_0 for which $k_{n_0} = 1$ and from Lemma 2 we have that $T^{k_0+\ldots+k_n+4}(I_0) = \Lambda$ concluding the proof. □

We conclude this subsection noting that the previous Theorem states the strong forward and backward *topological transitivity* of the Sawtooth map on Λ with parameters (17), described by the following

Corollary 2 *For any couple of open sets $J_1, J_2 \subseteq \Lambda$, there exist integers p_0, q_0 such that if $p \geq p_0$ and $q \geq q_0$ both $T^p(J_1) \cap J_2$ and $T^{-q}(J_1) \cap J_2$ are non-empty sets containing open intervals.*

4.2 The Frobenius-Perron operator for the Sawtooth map

For piecewise affine maps the Frobenius-Perron operator can be written in the simple form [14]

$$\Theta_\tau f(x) = \sum_{\tau(y)=x} \frac{f(y)}{\left|\frac{d\tau}{dy}(y)\right|}. \tag{19}$$

Recalling the definition of $I_1 = [-1, \alpha)$ and $I_2 = [\alpha, 1)$ for the Sawtooth map we have

$$\begin{aligned}\Theta_T f(x) = \frac{1}{B} f\left(\frac{x-1}{B} + \alpha\right) 1_{T(I_1)}(x) + \\ + \frac{1}{B} f\left(\frac{x+1}{B} + \alpha\right) 1_{T(I_2)}(x).\end{aligned} \tag{20}$$

If we calculate $T(I_1) = [-B(1+\alpha)+1, 1)$, $T(I_2) = [-1, B(1-\alpha)-1)$, and recalling the conditions (17) we finally obtain

$$\Theta_T f(x) = \begin{cases} \frac{1}{B} f\left(\frac{x+1}{B} + \alpha\right), & \text{if } x \in J_1, \\ \frac{1}{B}\left[f\left(\frac{x-1}{B} + \alpha\right) + f\left(\frac{x+1}{B} + \alpha\right)\right], & \text{if } x \in J_2, \\ \frac{1}{B} f\left(\frac{x-1}{B} + \alpha\right), & \text{if } x \in J_3, \end{cases} \tag{21}$$

where $J_1 = [-1, -B(1+\alpha)+1)$, $J_2 = [-B(1+\alpha)+1, B(1-\alpha)-1)$ and $J_3 = [B(1-\alpha)-1, 1)$.

Invariant a.c. measures can be found solving the functional equation $\Theta_T f^* = f^*$ a.e., searching for densities that are fixed points of the Frobenius-Perron operator (up to an a.e. equivalence). Unfortunately, performing this task is possible only for very simple cases and several methods have been proposed in the literature for computing approximations of invariant measures [2, 15–21]. In Sec. 8 we present an efficient method which is suitable for the Sawtooth map.

4.3 Exactness of the Sawtooth map

In this subsection we discuss the exactness of the Sawtooth map, and we start the study analyzing the support of a.c. invariant measures.

Lemma 3 *Let $\mu = f \cdot \lambda$ be an invariant measure for the iterated Sawtooth map with parameters (17) $T^n : \Lambda \to \Lambda$, with $n \in \mathbb{N}, n > 0$. Then $\Lambda = \mathcal{S}(\mu) = \overline{\Omega}_f$.*

Proof. Let us assume ab absurdo that there exists an open interval $I \subset \Lambda$ such that $\mu(I) = 0$. For any integer $n \geq 0$ we have $\mu(I) = \int_I f(x)dx = \int_I \Theta_T^n f(x)dx = \int_{T^{-n}(I)} f(x)dx = \mu(T^{-n}(I)) = 0$. Since $\lambda(T^{-n}(I)) > 0$ for all n then $f(x) = 0$ a.e. over $T^{-n}(I)$. From Corollary 2 for any open set J_2 there exist p_0 such that for all $p \geq p_0$ the set $T^{-p}(I) \cap J_2$ has positive Lebesgue measure. Since $\mu(T^{-p}(I) \cap J_2) = 0$ then $f = 0$ a.e. over $T^{-p}(I) \cap J_2$. From the arbitrariness of J_2 this implies that $f(x) = 0$ a.e. over Λ and this is impossible.

Therefore $\mathcal{S}(\mu) = \Lambda \setminus \varnothing = \Lambda$. If $\Omega_f \neq \Lambda$ then let us consider $x \in \mathcal{S}(\mu) \setminus \Omega_f$. In each interval $(x - \epsilon, x + \epsilon) \cap \Lambda$ we have points of Ω_f, otherwise there would be in Λ the interval $(x, x + \epsilon)$ with zero measure, and therefore $\mathcal{S}(\mu)$ is the closure of Ω_f. □

Theorem 6 *Any iteration T^n of the Sawtooth map with parameters (17) admits one unique probabilistic ergodic a.c. measure.*

Proof. From Theorem 4 we know that the Sawtooth map has a finite number of ergodic a.c. measures whose densities are of bounded variations. Let us assume $\mu_1 = f_1 \cdot \lambda$ and $\mu_2 = f_2 \cdot \lambda$ to be ergodic a.c. measures, and let f_1, f_2 be their related lower semicontinuos densities. Accordingly, μ_1 and μ_2 are concentrated on the open sets Ω_{f_1} and Ω_{f_2} respectively. From Lemma 3 we have that $\overline{\Omega}_{f_1} = \Lambda = \overline{\Omega}_{f_2}$ which implies $\Omega_{f_1} \cap \Omega_{f_2} \neq \varnothing$. On the other hand, if the two measures are different they must be mutually singular (Theorem 2): this is not possible since there exists an open interval $J \subset \Omega_{f_1} \cap \Omega_{f_2}$ such that $\mu_1(J), \mu_2(J) > 0$. Therefore $\mu_1 = \mu_2$. □

We conclude this subsection proving the following

Theorem 7 *The Sawtooth map with parameters (17) is an exact transformation with respect to its unique probabilistic ergodic a.c. measure $\mu = f \cdot \lambda$.*

Proof. Since the previous Theorem assures the uniqueness of the ergodic measure for T, we only have to prove that the periodic cycle described in eq. (14) has period one. Indeed, let us consider a finite collection of k disjoint sets $\{A_1, \ldots, A_k\}$ such that $\mathcal{S}(\mu) = \bigcup_{j=1}^{k} A_j$, and such that for $\phi_j = f 1_{A_j}$, $j = 1 \ldots, k$ we have $\Theta^k \phi_j = \phi_j$ a.e., i.e., $\mu_j = \phi_j \cdot \lambda$ is an invariant measure. Theorem 6 assures that T^k has one unique ergodic measure. On the other hand since the transformation $T^k : A_j \to A_j$ is exact, it is also ergodic, and T^k should have k different mutually singular ergodic measures, which is impossible. Therefore $k = 1$ and the Sawtooth map is exact over Λ with respect to its unique probabilistic ergodic a.c. measure. □

4.4 The special case with $B = 2$

The special case with $B = 2$ deserves to be analyzed in detail. In such case, according to (17), we have $|\alpha| \leq \frac{2}{B} - 1 \Rightarrow \alpha = 0$ and the Lebesgue measure is preserved. Indeed, from (18) we have $\lambda(T^{-1}(A)) = \lambda(A)/2 + \lambda(A)/2 = \lambda(A)$ and the uniform distribution over Λ, i.e., $f(x) = \frac{1}{2}$ is preserved. The same result can be obtained using (21) and verifying the functional equation $\Theta_T \frac{1}{2} = \frac{1}{2}$.

5 The Skorokhod metric on $T_{exp}(I)$

The set of maps $T_{exp}(I)$ previously introduced can be made a metric space. We introduce the Skorokod metric on $T_{exp}(I)$ which will be used for studying the robustness of invariant measures with respect to small parametric perturbations [13].

Definition 7 *Give two maps* $\tau_1, \tau_2 : I \to I$ *the* Skorokhod metric *is defined as* $d_s(\tau_1, \tau_2) =$
$\inf\{\varepsilon > 0 : \exists$ *a diffeomorphism*[8] $h : I \to I$ *and* $\exists A \subseteq I$ *such that* $\lambda(I) - \lambda(A) < \varepsilon\lambda(I)$, *and* $\forall x \in A$ *it results* $\tau_1(x) = \tau_2(h(x)), |h(x) - x| < \varepsilon, \left|\frac{1}{h'(x)} - 1\right| < \varepsilon\}$.

We are interested in estimating the Skorokhod distance between two maps (16) with different parameters. In detail, we assume $T_1, T_2 : \Lambda \to \Lambda$ as

$$T_1(x) = \begin{cases} B_1(x - \alpha_1) + 1 & \text{if } x < \alpha_1, \\ B_1(x - \alpha_1) - 1 & \text{if } x \geq \alpha_1 \end{cases} \tag{22}$$

and

$$T_2(x) = \begin{cases} B_2(x - \alpha_2) + 1 & \text{if } x < \alpha_2, \\ B_2(x - \alpha_2) - 1 & \text{if } x \geq \alpha_2, \end{cases} \tag{23}$$

and we want to estimate $d_S(T_1, T_2)$. To this aim we will compute three quantities K_1, K_2, K_3 such that $d_S(T_1, T_2) \leq \max\{K_1, K_2, K_3\}$.

Accordingly, let us define two points $p_1, p_2 \in [-1, 1]$ as

$$p_1 = \max\left\{-1, \alpha_1 - \frac{B_2}{B_1}(1 + \alpha_2)\right\}, \tag{24}$$

and

$$p_2 = \min\left\{\alpha_1 + \frac{B_2}{B_1}(1 - \alpha_2), 1\right\}. \tag{25}$$

It can be proved that the definition of p_1, p_2 is correct since if conditions (17) are satisfied then $p_1, p_2 \in [-1, 1]$. We can therefore define the open interval $A = (p_1 + \delta, p_2 - \delta) \subset \Lambda$, where $0 < \delta < \frac{p_2 - p_1}{2}$ is arbitrary. We now have to construct a diffeomorphism h such that $T_1(x) = T_2(h(x))$, $\forall x \in A$. It can be easily verified that on the set A h has the linear form

$$h(x)|_A = \frac{B_1}{B_2}(x - \alpha_1) + \alpha_2, \tag{26}$$

whereas on $\Lambda \setminus A$ one can extend h such to make it a diffeomorphism in several ways, e.g., with two polynomial functions: the extension method of h over $\Lambda \setminus A$ does not matter (See Fig. 4).

By construction we have $\lambda(\Lambda) = 2$, $\lambda(A) = p_2 - p_1 - 2\delta = L_{\min} - 2\delta$, where $0 < L_{\min} \leq 2$ is determined by

$$\begin{aligned} L_{\min} \quad = \min\Bigg\{&2, \frac{2B_2}{B_1}, \\ &1 - \alpha_1 + \frac{B_2(1 + \alpha_2)}{B_1}, 1 + \alpha_1 + \frac{B_2(1 - \alpha_2)}{B_1}\Bigg\}. \end{aligned} \tag{27}$$

The four values in (27) are obtained considering the four permutations for the couple of points p_1 and p_2, according to (24)-(25), and computing the general expression for $p_2 - p_1$. Therefore we can write

$$K_1 = \frac{\lambda(\Lambda) - \lambda(A)}{\lambda(\Lambda)} = \frac{2 - L_{\min}}{2} + \delta. \tag{28}$$

[8] A function $h : I \to I$ is a diffeomorphism on I if it is invertible and differentiable together with its inverse.

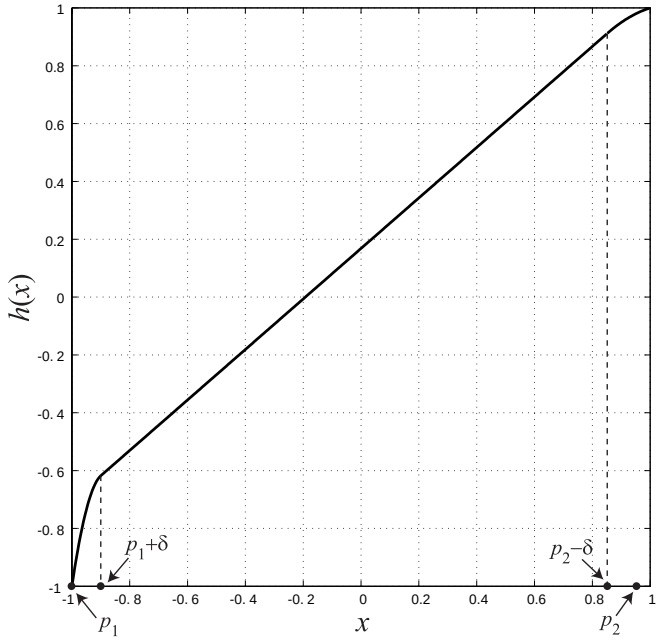


Figure 4: A diffeomorphism $h : \Lambda \to \Lambda$ that works for $\tilde{B} = 1.47, \tilde{\alpha} = 0.16$, $B = 1.72, \alpha = 0.20$, $\delta = 0.1$. For $x > p_2 - \delta$ and $x < p_1 + \delta$ the function is completed with two parabolas.

If $x \in A$ then

$$g(x) = |h(x) - x| = \left| x\left(\frac{B_1}{B_2} - 1\right) + \alpha_2 - \frac{B_1\alpha_1}{B_2} \right| \tag{29}$$

which implies that

$$\begin{aligned} \sup_{x\in A} g(x) \leq \max \Bigg\{ & \left| \left(\frac{B_1}{B_2} - 1\right) + \alpha_2 - \frac{B_1\alpha_1}{B_2} \right|, \\ & \left| \left(1 - \frac{B_1}{B_2}\right) + \alpha_2 - \frac{B_1\alpha_1}{B_2} \right| \Bigg\} = K_2. \end{aligned} \tag{30}$$

Finally,

$$K_3 = \sup_{x\in A} \left| \frac{1}{h'(x)} - 1 \right| = \left| \frac{B_2}{B_1} - 1 \right|. \tag{31}$$

We have obtained an estimation for $d_S(T_1, T_2)$ since

$$d_S(\tilde{T}, T) \leq \max\{K_1, K_2, K_3\}. \tag{32}$$

It is worth noting that since the Skorokhod distance is defined in (7) as an infimum, in the estimation (32) the arbitrary value of δ derived from (28) can be set equal to zero. We will use this result for analyzing the Sawtooth map with parameter perturbations.

5.1 Converging sequences of Sawtooth maps

Let us consider a sequence of parameters $\{B_n, \alpha_n\}$ such that $\forall n \in \mathbb{N}$ the conditions (17) are satisfied. We denote with $T_n : \Lambda \to \Lambda$ the Sawtooth map (16) defined by the parameters B_n and α_n.

With reference to the metric space $(\mathcal{T}_{exp}(\Lambda), d_S)$ we say that the sequence of maps $\{T_n\}$ converges to $\tilde{T}$ if

$$\lim_{n\to\infty} d_S(\tilde{T}, T_n) = 0.$$

The following proposition assures the existence of a converging sequence of Sawtooth maps.

Proposition 2 *Assuming conditions (17) satisfied, let us consider the parameters $\tilde{B}, \tilde{\alpha}$ and the sequence $\{B_n, \alpha_n\}$. If*

$$\lim_{n\to\infty} B_n = \tilde{B}, \qquad \lim_{n\to\infty} \alpha_n = \tilde{\alpha}, \tag{33}$$

then $\{T_n\}$ converges to $\tilde{T}$.

Proof. From (28)-(31) we have that $\lim_{n\to\infty} K_2 = \lim_{n\to\infty} K_3 = 0$. Since δ in (28) is arbitrary the same result holds for K_1, and inequality (32) assures the convergence. □

6 Robustness of the invariant measure

In this section we study the robustness of the unique a.c. ergodic measure for the Sawtooth map under parameter perturbations. Toward this aim, following the approach of Keller, Boyarsky and Góra [13, 28], we first introduce a theoretical framework valid for a large class of piecewise expanding transformations, then we prove the robustness of the Sawtooth map.

Let us consider a sequence of maps $\{\tau_n\}$, with $\tau_n \in \mathcal{T}_{exp}(I)$, and the associated sequence of Frobenius-Perron operators $\{\Theta_{\tau_n}\}$.

Definition 8 *The sequence $\{\Theta_{\tau_n}\}$ is called S-bounded if there are $0 < C < 1$ and $D > 0$ and for each $n \in \mathbb{N}$ there exist $k_n \in \mathbb{N}$ such that $\left\|\Theta^{k_n}_{\tau_n} f\right\|_{BV} \leq C \|f\|_{BV} + D \|f\|_1$, $f \in BV(I)$.*

About the robustness of invariant measures, the following fundamental result holds [13]

Theorem 8 *Let $\tilde{\tau} \in \mathcal{T}_{exp}(I)$ be a map that admits one unique ergodic a.c. probabilistic measure, and let us consider a sequence of maps $\{\tau_n\}$, with $\tilde{\tau}, \tau_n \in \mathcal{T}_{exp}(I)$. If the sequence $\{\Theta_{\tau_n}\}$ is S-bounded and $\lim_{n\to\infty} d_s(\tilde{\tau}, \tau_n) = 0$ then for sufficiently large n the map τ_n admits one unique ergodic a.c. probabilistic measure, and the sequence of densities $\{f_n\}$ related to the sequence of ergodic probabilistic measures converges in $L^1(I, \mathcal{B}_{\mathbb{R}}, \lambda)$ to the ergodic density $\tilde{f}$ of $\tilde{\tau}$, i.e.,*

$$\lim_{n\to\infty} \left\|f^*_n - \tilde{f}^*\right\|_1 = \int_I |f^*_n(x) - \tilde{f}^*(x)| dx = 0. \tag{34}$$

6.1 Robustness of the Sawtooth map

We now focus on the Sawtooth map and we consider a generic sequence of parameters $\{B_n, \alpha_n\}$ with a slightly stronger restriction on than (17). In particular, it is required that two positive constants $B_{\min}$ and $\alpha_{\max}$ exist such that $\forall n \in \mathbb{N}$

$$0 \leq |\alpha_n| \leq \alpha_{\max} < \frac{\sqrt{2}-1}{1+\sqrt{2}}, \tag{35}$$

and

$$\sqrt{2} < B_{\min} \leq B_n \leq \frac{2}{1+|\alpha_n|}. \tag{36}$$

The restriction avoid us to take into account not interesting particular values for the parameters.

If we denote with $T_n : \Lambda \to \Lambda$ the Sawtooth map (16) defined by parameters (B_n, α_n), we can refer to the related sequence of Frobenius-Perron operators $\{\Theta_{T_n}\}$.

Lemma 4 *If (35)-(36) are satisfied then the sequence $\{\Theta_{T_n}\}$ is S-bounded.*

Proof. We show that there exist $0 < C < 1$ and $D > 0$ such that for each $n \in \mathbb{N}$ we have $\left\|\Theta^2_{T_n} f\right\|_{BV} \leq C \|f\|_{BV} + D \|f\|_1$, $f \in BV(I)$. Let us consider, for each n, the map T_n^2 defined as

$$T_n^2(x) = \\ = \begin{cases} B_n^2(x-\alpha_n) + B_n + 1, & \text{if } -1 \leq x < \frac{\alpha_n + B_n\alpha_n - 1}{B_n}, \\ B_n^2(x-\alpha_n) + B_n - 1, & \text{if } \frac{\alpha_n + B_n\alpha_n - 1}{B_n} \leq x < \alpha_n, \\ B_n^2(x-\alpha_n) - B_n + 1, & \text{if } \alpha_n \leq x < \frac{\alpha_n + B_n\alpha_n + 1}{B_n}, \\ B_n^2(x-\alpha_n) - B_n - 1, & \text{if } \frac{\alpha_n + B_n\alpha_n + 1}{B_n} \leq x < 1. \end{cases} \tag{37}$$

The map T_n^2 belongs to $\mathcal{T}_{exp}(\Lambda)$, and its associated Frobenious-Perron operator satisfies the equation $\Theta_{T_n^2} = \Theta^2_{T_n}$. If conditions (35)-(36) are satisfied then the four intervals partitioning Λ have positive measure, and the smallest interval has length greater than $\lambda_{\min} = (B_{\min} - \alpha_{\max} - B_{\min}\alpha_{\max} - 1)/B_{\min}$.

From Theorem 1 we have that $V_\Lambda(\Theta^2_{T_n} f) = V_\Lambda(\Theta_{T_n^2} f) \leq \delta_{1_n} \tilde{V}_\Lambda(f) + \frac{\delta_{1_n}}{\delta_{2_n}} \|f\|_1$, where for the map T_n^2 we have $\delta_{1_n} = \frac{2}{B_n^2}$ and $\delta_{2_n} > \lambda_{\min}$. Accordingly, $\left\|\Theta^2_{T_n} f\right\|_{BV} = \|f\|_1 + V_\Lambda(\Theta^2_{T_n} f) \leq \frac{2}{B_n^2} V_\Lambda(f) + \frac{2}{B^2 \lambda_{\min}} \|f\|_1 + \|f\|_1 = \frac{2}{B^2} \|f\|_{BV} + (1 - \frac{2}{B_n^2} + \frac{2}{B_n^2 \lambda_{\min}}) \|f\|_1$, concluding the proof. □

Referring to Theorem 8 and Lemma 4, and on the previous results, we can now state the robustness of the invariant measure for the Sawtooth map with the following

Theorem 9 *Assuming conditions (35)-(36) satisfied, let us consider the parameters $\tilde{B}, \tilde{\alpha}$ and the sequence $\{B_n, \alpha_n\}$. If*

$$\lim_{n\to\infty} B_n = \tilde{B}, \qquad \lim_{n\to\infty} \alpha_n = \tilde{\alpha}, \tag{38}$$

then the sequence of unique ergodic probability densities $\{f_n^*\}$ of $\{T_n\}$ strongly converges in $L_1(\Lambda)$ to the unique ergodic probability density $\tilde{f}^*$ of $\tilde{T}$, i.e.,

$$\lim_{n\to\infty} \left\| f_n^* - \tilde{f}^* \right\|_1 = 0. \tag{39}$$

It is worth remarking that (39) implies the convergence of measures, i.e. $\lim_{n\to\infty} \mu_n^* = \tilde{\mu}$, where $\mu_n^* = f_n^* \cdot \lambda$ and $\tilde{\mu} = \tilde{f}^* \cdot \lambda$.

7 Invariant Measures for Nontrivial Parameter Values

We have proved that the Sawtooth map with parameters satisfying (17) admits one unique ergodic pdf and, as already discussed in Section 4.4, the uniform distribution defines the unique ergodic a.c. measure for the case $B = 2$. The analytical expression of the invariant densities for the Sawtooth map with $B < 2$ can be written only for few special values of the system parameters.

In this section we restrict our attention to the set of *piecewise affine expanding maps* $\mathcal{T}_{aex}(I) \subset \mathcal{T}_{exp}(I)$ for which there exists a partition of intervals $M \in \mathcal{P}_I$, $M = \{I_i, i = 1, \ldots, m, \text{with } m > 0\}$, such that $\tau|_{I_i}$ is a function of the form $k_1 x + k_2$, with $|k_1| > 1$ and $k_1, k_2 \in \mathbb{R}$. The Sawtooth map belongs to this set.

In the following we assume the interval I bounded by its endpoints $a, b \in \mathbb{R}$, $a < b$. If the endpoints a, b of I do not belong to I, we can extend τ to $\overline{I}$, and we have $\tau(a), \tau(b) \in \overline{I}$. We denote with $\tau_+(x)$ and $\tau_-(x)$ the limits $\lim_{y\to x^+} T(y)$ and $\lim_{y\to x^-} T(y)$, respectively, and we assume for notation convenience $\tau_+(x) = \tau_-(x) = \tau(x)$, for $x = a, b$.

7.1 Markovian Transformations

Markovian transformations are a special class of piecewise monotonic transformations. In these systems there exists a nontrivial defining partition[9] of intervals M such that each one of them is mapped onto a connected union of intervals of the same partition. More in detail, we have the

Definition 9 *Let us denote with* $\mathcal{M}_{aex}(I) \subset \mathcal{T}_{aex}(I)$ *the class of* Markovian piecewise expanding affine maps. *The map* $\tau \in \mathcal{M}_{aex}(I)$ *if and only if there exists a partition* $M \in \mathcal{P}_I$ *with endpoints* $\mathcal{E}_M = \{s_0, \ldots, s_m\}$ *such that for any open interval* (s_i, s_{i+1}) *there are two endpoints* $s_j < s_k$ *in* $\mathcal{E}_M$ *and* $\tau : (s_i, s_{i+1}) \to (s_j, s_k)$ *is an affine homeomorphism*[10].

It is worth noting that in the previous definition s_j and s_k are determined computing $\tau_+(s_i)$ and $\tau_-(s_{i+1})$. Moreover, $\tau(\mathcal{E}_M) \subseteq \mathcal{E}_M$. It is well known that for Markovian transformations the Frobenius-Perron linear operator can be related to the transition probability matrix $W_\tau \in \mathbb{R}^{m\times m}$ of a finite Markov chain, in which each state is related to an element of the partition M. The entries of the stochastic matrix W_τ are defined as [13]

$$w_{i,j} = P(x_{n+1} \in I_j | x_n \in I_i) = \frac{\lambda(I_i \cap \tau^{-1}(I_j))}{\lambda(I_i)}, \tag{40}$$

and invariant densities for the associated Markov chain are described by *invariant row vectors* $v^* \in \mathbb{R}^{+m}$ such that $v^* W_\tau = v^*$. These invariant vectors define

[9] For example, the partition Q in the Definition 2 given for piecewise expanding maps.

[10] A continuous function f between two topological spaces X and Y is called a homeomorphism if it is a bijection and if the inverse function f^{-1} is continuous.

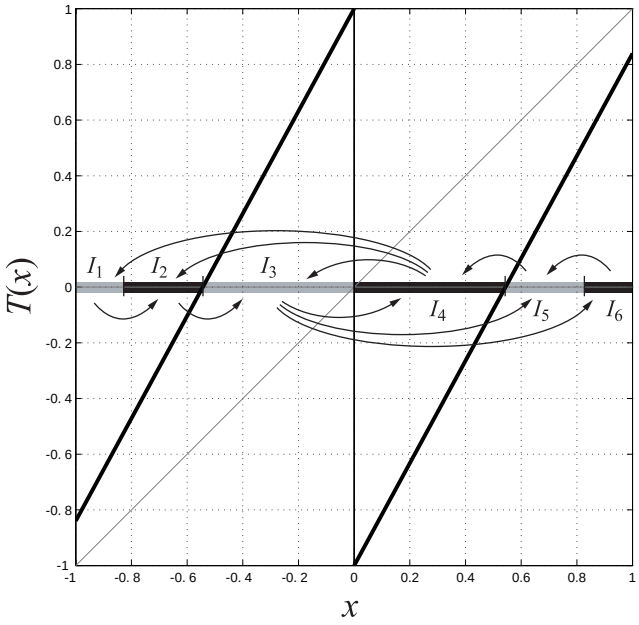


Figure 5: The Markovian Sawtooth map with $\alpha = 0$ and B defined in (42) and its defining partition M.

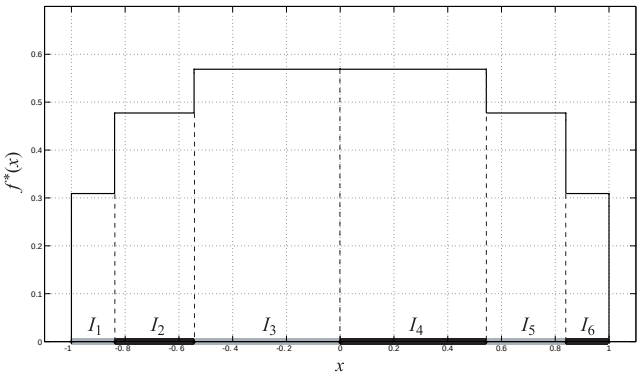


Figure 6: The invariant pdf associated with the Sawtooth map of Fig. 5.

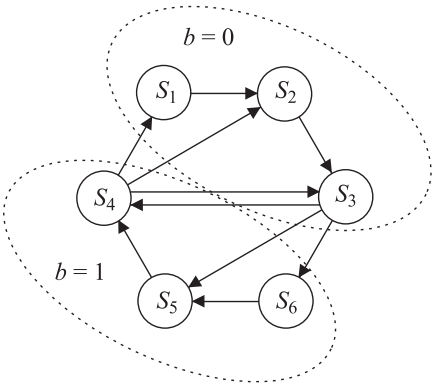


Figure 7: The Markov chain associated with the Sawtooth map of Fig. 5.

the levels of piecewise constant invariant densities for the transformation τ, i.e., there exist m levels $\{l_1, \ldots, l_m\}$ (the entries of the invariant vector v^*) such that the piecewise constant density

$$f^*(x) = \sum_{1 \le i \le m} \frac{l_i}{\lambda(I_i)} \, 1_{I_i}(x), \tag{41}$$

defines an a.c. invariant measure for the transformation τ. Accordingly, solving the linear system $v^* W_\tau = v^*$ such that $\sum_{1 \le i \le m} l_i = K$, with $l_i \ge 0$, provides solutions for the functional equation $\Theta_\tau f^* = f^*$ a.e.. Note that if $K = 1$ we obtain probability vectors and, consequently, (41) yields probability densities.

Example 1

The Sawtooth map with parameters $B = 2, \alpha = 0$ is a trivial case of Markovian transformation. In such case the induced Markov chain has two states (associated with the partitioning intervals $[-1, 0)$ and $[0, 1)$). The Markov chain process is a memoryless unbiased Bernoulli process, and the piecewise constant invariant density for T given by (41) is the uniform distribution. This fact suggested the use of the Sawtooth map for random bit generation, being the random binary sequence obtained recording the digital sequence $\{b_n\}$ at the output of the comparator in the block diagram of Fig. 1. There are several simple well known piecewise affine maps sharing the Bernoulli property with the Sawtooth map [2, 3, 13].

Example 2

A more interesting example is obtained considering the parameters $\alpha = 0$ and

$$B = \frac{\theta^2 + \theta + 4}{3\theta} \approx 1.84, \tag{42}$$

where $\theta = \sqrt[3]{19 + 3\sqrt{33}}$. In such special case we have a Markov chain induced by the partition M depicted in Fig. 5, which is a *refinement* of the smallest defining partition Q introduced in Def. 2. The endpoints of the partition M are

the element of the set $\{\pm\, T^k(-1), 0 \le k \le 3\}$, where for the particular chosen values of parameters we have $T^3(1) = T^3(-1) = 0$.

Since $\lambda(I_1) = \lambda(I_6) = 2 - B$, $\lambda(I_2) = \lambda(I_5) = 2B - B^2$, $\lambda(I_3) = \lambda(I_4) = 2B^2 - B^3$, using (40) one can obtain the transition probability matrix

$$W_T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda(I_3) & \lambda(I_2) & \lambda(I_1) \\ \lambda(I_1) & \lambda(I_2) & \lambda(I_3) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{43}$$

Solving the linear system $v^* W_T = v^*$ such that $\sum_{1 \le i \le m} l_i = 1$, with $l_i \ge 0$, one can compute the analytical expression of the invariant density (41), where

$$\begin{aligned} l_1 &= l_6 = \frac{2 - B}{2(5 - B^2)}, \\ l_2 &= l_5 = \frac{2 + B - B^2}{2(5 - B^2)}, \\ l_3 &= l_4 = \frac{1}{2(5 - B^2)}, \end{aligned} \tag{44}$$

obtaining the invariant probability density depicted in Fig.6.

If used for random bit generation, this Markovian map defines a binary process $\{b_n\}$ ruled by the hidden Markov chain of Fig. 7.

7.2 The general case

The main problem when dealing with piecewise affine maps with nontrivial parameter values is to determine their invariant measures. The issue may appear simplified for Markovian transformations, but in practical cases the problem of determining if a given map is Markovian (and therefore the problem of finding the related defining partition M) is unfeasible, and the estimation of invariant measures is achieved resorting to numerical calculations. About this point we provide the following

Proposition 3 *Referring to the notation in Def. 2, let $\tau \in \mathcal{T}_{aex}(I)$ and let $\mathcal{E}_Q = \{p_0, \ldots, p_q\}$ be the endpoints of its defining partition Q.*[11] *The map τ is Markovian if and only if there exists an integer n such that*

$$\tau^n(\Sigma) \subseteq \cup_{k=0}^{n-1} \tau^k(\Sigma), \tag{45}$$

where $\Sigma = \mathcal{E}_Q \cup \tau_-(\mathcal{E}_Q) \cup \tau_+(\mathcal{E}_Q)$.

Proof. If (45) holds, then let us consider the partition $M \in \mathcal{P}_I$ with endpoints $\mathcal{E}_M = \cup_{k=0}^{n-1} \tau^k(\Sigma)$. Since $\mathcal{E}_Q \subseteq \mathcal{E}_M$ then M is a refinement of Q and τ is affine over each element of the partition M. Let us consider two contiguous endpoints $s_i, s_{i+1} \in \mathcal{E}_M$. If $s_i \in \mathcal{E}_Q \subseteq \Sigma$ then $\tau_-(s_i), \tau_+(s_i) \in \Sigma \subseteq \mathcal{E}_M$, and the same holds for s_{i+1}. If otherwise $s_i \notin \mathcal{E}_Q$ then τ is continuous in s_i and $\tau_-(s_i) = \tau_+(s_i) = \tau(s_i) \in \mathcal{E}_M$. Accordingly, for any open interval (s_i, s_{i+1})

[11] Recall that we assume the defining partition Q to be the smallest one.

there are two endpoints $s_j < s_k$ in $\mathcal{E}_M$ such that $\tau : (s_i, s_{i+1}) \to (s_j, s_k)$ is an affine homeomorphism, and $\tau \in \mathcal{M}_{aex}(I)$.

On the other hand, let $\tau \in \mathcal{M}_{aex}(I)$, let M be its Markovian defining partition. Any endpoint p of Q must belong to $\mathcal{E}_M$, otherwise there would be an interval (s_i, s_{i+1}), $s_i, s_{i+1} \in \mathcal{E}_M$, in which the map is not affine. From the Markovian property $\tau((s_i, s_{i+1}))$ is an open interval bounded by $\tau_+(s_i), \tau_-(s_{i+1}) \in \mathcal{E}_M$ and the set $\mathcal{E}_M$ must contain $\tau_+(p), \tau_-(p)$, for each endpoint $p \in \mathcal{E}_Q$, i.e., $\Sigma \subseteq \mathcal{E}_M$. Since $\tau(\mathcal{E}_M) \subseteq \mathcal{E}_M$, (45) holds. □

On the basis of the previous proposition, a map $\tau \in \mathcal{T}_{exp}(I)$ is Markovian if and only if any initial condition belonging to the set $\Sigma = \mathcal{E}_Q \cup \tau_-(\mathcal{E}_Q) \cup \tau_+(\mathcal{E}_Q)$ triggers an eventually periodic orbit. Typically, this fact is true for a zero-measure countable subset of parameter values which is dense in the chaotic parameter space [27]. As a result, any map τ belonging to $\mathcal{T}_{exp}(I)$ can be studied taking into account a Markovian map τ' with a dynamical behavior "arbitrarily close" to the one of the original transformation τ. This theory is known as the *Approximation by Markov Transformations Method* [13], and on this basis some numerical algorithms have been proposed for the estimation of invariant measures (see, e.g., [2, 15–21] and the references cited therein).

Besides the numerical methods based on the above mentioned approach, a variety of algorithms have been proposed in the literature for the estimation of invariant measures in chaotic systems. Basically, the known methods can be grouped in different classes that, depending on the chaotic system, involve different amounts of both algorithmic complexity and resulting accuracy of the calculations: the Monte-Carlo methods or other methods based on the discretization of the Frobenius-Perron operator [2, 15–21].

8 An alternative efficient method

In this Section we discuss an alternative efficient method for estimating the invariant measure of piecewise affine chaotic maps defining stochastically stable dynamical systems. We first prove the following

Proposition 4 *Let $\tau \in \mathcal{T}_{aex}(I)$, and let Q be its defining partition. If $f \in L^1(I)$ is piecewise constant a.e.*[12]*, then $\Theta_\tau f$ is piecewise constant a.e..*

Proof. Let us assume f a.e. equal to a piecewise constant function $g \in L^1(I)$, and let $J \in \mathcal{P}_I$ be a partition of intervals of I such that the restriction of g over each element of the partition is constant. Let us consider the refinement $R = J \vee Q = \{R_1, \ldots, R_m\}$. The function g can be redefined on a finite number of endpoints of J such that each element $R_k \in R$ is an interval delimited by two endpoints $c_k < d_k$ belonging to $\overline{I}$. It follows that the restriction of $\tau|_{R_k}$ is affine, with the form $\alpha_k x + \beta_k$, whereas the restriction $g|_{R_k}$ is constant and equal to l_k. Each interval R_k is mapped to the interval $\tau(R_k) \subseteq I$ delimited by the couple of endpoints $v_k < w_k$ belonging to $\overline{I}$. It is worth noting that the endpoints of I, a and b, must belong to the set $\cup_{k=1}^m \{v_k, w_k\}$, since we have $\tau(I) = I$. We can therefore consider the new partition $R' = \{L_1, \cdots, L_s\}$ of intervals in I with endpoints $\mathcal{E}_{R'}$ obtained ordering the set $\cup_{k=1}^m \{v_k, w_k\}$. By construction we have

[12] We refer to functions a.e. equal to simple functions with a finite number of discontinuities and constant over positive measure intervals partitioning I.

that $\tau^{-1}(L_i) = \cup_{j=1}^{m} K_{i,j} \neq \varnothing$, where $K_{i,j} = \{x \in R_j : \tau(x) \in L_i\}$. It follows from the expression of the Frobenius-Perron operator for piecewise affine maps (19) that $\forall x \in L_i$ we have

$$\Theta_\tau g(x) = \sum_{j=1}^{m} \frac{l_j}{|\alpha_j|} u(K_{i,j}), \tag{46}$$

where the function $u : \mathcal{B}_I \to \{0, 1\}$ is defined as

$$u(A) = \begin{cases} 0, & \text{if } A = \varnothing, \\ 1, & \text{otherwise.} \end{cases} \tag{47}$$

Accordingly, the restriction of the pdf $\Theta_\tau g$ over each element of the partition R' is constant and $\Theta_\tau f$ is constant a.e..[13] □

A piecewise constant pdf $f \in D_1(I)$ can be easily described with a matrix

$$H = \begin{pmatrix} x_0 & l_0 \\ \vdots & \\ x_N & l_N \\ x_{N+1} & 0 \end{pmatrix}, \tag{48}$$

such that $x_0 = a$, $x_{N+1} = b$ and if $x_i \leq x < x_{i+1}$ then $f(x) = l_i$. Furthermore, due to the normalization property of pdfs, $\sum_{k=0}^{N} l_k(x_{k+1} - x_k) = 1$. The matrix H has 2 columns and $N + 2$ rows, and the entries of its first column are the N discontinuity points of f, plus the two endpoints of I (a and b).

We can prove the following

Proposition 5 *Let f be piecewise constant, and let N be its number of discontinuities in I. The pdf $\Theta_T f$ has at most $N + 2(q - 1)$ discontinuities, where q is the number of elements of the partition Q.*

Proof. Let us refer again to the partitions $R = \{R_1, \ldots, R_m\}$ and $R' = \{L_1, \ldots, L_s\}$ previously introduced. The partition R is made of $N + q$ intervals in the worst case, that is if none of the N discontinuity points of f agrees with any of the $q+1$ endpoints of Q. [14] The endpoints $\mathcal{E}_{R'}$ of R' are the endpoints of the intervals $\tau(R_k)$, that are no more than $2(N+q)$ points. Nevertheless, since in the worst case the N discontinuity points fall inside the sub-intervals over which τ is affine, we have that the intervals $\tau(R_j)$ and $\tau(R_{j+1})$ share one endpoint, for N different values of j such that $1 \leq j < m$. Accordingly, the maximum number of elements of $\mathcal{E}_{R'}$ is $2(N + q) - N = N + 2q$. Since two endpoints of R' agree with a and b, $\Theta_\tau f$ has at most $N + 2(q - 1)$ discontinuities. □

Using the Frobenius-Perron operator, one can obtain a sequence of pdfs using the rule $f_{n+1} = \Theta_\tau f_n = \Theta_\tau^{n+1} f_0$. Proposition 4 assures that if the starting pdf f_0 is piecewise constant, such sequence can be unambiguously described by a sequence of matrices $\{H_n, n \in \mathbb{N}\}$. In general, if $N(n)$ is the number of discontinuities of f_n, the number of rows of the matrix H_n is equal to $N(n)+2$. As a direct consequence of the previous results, we have the following

[13] We recall that if $f = g$ a.e. then $\Theta_\tau f = \Theta_\tau g$ a.e..

[14] If we add an arbitrary endpoint to Q, one of its interval is split in two parts, adding one element to the partition.

Corollary 3 *Let $x_0(n), \ldots, x_{N(n)+1}(n)$ be the entries of the first column of H_n, and let $\{p_i, 0 \leq i \leq q\}$ be the endpoints of Q. The entries of the first column of H_{n+1} are the elements of the set $\tau(\{x_0(n), \ldots, x_{N(n)+1}(n)\}) \cup \tau_-(\{p_i, 0 \leq i \leq q\}) \cup \tau_+(\{p_i, 0 \leq i \leq q\})$.*

8.1 The proposed algorithm

The following method for the estimation of the invariant measure for a given transformation $\tau \in \mathcal{T}_{aex}(I)$ is valid for those maps that are stochastically stable, i.e., for which the strong mixing property (11) holds (as stated by Theorem 3 exact maps have this properties [13,14]). In particular, we recall from Theorem 3 that if $\tau \in \mathcal{T}_{exp}$ is exact then one unique invariant density $f^* \in D_{BV}(I)$, a positive constant K and a constant $0 < r < 1$ exist such that for any p with $1 \leq p \leq \infty$ and for any $n \in \mathbb{N}$

$$\left\|\Theta_\tau^n g - f^*\right\|_p < K r^n \left\|g\right\|_{BV}, \qquad \forall g \in D_{BV}(I). \tag{49}$$

The idea is to study the evolution of the uniform distribution for computing an estimation of the invariant density f^*. Accordingly, we first remark that if we consider f_0 as the uniform distribution over I, described by the matrix

$$H_0 = \begin{pmatrix} x_0(0) = a & l_0(0) = \frac{1}{b-a} \\ x_1(0) = b & l_1(0) = 0 \end{pmatrix}, \tag{50}$$

directly from Proposition 5 we have the following

Corollary 4 *Let f_0 be uniform. The matrix H_n has at most $2+2n(q-1)$ rows.*

The stochastic stability (11) of the system assures that the evolution of the uniform distribution $\{\Theta_\tau^n f_0\}$ is a converging sequence of densities in $BV(I)$. Since a converging sequence is always a Cauchy sequence, if we define $E_n = \operatorname{ess\,sup}_{x \in I} \left|\Theta_\tau^n f - \Theta_\tau^{n-1} f\right| = \left\|f_n - f_{n-1}\right\|_\infty$, we have that $\lim_{n\to\infty} E_n = 0$. Since the invariant density f^* is unknown a priori, we can not directly estimate the quantity $\left\|\Theta_\tau^n f - f^*\right\|_p$, and we refer to the quantity E_n for determining a criterion to stop the following algorithm:

1. set $n := 1$ and H_0 as in (50) (the uniform pdf);
2. compute the entries of the first column of H_1 by sorting the elements of the set $\{x_0(1), \ldots, x_{N(1)+1}(1)\} =$ $\tau_-(\{p_i, 0 \leq i \leq q\}) \cup \tau_+(\{p_i, 0 \leq i \leq q\})$;
3. compute the levels $l_0(n), \ldots, l_{N(n)}(n)$ by means of (19);
4. compute E_n and if $E_n \leq \varepsilon$ then `STOP`;
5. compute the entries of the first columns of H_{n+1} by sorting the elements of the set $\tau(\{x_0(n), \ldots, x_{N(n)+1}(n)\}) \cup \tau_-(\{p_i, 0 \leq i \leq q\}) \cup$ $\tau_+(\{p_i, 0 \leq i \leq q\})$;
6. set $n := n + 1$ and `REPEAT` from step 3);

Each step of this algorithm requires basic low-complexity computations, and in step 4) the quantity ε is the accepted threshold for E_n, introduced for stopping the algorithm. The number of rows of the involved matrices grows linearly with n, whereas the convergence rate is assured from (49) to be at least exponential[15].

[15] It is easy to prove that E_n approaches 0 faster than $2Kr^n \left\|g\right\|_{BV}$.

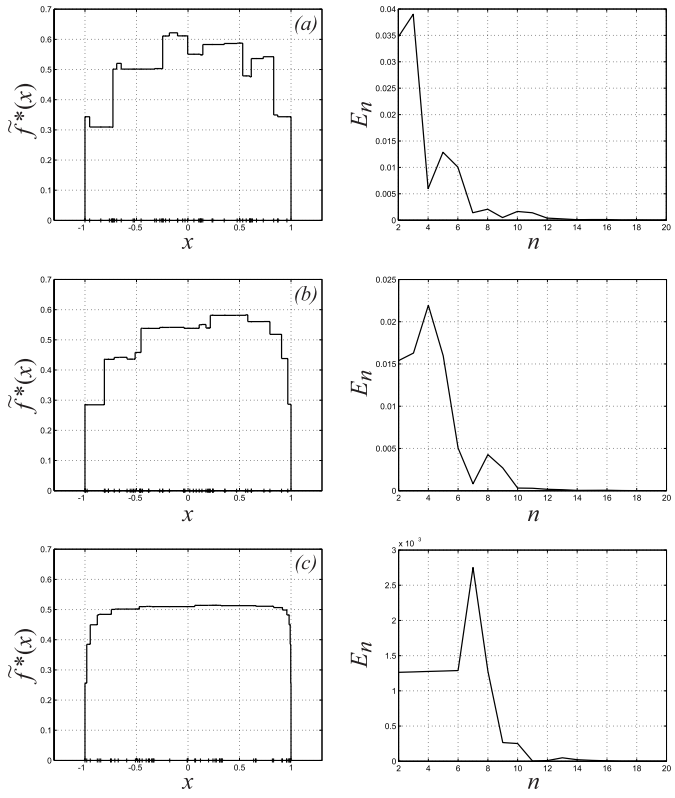


Figure 8: In the left column, the computed estimation of the invariant pdf f^* for system (16), when (a) $B = 1.78, \alpha = -0.0288$, (b) $B = 1.89, \alpha = -0.0414$ and (c) $B = 1.99, \alpha = 0.0036$, setting $\epsilon = 10^{-4}$. In the right column, the three related sequences $E_n = \operatorname{ess\,sup}_{x\in\Lambda} \left|\Theta_\tau^n f - \Theta_\tau^{n-1} f\right|$ are reported for $n = 2, \ldots, 20$.

Exploiting the proposed algorithm a piecewise constant pdf $\tilde{f}^*$ can be computed for approximating f^*, with an accuracy that can be – in principle – arbitrary.

8.2 Speed of convergence and estimation accuracy

The results of the proposed algorithm for the estimation of the invariant pdf are reported in Fig. 8 for three different parameter values of the Sawtooth map (16). In the left column the estimated pdfs are depicted, whereas in the right column the sequences E_n, for $n = 2, \ldots, 20$ are reported for each case. The E_1 values are omitted in the graphs to allow a magnification of the plots for small values of E_n (the omitted values are (a) $E_1 \approx 0.22$, (b) $E_1 \approx 0.24$, (c) $E_1 \approx 0.25$, i.e., they are much greater than the respective values of E_2). It is interesting noting that if $B \to 2$ the estimated pdf approaches the uniform distribution, according to the robustness property of the invariant measure for

the Sawtooth map, discussed in Section 6.1[16].

The accuracy and the speed of convergence of the algorithm is hereafter discussed; in particular we discuss how the halting threshold ϵ determines the accuracy of the obtained estimation of f^*. The problem of estimating the invariant density f^* is equivalent to finding an approximated solution of the functional equation

$$\Theta_\tau f^* = f^*, \qquad f^* \in D_{BV}(I). \tag{51}$$

In general, if $f \in D_{BV}(I)$ we can write $\Theta_\tau f = f + e$, where $e \in BV(I)$, and $E = \|\Theta_\tau f - f\|_\infty = \|e\|_\infty$. Accordingly, the halting threshold ϵ in the algorithm represents a direct measure of the accuracy with which the approximated solution of the functional equation (51) is calculated. Therefore, if n is the smallest integer such that $E_n < \epsilon$ then, by denoting $g = \Theta_\tau^{n-1} f_0$, we have

$$\Theta_\tau g = g + e, \qquad \text{with } \|e\|_\infty < \epsilon. \tag{52}$$

Depending on the accuracy of the required application, if ϵ is chosen sufficiently small we can assume $\Theta_\tau g \approx g$.

A further remark on the theoretical meaning of the sequence E_n is given by noting that we can write $f_n = \Theta_\tau^n(f_0) = f^* + e_n$, where $e_n \in BV(I)$ for all $n \in \mathbb{N}$ and the sequence $\{e_n\}$ is such that for any $1 \leq p \leq \infty$

$$\lim_{n\to\infty} \|e_n\|_p = \lim_{n\to\infty} \|e_n\|_{BV} = 0. \tag{53}$$

Accordingly, if $E_n = \|f_n - f_{n-1}\|_\infty$, due to the linearity of the Frobenius-Perron operator, we have $E_n = \|\Theta_\tau e_{n-1} - e_{n-1}\|_\infty$.

As it can be observed from the three examples in Fig. 8, the sequence E_n is not monotonically decreasing: we only know from (49) that it is bounded by a decreasing exponential curve depending on the two constants K and r. According to our experience and taking into account the piecewise affine systems considered in the literature for ICT applications [2–4,8], the number of required iterations is typically small (e.g., less than 20 iterations setting $\epsilon = 10^{-4}$). The proposed method converges not slower than exponentially, it requires basic and low-complexity calculations and it has a complexity that grows linearly with the iterations, resulting particularly suitable for performing the huge amount of computer simulations commonly required for evaluating the robustness of chaotic sources with respect to parameter random deviations (e.g., due to the hardware implementation). We have proposed such an application example of this algorithm in [29].

The ignorance of the constants K and r in (49) does not allow for setting *a priori* the number of the iterates to be executed before halting the algorithm. As a comment to this issue, it is worth recalling that the Frobenius-Perron operator is an infinite dimensional operator, and the problem of determining analytically the constants K and r is unfeasible in the majority of cases [2, 13]. These constants can be estimated numerically using the finite dimensional approximation of the Frobenius-Perron operator, but involving an algorithmic complexity that makes useless the method proposed in this paper. Indeed, the Frobenius-Perron operator is approximated with the transition probability matrix of a finite Markov chain, and the problem of finding an estimation for

[16] In this case the limit of the sequence of invariant measures associated with the converging sequence of Sawtooth maps is described by the uniform distribution.

the invariant density is related to quantifying the accuracy of the approximating Markov chain [2].

9 Conclusions

In this paper a theoretical approach for studying the effects of parameter perturbations on the chaotic statistics for piecewise affine expanding transformations is provided. In detail, the ergodic behavior of these systems is discussed adopting the approach developed by Boyarsky and Góra, with particular reference to the family of Sawtooth maps, and the robustness of their invariant measures is studied. Although the study is particularly focused on this specific family of maps, the authors stress that the proposed approach can be easily generalized to any family of piecewise affine maps considered in the literature. Moreover, in the paper an efficient method is proposed for estimating the unique invariant density of stochastically stable piecewise affine maps. The method converges not slower than exponentially, it requires basic and low-complexity calculations, and it has a complexity that grows linearly with the iterations. Therefore, it results particularly suitable for performing the huge amount of computer simulations commonly required for evaluating the robustness of chaotic stochastic sources with respect to parameter random deviations (e.g., due to the hardware implementation). The accuracy of the method is theoretically guaranteed.

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