# On the advantage of centralized strategies in the three-pursuer single-evader game


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**Abstract**

Cooperation in multi-pursuer games is known to be useful. However, it is not easy to quantify how much it is convenient for the pursuers to play according to a centralized strategy with respect to a decentralized one. This paper provides an answer to this question, for the problem of three pursuers chasing a single evader in a planar environment. It is shown that centralized pursuit algorithms can halve the time required to capture the evader, with respect to decentralized pursuit strategies. Moreover, this limit is proven to be tight. Numerical computations of lower bounds to the ratio between the capture times of centralized and decentralized strategies, show that for several game initial conditions the benefit of playing in a centralized way may be significantly less than halving the game duration.



*Keywords:* pursuit-evasion games, autonomous agents, cooperative control.


## 1. Introduction

Pursuit-evasion games have attracted researchers since long time, due to both the intriguing theoretical questions they pose and the wide variety of applications. The interested reader can refer to [1] for a historical perspective and to [2] for a taxonomy of the many different problem settings and solutions proposed in the literature. The classical approach to these problems is to cast them as differential games [3, 4] and then apply the vast body of results available in this context (see, e.g., [5, 6] and references therein). However, it is well known that the characterization of optimal solutions can be very difficult even for extremely simple problem formulations. A celebrated example is David Gale's lion and man problem [7], for which several suboptimal solutions have been devised [8, 9, 10], but an optimal one is yet to be found.

The complexity of pursuit-evasion games is further increased when multiple pursuers and/or evaders are involved. One reason why it is difficult to characterize the optimal solution of such problems is that the reachable sets of the pursuers and the evaders are typically nonconvex or disconnected sets. It may seem surprising that even in the basic setting of three pursuers chasing an evader in the plane, devising successful pursuit algorithms is far from being trivial and the strategy guaranteeing capture in minimum time is still an open problem [11]. Recent works have addressed multi-pursuer settings in which players move with different velocities [12, 13, 14]. In particular, results in [13, 14] further motivate the study of


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the three-pursuers against one-evader game, as it captures fundamental features of more general multi-pursuer settings. Pursuit problems in environments including obstacles or boundaries call for even more complex strategies, relying on quite involved geometric conditions [15, 16].

A fundamental question that has not been deeply addressed so far, concerns the value of adopting strategies based on the knowledge of the full game state (which we will refer to as "centralized"), with respect to algorithms based only on the local information available to each player. In the former case, information can be gathered by a supervisor or can be shared among pursuers, while in the latter, each pursuer has to rely only on its own local knowledge. In recent years, motivated by the widespread diffusion of new technologies allowing networks of autonomous agents to collaborate in the execution of complex tasks, a number of centralized solutions to multi-agent pursuit-evasion games have been proposed (see, e.g., [17, 18, 19, 20]). While it is widely recognized that the knowledge of the game state is helpful to achieve the common goal of the team, possibly at the price of a significant communication overhead, it is not easy to quantify the actual benefit for the agents to share information within a pursuit-evasion game.

For games involving many pursuers against a single evader, several centralized strategies have been proposed in the literature. Although the settings may be different, the common feature is that each pursuer's move depends on the entire (or partial) game state. In [21], asymptotic capture is guaranteed by collective minimization of the area of the evader's Voronoi cell. This approach has been extended to non-holonomic agents in [22], while uncertainty affecting the knowledge of the evader position has been considered in [23]. The cooperative chasing of a faster evader has been addressed in [24, 25]. The concept of dominance region has been employed in [26] to tackle a pursuit-evasion game in the presence of obstacles. In all these works, it is apparent that the knowledge of the whole game state is a key feature that allows the pursuers to capture the evader, but it is not clear *how much* it is useful. On the other hand, it is well known that in several multi-pursuer settings the evader can be captured by adopting decentralized strategies, like the one proposed in [27]. To the best of the authors knowledge, a quantitative evaluation of the advantage of centralized strategies with respect to decentralized ones, has not been provided yet, even in simple problem settings.

The objective of this paper is to investigate the benefit of adopting centralized strategies in a game involving three pursuers chasing one evader in the plane, in terms of the time required by the pursuers to capture the evader. The main result consists in showing that centralized algorithms can reduce the capture time up to one half of that required by the decentralized strategy proposed in [27]. This is achieved by first deriving the optimal evader strategy, and hence the maximum game length, for the problem studied in [27]. Then, upper and lower bounds on the game length are obtained for a generic centralized pursuer strategy. A further contribution is to present specific games in which such bounds are actually achieved. The numerical evaluation of the bounds for several initial game settings shows that the advantage of centralized strategies may be much smaller than halving the game duration. Although the considered setup is quite simple, these results provide a first assessment of the benefits of centralized algorithms in a pursuit-evasion game and may pave the way to further research in more complex settings. A preliminary version of this work has appeared in [28].

The paper is organized as follows. The formulation of the three-pursuer one-evader game is given in Section 2. Section 3 presents the optimal evader's strategy to counteract the decentralized pursuit algorithm proposed in [27]. A general lower bound on the game length, independent on the strategies adopted by the players, is obtained in Section 4. This is instrumental to derive the main result on the advantage of centralized strategies with respect to decentralized ones, in Section 5. Numerical evaluations of this comparison are reported in Section 6 for several game examples. Conclusions and future developments are discussed in Section 7.

## 2. Pursuit-evasion game

### *2.1. Notation*

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space and $\|\cdot\|$ be the Euclidean norm. The transpose of a vector $v$ is denoted by $v'$. For $V, W \in \mathbb{R}^2$, we denote by $\overline{VW}$ the segment with $V$ and $W$ as endpoints. Let $V, v \in \mathbb{R}^2$, then $\mathcal{L}(V, v)$ denotes the line passing through $V$ with direction $v$, i.e.,

$$\mathcal{L}(V, v) = \{X \in \mathbb{R}^2 \colon X = V + \alpha v,\ \alpha \in \mathbb{R}\}.$$

### *2.2. Problem formulation*

A pursuit-evasion game involving three pursuers is considered. It is assumed that the players move in an open and empty two-dimensional environment. Let $E(t) \in \mathbb{R}^2$ and $P_i(t) \in \mathbb{R}^2,\ i = 1, 2, 3$, denote the evader and pursuers' locations at time $t$, respectively. A first-order continuous-time motion model is assumed for the players

$$\begin{cases} \dot{E}(t) = e(t)\,, \\ \dot{P}_i(t) = w_i(t)\,,\ i = 1, 2, 3\,, \end{cases} \tag{1}$$

where $e(t) \in \mathbb{R}^2$ and $w_i(t) \in \mathbb{R}^2$, $i = 1, 2, 3$. Moreover, we assume that the pursuers and the evader have the same speed, set to 1 without loss of generality, i.e., $\|e(t)\| = 1$ and $\|w_i(t)\| = 1$.

The aim of the pursuers is to capture the evader, i.e., to achieve $P_i(t) = E(t)$ for at least one $i \in \{1, 2, 3\}$, at some finite time $t$. The following assumption is enforced throughout the paper.

**Assumption 1.** *The initial evader position is strictly inside the convex hull of the pursuers.*

If the players move at the same speed, enforcing Assumption 1 is standard, otherwise the evader may easily escape going straight along a direction opposite to the convex hull of the pursuers [29]. On the contrary, if Assumption 1 holds, there exist pursuers' strategies which guarantee capture of the evader in finite time [29, 27].

In pursuit-evasion games, the game cost is in general a function of the entire game state (see, e.g., [5]). In this work, we assume as cost function the time at which capture occurs: the pursuers aim at minimizing it, while the evader tries to maximize it. Thus, a pursuers' strategy is said optimal (for a given evader's strategy) when it guarantees capture in minimum time, while an evader's strategy is optimal (for a given pursuers' strategy) if it guarantees survival of the evader for the longest time. The choice of the time-to-capture as the game cost is common in pursuit-evasion games and provides a meaningful way to assess the performance of different pursuit strategies.

For a given configuration of the players at time $t$, let us define as $\mathcal{V}(t)$ the Voronoi cell associated to the evader, i.e., the region of the plane closer to the evader than to the pursuers, at time $t$. Under Assumption 1, $\mathcal{V}(t)$ turns out to be a triangle; let us denote by $V_i(t),\ i = 1, 2, 3$, its vertices. For a given triangle $\mathcal{V}$, we denote by $l, m, s$ the longest, medium and shortest edge of $\mathcal{V}$, respectively. Moreover, we name the vertices of $\mathcal{V}$ such that $V_1$ is the vertex joining the longest and medium edges, while pursuers are labeled such that $P_i$ is the pursuer farthest from $V_i$. It can be easily observed that (see Fig. 1)

$$\|V_i - P_i\| > \|V_i - E\| \ , \ i = 1, 2, 3 \, . \tag{2}$$

In this paper, two classes of pursuers' strategies are considered: *centralized* and *decentralized*. A pursuers' strategy is said decentralized if each pursuer does not have information about the other pursuers and chooses its motion solely on the base of its own position $P_i(t)$ and the evader position and control, i.e.

$$w_i(t) = \pi_i(P_i(t), E(t), e(t)).$$

On the contrary, in the centralized case, each pursuer chooses its motion as a function of the full game state and the control of the evader, i.e.

$$w_i(t) = \pi_i(P_1(t), P_2(t), P_3(t), E(t), e(t)). \tag{3}$$

In both cases, we assume the evader has a complete knowledge of the state of the game and we denote its strategy as

$$e(t) = \sigma(P_1(t), P_2(t), P_3(t), E(t)).$$

Throughout the paper, only pursuit strategies $\pi$ that guarantee capture of the evader in finite time will be considered. The time at which capture occurs is clearly a function of the pursuers' and evader's strategies and will be denoted by $M(\pi, \sigma)$.

## 3. Decentralized pursuit strategy

In this section, a decentralized pursuers' strategy is recalled and the corresponding optimal evader's strategy is derived. The pursuers' strategy has been proposed in [30] for the continuous-time framework considered in this paper, and in [27] (under the name "Planes") within a discrete-time setting. Such a strategy is designed in $\mathbb{R}^n$ and it guarantees capture in finite time under Assumption 1. In this paper, we will restrict the analysis to the two-dimensional space.

Let $C_i(t) = (P_i(t) + E(t))/2$ and $z_i(t) = P_i(t) - E(t)$, $i = 1, 2, 3$. Denote by $z_i(t)^{\perp}$ a vector orthogonal to $z_i(t)$ and set

$$B_i(t) = \mathcal{L}(E(t), e(t)) \cap \mathcal{L}(C_i(t), z_i(t)^{\perp}) \, .$$

Let us define the pursuers' motion $w_i(t)$, $i = 1, 2, 3$, as follows (see Fig. 1)

$$w_i(t) = \pi_{\mathcal{D},i}(P_i(t), E(t), e(t)) = \begin{cases} e(t) & \text{if } z_i(t)' e(t) \leq 0 \, , \qquad \text{(4a)} \\ \dfrac{B_i(t) - P_i(t)}{\|B_i(t) - P_i(t)\|} & \text{if } z_i(t)' e(t) > 0 \, . \qquad \text{(4b)} \end{cases}$$

We refer to the decentralized pursuers' strategy in (4a)-(4b) as $\mathcal{D}$-strategy. In words, when the evader moves towards an edge of the Voronoi cell (condition (4b)), the corresponding pursuer makes a specular move which leaves the edge unchanged. Conversely, if the evader moves away from an edge of the Voronoi cell (condition (4a)), the pursuer makes the same move of the evader, thus causing a shrinking of the Voronoi cell (in Fig. 1, this occurs for pursuer $P_3$). It is worth remarking that in the latter case, the

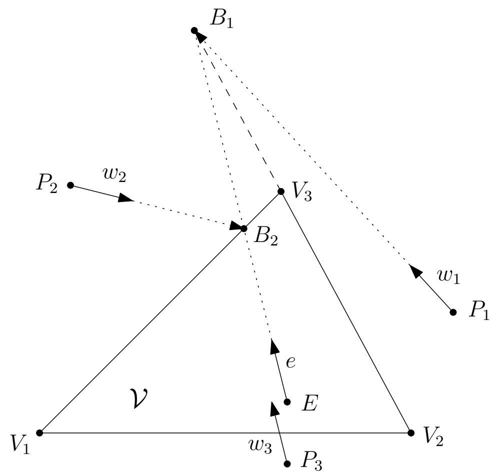


Figure 1: Example of pursuers' moves based on the $\mathcal{D}$-strategy. According to (4b), $P_1$ and $P_2$ move towards $B_1$ and $B_2$, respectively. Pursuer $P_3$ moves parallel to $e(t)$, obeying to (4a).

direction of the edge of the Voronoi cell does not change. Therefore, all the Voronoi cells throughout the entire game are similar triangles, no matter of the path followed by the evader.

In the sequel, an optimal evader's strategy is devised for games in which the pursuers play the $\mathcal{D}$-strategy. Let us name such strategy as $\mathcal{E}$. It is worthwhile to notice that there exist several evader strategies which lead to the same optimal capture time; we will just focus on one of them.

Let $E(0)$, $P_i(0)$, $i = 1, 2, 3$, be given, and let $\mathcal{V}(0)$ be the corresponding Voronoi cell. Without loss of generality let us assume the longest edge of $\mathcal{V}(0)$ be $\|V_1(0) - V_2(0)\| = l$. Define the unitary vectors connecting the vertices of $\mathcal{V}(0)$ as

$$v_{ij} = \frac{V_i(0) - V_j(0)}{\|V_i(0) - V_j(0)\|}\ , \quad i \neq j\,. \tag{5}$$

Recalling that, by definition of $V_1$, $||V_1(0) - V_3(0)|| > ||V_2(0) - V_3(0)||$, denote by $Q$ and $S$ the intersection points between the line passing through $E(0)$ parallel to $v_{12}$ and $V_1(0) - V_3(0)$ and $V_2(0) - V_3(0)$, respectively (see Fig. 2), i.e.,

$$Q = \overline{V_1(0)V_3(0)} \cap \mathcal{L}(E(0), v_{12})\,, \tag{6}$$

$$S = \overline{V_2(0)V_3(0)} \cap \mathcal{L}(E(0), v_{12})\,. \tag{7}$$

Let us define

$$v_{QE} = \frac{Q - E(0)}{\|Q - E(0)\|}\ ,\ v_{V_1Q} = \frac{V_1(0) - Q}{\|V_1(0) - Q\|}\,.$$

Let us now formulate the evader's strategy $\mathcal{E}$ as follows (see Fig. 2).

a) From $E(0)$ the evader moves along $v_{QE}$ to $Q$.

b) Once in $Q$, it moves along $v_{V_1Q}$ to $V_1(0)$.

c) Once in $V_1(0)$, it moves towards $V_2(0)$, until it reaches the farthest vertex of the current Voronoi cell, where it is captured.

**Remark 1.** *In the formulation of the $\mathcal{E}$-strategy and in the rest of the paper, we adopt a slight abuse of terminology in order to simplify the exposition. When it is stated that the evader moves to a point which lies on the boundary of $\mathcal{V}$, it is actually meant that it moves to an interior point of $\mathcal{V}$ which is arbitrarily close to the boundary. Indeed, such a move is always feasible and safe, due to the fact that the evader can reach any point inside $\mathcal{V}$ without being captured, by definition of the Voronoi cell. For instance, referring to stage a) of the $\mathcal{E}$-strategy, the evader will actually move along $v_{QE}$ to a point $\widetilde{Q} \in \mathcal{V}$ such that $\|\widetilde{Q} - Q\| < \delta$, for a small $\delta > 0$, and then switch to stage b). Thus, the resulting path traveled by the evader before being captured (and consequently also the game length) will be functions of $\delta$ and the supremum of such values can be achieved by letting $\delta$ tend to 0. In this respect, all the results presented in the paper must be intended as limit results, for $\delta \to 0$.*

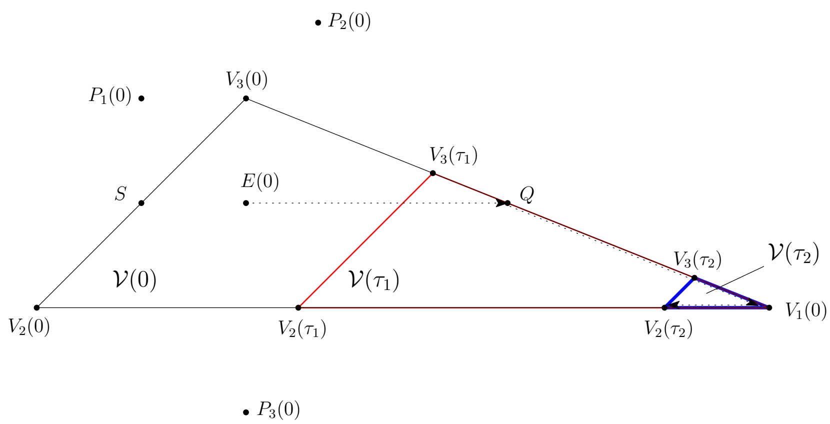


Figure 2: Sketch of the evader's strategy $\mathcal{E}$. The evader follows the dotted arrows performing a 3-step path. In the first step, it moves from $E_0$ to $Q$, while in the second one it goes towards $V_1(0)$. Finally, in the third step, it reaches $V_2(\tau_2)$ where the capture occurs. The Voronoi cells at the beginning of steps 1 ($\mathcal{V}(0)$), 2 ($\mathcal{V}(\tau_1)$) and 3 ($\mathcal{V}(\tau_2)$) are depicted in black, red and blue, respectively.

The next theorem states that the $\mathcal{E}$-strategy is optimal for the evader when the pursuers play the $\mathcal{D}$-strategy, and provides the related game length $M_{\mathcal{D}}$.

**Theorem 1 (Game length for decentralized strategy).** *Let the pursuers play the $\mathcal{D}$-strategy and the evader play the $\mathcal{E}$-strategy. Then, the game will terminate after a time*

$$M_{\mathcal{D}} = \|S - Q\| + \|Q - V_1(0)\| . \tag{8}$$

*Moreover, the $\mathcal{E}$-strategy is optimal for the evader, i.e.*

$$M_{\mathcal{D}} = \sup_{\sigma} M(\pi_{\mathcal{D}}, \sigma) \tag{9}$$

*where the supremum in* (9) *is taken over all possible evader strategies.*

*Proof:* See appendix. □

**Remark 2.** *In [30], an upper bound to the capture time is reported for the generic game played in $\mathbb{R}^n$ involving $m$ pursuers. Let $z_i = P_i - E$, by fixing $n = 2$ and $m = 3$, such bound turns out to be*

$$B_P = \frac{\max\limits_{i=1,2,3} \|z_i\|}{\delta_0},$$

*with*

$$\delta_0 = \min_{\|p\|=1} \max_{i=1,2,3} \frac{p' z_i}{\|z_i\|}\,.$$

*It can be shown that $B_P$ is in general much larger than the exact capture time given by Theorem 1. In fact, for $10^6$ randomly generated game initial conditions, the ratio between $B_P$ and $M_{\mathcal{D}}$ turned out ranging from about 1.2 to over 3000.*

## 4. A general lower bound on capture time

Assume now that the pursuers have access to the full game state and play a centralized strategy (3). The next theorem gives a lower bound $\underline{B}$ on the capture time, i.e., the evader, playing a suitable strategy, may avoid capture for at least a time $\underline{B}$, for any possible pursuers' strategy.

**Theorem 2 (Lower bound on time-to-capture).** *No matter which strategy is played by the pursuers, the evader is able to survive for at least a time $\underline{B}$, where*

$$\underline{B} = \max_{i=1,2,3} \frac{1}{2} \left(\|V_i(0) - P_i(0)\| + \|V_i(0) - E(0)\|\right) \tag{10}$$

$$= \max_{i=1,2,3} \left(\|V_i(0) - E(0)\| + \frac{1}{2} \left(\|V_i(0) - P_i(0)\| - \|V_i(0) - E(0)\|\right)\right). \tag{11}$$

*Proof:* Let us consider the following evader's strategy. From its initial position $E(0)$, it moves straight to $V_i(0)$ for a time $\tau_1 = \|E(0) - V_i(0)\|$ for a given $i \in \{1, 2, 3\}$. Since $V_i(0) \in \mathcal{V}(0)$, it can be arbitrarily approached, irrespectively of the pursuers' strategy.

Let $d_0 = \|P_i(0) - V_i(0)\|$. By (2), one has

$$d_0 = \|P_i(0) - V_i(0)\| > \|E(0) - V_i(0)\| = \tau_1\,. \tag{12}$$

Since the speed of the pursuers is set to 1, by (12) the distance between the evader and the pursuer $P_i$ at time $\tau_1$ is such that

$$\|P_i(\tau_1) - E(\tau_1)\| = \|P_i(\tau_1) - V_i(0)\| \geq \|P_i(0) - V_i(0)\| - \tau_1 = d_0 - \tau_1 > 0\,.$$

Define $d_{\tau_1} = d_0 - \tau_1 > 0$. Let $Z = (E(\tau_1) + P_i(\tau_1))/2$ be the midpoint between $E(\tau_1)$ and $P_i(\tau_1)$. By the definition of Voronoi cell, $Z$ lies on the boundary of $\mathcal{V}(\tau_1)$. Assuming the evader goes straight to $Z$, it covers a distance

$$\|Z - E(\tau_1)\| = \frac{\|P_i(\tau_1) - E(\tau_1)\|}{2} \geq \frac{d_{\tau_1}}{2}\,,$$

and then it is captured in $Z$. Hence, the time needed to cover the entire path turns out to be

$$T \geq \tau_1 + \frac{d_{\tau_1}}{2} = \tau_1 + \frac{d_0 - \tau_1}{2} = \frac{1}{2}(\tau_1 + d_0) = \frac{1}{2}(\|V_i(0) - E(0)\| + \|V_i(0) - P_i(0)\|)\,. \tag{13}$$

Therefore, the right hand side of (13) is a lower bound to the evader's survival time. By taking the maximum with respect to $i = 1, 2, 3$, one gets the lower bound $\underline{B}$ in (10). The expression (11) follows from straightforward manipulations. □

## 5. Advantages of centralized pursuit strategies

The aim of this section is to assess the potential advantage of centralized pursuit strategies with respect to decentralized ones, and in particular to the $\mathcal{D}$-strategy introduced in Section 3. Let us denote by $\mathcal{C}$ the optimal centralized pursuers' strategy and by $M_{\mathcal{C}}$ the related maximum capture time, i.e.

$$M_{\mathcal{C}} = \inf_{\pi} \sup_{\sigma} M(\pi, \sigma) \tag{14}$$

The main result of this section consists in showing that $M_{\mathcal{C}}$ cannot be smaller than $\frac{1}{2}M_{\mathcal{D}}$, thus meaning that the maximum advantage of centralized pursuit strategies amounts to halving the capture time, with respect to decentralized ones. The following lemmas are instrumental to prove the main results. Hereafter, the time dependence is omitted when it is clear from the context.

**Lemma 1.** *Let the pursuers play the $\mathcal{D}$-strategy and let $l$ denote the longest edge of $\mathcal{V}$. Then*

$$M_{\mathcal{D}} \leq l\,. \tag{15}$$

*Proof:* Let $Q$ and $S$ be defined as in (6)-(7) and assume $l = \|V_1 - V_2\|$, see Fig. 3. It holds

$$\frac{\|V_1 - V_3\|}{\|V_1 - V_2\|} = \frac{\|Q - V_3\|}{\|Q - S\|} = \frac{\|V_1 - V_3\| - \|Q - V_3\|}{\|V_1 - V_2\| - \|Q - S\|}\,.$$

Since $\|V_1 - V_3\| \leq \|V_1 - V_2\|$ one has

$$\|V_1 - V_3\| - \|Q - V_3\| \leq \|V_1 - V_2\| - \|Q - S\|\,,$$

or equivalently

$$\|Q - V_1\| \leq \|V_1 - V_2\| - \|Q - S\|\,.$$

So, by Theorem 1,

$$M_{\mathcal{D}} = \|Q - S\| + \|Q - V_1\| \leq \|Q - S\| + \|V_1 - V_2\| - \|Q - S\| = l\,.$$

□

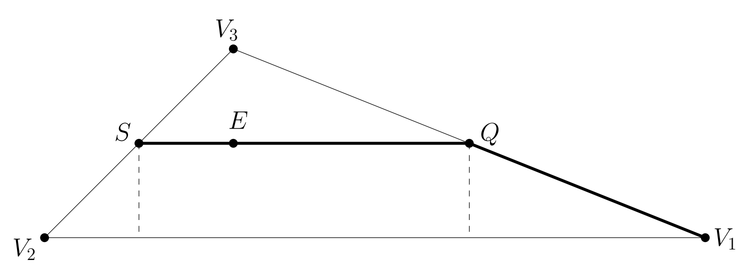


Figure 3: Illustration of the maximum game length $M_{\mathcal{D}}$ when pursuers play the $\mathcal{D}$-strategy. Since players' speed is set to 1, $M_{\mathcal{D}}$ corresponds to the length of the bold line, i.e., $M_{\mathcal{D}} = \|Q - S\| + \|Q - V_1\|$. It also coincides with the length of the evader path depicted in Fig. 2.

**Lemma 2.** *Let $\underline{B}$ be given as in Theorem 2 and let $l$ denote the longest edge of $\mathcal{V}$. Then,*

$$\underline{B} \geq l/2\,. \tag{16}$$

*Proof:* Let $l = \|V_1 - V_2\|$. By the triangle inequality,

$$2\max\{\|E - V_1\|, \|E - V_2\|\} \geq \|E - V_1\| + \|E - V_2\| \geq \|V_1 - V_2\| = l\,.$$

By (2) and (11), one has

$$\underline{B} \geq \max_{i=1,\ldots,3} \|V_i - E\| \geq \max\{\|E - V_1\|, \|E - V_2\|\} \geq l/2\,.$$

□

We are now ready to prove the main result of the section.

**Theorem 3 (Advantage of centralized strategies).** *Let $M_{\mathcal{D}}$ be given by (8) and $M_{\mathcal{C}}$ be the optimal game length in a centralized pursuers' setting. Then,*

$$M_{\mathcal{C}} \leq M_{\mathcal{D}} \leq 2M_{\mathcal{C}}\,. \tag{17}$$

*Proof:* According to (9) and (14), one has $M_{\mathcal{C}} \leq M_{\mathcal{D}}$. By Lemmas 1 and 2, it turns out that

$$M_{\mathcal{D}} \leq l \leq 2\underline{B} \leq 2M_{\mathcal{C}}\,, \tag{18}$$

where the last inequality comes from the fact that $\underline{B}$ is a lower bound on the game length for any pursuers' strategy, as stated by Theorem 2. □

In the sequel, we show that there indeed exist games in which

$$M_{\mathcal{C}} = M_{\mathcal{D}}\,, \tag{19}$$

and others in which

$$M_{\mathcal{C}} = \frac{1}{2}M_{\mathcal{D}}\,, \tag{20}$$

thus meaning that both bounds in (17) are tight.

**Theorem 4 (Tightness of lower bound).** *There exist games such that $M_{\mathcal{C}} = M_{\mathcal{D}}$.*

*Proof:* To prove the theorem, we show that there exist games for which $M_{\mathcal{D}} = \underline{B}$. In fact, being $\underline{B} \leq M_{\mathcal{C}} \leq M_{\mathcal{D}}$, then $M_{\mathcal{D}} = \underline{B}$ implies $M_{\mathcal{C}} = M_{\mathcal{D}}$. Let us choose a game initial condition such that $\mathcal{V}$ is a right triangle and let us adopt the notation shown in Fig. 4 in which $M_{\mathcal{D}} = \|S - Q\| + \|Q - V_1\|$. Since $\overline{SQ}$ is the hypotenuses of the triangle with vertices $S$, $Q$ and $V_3$, by Theorem 1, one easily gets $M_{\mathcal{D}} \geq \|V_1 - V_3\| = m$. Moreover, by (15), it holds

$$m \leq M_{\mathcal{D}} \leq l\,.$$

Let the smallest edge $s$ shrink to 0. One has

$$\lim_{s\to 0} l = \lim_{s\to 0} \sqrt{m^2 + s^2} = m\,,$$

and hence

$$\lim_{s\to 0} M_{\mathcal{D}} = m\,. \tag{21}$$

Moreover, it is easy to show that as $s \to 0$, $P_1, E$ and $V_1$ tend to be collinear. This implies that

$$\lim_{s\to 0} (\|V_1 - P_1\| - \|V_1 - E\|) = \lim_{s\to 0} \|P_1 - E\|\,.$$

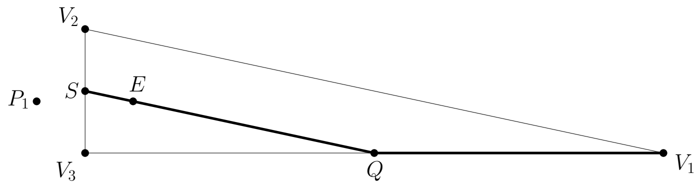


Figure 4: Proof of Theorem 4. The length of the bold line is equal to $M_{\mathcal{D}}$. As $\|V_2 - V_3\| \to 0$, both $M_{\mathcal{D}}$ and $\underline{B}$ tend to $\|V_1 - V_3\|$ and then $M_{\mathcal{D}} \to \underline{B}$, thus implying $M_{\mathcal{D}} = M_{\mathcal{C}}$.

By (11), one has

$$\lim_{s\to 0} \underline{B} \geq \lim_{s\to 0} \left( \|V_1 - E\| + \frac{1}{2}(\|V_1 - P_1\| - \|V_1 - E\|) \right) = \lim_{s\to 0} \left( \|V_1 - E\| + \frac{1}{2}\|P_1 - E\| \right) = m\,. \tag{22}$$

Since $M_{\mathcal{D}} \geq \underline{B}$, by (21) and (22) one has

$$\lim_{s\to 0} M_{\mathcal{D}} = \lim_{s\to 0} \underline{B} = m\,,$$

which concludes the proof. □

**Theorem 5 (Tightness of upper bound).** *There exists games such that $M_{\mathcal{D}} = 2M_{\mathcal{C}}$.*

*Proof:* We show that there exist games in which $M_{\mathcal{C}} = \underline{B}$ and $M_{\mathcal{D}} = 2\underline{B}$. Let $\mathcal{V}(0)$ be the isosceles triangle depicted in Fig. 5. Define $H, T_1, T_2$ the projections of $V_3(0), P_1(0), P_2(0)$ on $\overline{V_1(0)V_2(0)}$, respectively. Let the evader initially lie on the segment $\overline{V_3(0)H}$, define $\varepsilon = \|V_3(0) - H\|$ and $K = E(0) + 2(V_3(0) - E(0))$. Since $\|V_3(0) - P_1(0)\| = \|V_3(0) - E\| = \|V_3(0) - K\|$ one has

$$\begin{aligned}
\|P_1(0) - T_1\| &\leq \|P_1(0) - V_3(0)\| + \|V_3(0) - H\| \\
&= \|V_3(0) - K\| + \|V_3(0) - H\| \\
&\leq 2\|V_3(0) - H\| = 2\varepsilon\,.
\end{aligned} \tag{23}$$

Now, let $\varepsilon$ tend to 0. In Fig. 5, $M_{\mathcal{D}}$ corresponds to the length of the bold line. By following a similar reasoning as in the proof of Theorem 4, one has

$$\lim_{\varepsilon\to 0} M_{\mathcal{D}} = \lim_{\varepsilon\to 0} \|S - Q\| + \|Q - V_1(0)\| = \|V_2(0) - V_1(0)\|\,. \tag{24}$$

Let us now introduce a two-step centralized pursuers' strategy, denoted by $\widehat{\mathcal{C}}$. Let $v_{QE} = Q - E(0)$. Assume first that the evader starts moving in the half-plane $v_{QE}' e(t) \geq 0$, i.e., the evader moves to the right in Fig. 5 (the case in which the evader starts moving in the opposite direction is similar). The strategy $\widehat{\mathcal{C}}$ is defined as follows.

a) Initially, $P_1$ moves towards $T_1$, while $P_2$ and $P_3$ move symmetrically to the evader with respect to $\overline{V_1(0)V_3(0)}$ and $\overline{V_1(0)V_2(0)}$, respectively, see Fig. 6.

b) As soon as $(P_1 - E)$ is parallel to $(V_1 - V_2)$, pursuers play the decentralized strategy $\mathcal{D}$ until capture occurs.

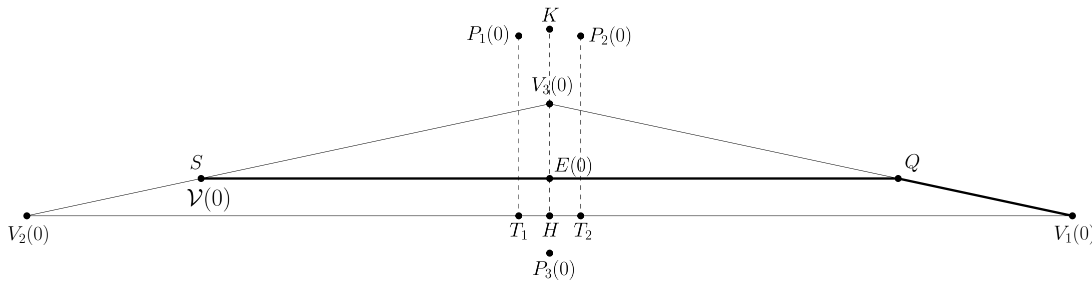


Figure 5: Proof of Theorem 5. Illustration of the Voronoi cell $\mathcal{V}$ at time 0 and of projections $T_1$, $T_2$ and $H$. The length of the bold line corresponds to $M_{\mathcal{D}}$.

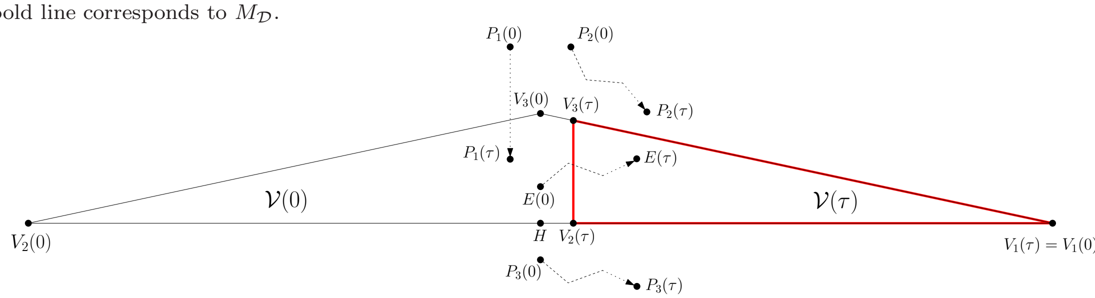


Figure 6: Proof of Theorem 5. First step of strategy $\widehat{\mathcal{C}}$. Players move along the dotted arrows. Pursuer $P_1$ moves downward, while $P_2$ and $P_3$ move symmetrically to the evader w.r.t. the segments $\overline{V_1(0)V_3(0)}$ and $\overline{V_1(0)V_2(0)}$, respectively. Voronoi cells $\mathcal{V}$ at time 0 and at time $\tau$ are depicted in black and red, respectively. As the triangle get squeezed, due to $\|V_3 - H\| \to 0$, one has $M_{\mathcal{D}} \to 2M_{\mathcal{C}}$, i.e., the decentralized $\mathcal{D}$-strategy needs a double time to capture the evader, w.r.t. the $\widehat{\mathcal{C}}$-strategy.

Notice that in step a), $P_2$ and $P_3$ move in such a way to guarantee that $V_1$, $v_{12}$ and $v_{13}$ remain the same, where $v_{12}$ and $v_{13}$ are defined as in (5). Therefore, there exists a finite time $\tau$ at which the $\widehat{\mathcal{C}}$ strategy switches from step a) to step b). Since, by (23), $\|P_1(0) - T_1\| \le 2\varepsilon$, one has $\tau \le 2\varepsilon$. Notice that, by letting $\varepsilon$ tend to 0, the duration of step a) can be made arbitrarily short. Now, let $\eta$ be the duration of step b), until capture occurs. Hence, $M_{\widehat{\mathcal{C}}} = \tau + \eta \le 2\varepsilon + \eta$. Observe that, due to the switching condition to step b), at time $\tau$ the Voronoi cell becomes a right triangle like the one in the proof of Theorem 4, see Fig. 6. So, by using the same argument as in the proof of Theorem 4, one has $\lim_{\varepsilon\to 0} \eta = \|V_1(\tau) - V_2(\tau)\|$. Then, it holds

$$\begin{aligned}\lim_{\varepsilon\to 0} M_{\widehat{\mathcal{C}}} &= \lim_{\varepsilon\to 0} \tau + \|V_1(\tau) - V_2(\tau)\| \\ &\le \lim_{\varepsilon\to 0} 2\varepsilon + \|V_1(\tau) - V_2(\tau)\| \\ &= \lim_{\varepsilon\to 0} \|V_1(\tau) - V_2(\tau)\| \\ &\le \|V_1(0) - H\|\,,\end{aligned}$$

where the last inequality comes from the fact that $V_1(\tau) = V_1(0)$. Thus, by (24) and by the fact that the optimal centralized strategy is such that $M_{\mathcal{C}} \le M_{\widehat{\mathcal{C}}}$, one has

$$\lim_{\varepsilon\to 0} M_{\mathcal{C}} \le \lim_{\varepsilon\to 0} M_{\widehat{\mathcal{C}}} \le \|V_1(0) - H\| = \frac{1}{2}\|V_2(0) - V_1(0)\| = \frac{1}{2}\lim_{\varepsilon\to 0} M_{\mathcal{D}}\,.$$

Since by (17), $M_{\mathcal{C}} \ge M_{\mathcal{D}}/2$, one gets

$$\lim_{\varepsilon\to 0} M_{\mathcal{C}} = \frac{1}{2}\lim_{\varepsilon\to 0} M_{\mathcal{D}}\,,$$

which conclude the proof. □

**Remark 3.** *The two-step pursuers' strategy* $\widehat{\mathcal{C}}$ *adopted in the proof of Theorem 5 is centralized because during step a),* $P_1$ *moves orthogonally to the largest edge of* $\mathcal{V}(0)$*. Since the vertices of* $\mathcal{V}(0)$ *are defined by the position of all the pursuers (and the evader), it is apparent that the* $\widehat{\mathcal{C}}$ *strategy requires that the pursuers have full knowledge of the game state.*

By Theorem 3, it follows that pursuers playing in a centralized way may reduce the game duration by at most a factor two, with respect to pursuers adopting a decentralized strategy. For a given game, the improvement of using a centralized strategy instead of a decentralized one can be summarized by the index

$$\delta = \frac{M_{\mathcal{C}}}{M_{\mathcal{D}}}\,. \tag{25}$$

By Theorem 3, it follows that

$$\frac{1}{2} \leq \delta \leq 1\,. \tag{26}$$

Moreover, as stated in Theorems 4 and 5, these bounds are tight in the sense that there exist games in which they can be achieved.

For specific game initial conditions, the index $\delta$ in (25) is difficult to evaluate, because the optimal centralized pursuers' strategy is unknown and therefore the actual value of $M_{\mathcal{C}}$ cannot be computed. Hence, it is useful to introduce the lower bound

$$\underline{\delta} = \frac{\underline{B}}{M_{\mathcal{D}}}\,. \tag{27}$$

Since $M_{\mathcal{C}} \geq \underline{B}$, one has $\delta \geq \underline{\delta}$. Moreover, by (18), one has $M_{\mathcal{D}} \leq 2\underline{B}$ and hence $\underline{\delta} \geq 1/2$. So, the bounds in (26) can be refined as

$$\frac{1}{2} \leq \underline{\delta} \leq \delta \leq 1\,. \tag{28}$$

Notice that, for a specific game, the lower bound $\underline{\delta}$ can be easily computed, since both $M_{\mathcal{D}}$ and $\underline{B}$ can be readily evaluated by (8) and (10), respectively.

## 6. Numerical evaluation of lower bounds

In this section, the lower bound $\underline{\delta}$ is computed for a number of different game initial conditions. Clearly, the larger is $\underline{\delta}$, the smaller improvement can be obtained by playing in a centralized manner.

In Fig. 7, some examples of initial Voronoi cells $\mathcal{V}(0)$ are reported. For each initial position of the evader $E(0)$ inside $\mathcal{V}(0)$, the corresponding color denotes the value of $\underline{\delta}$. The positions of the pursuers are not reported since they depend on $E(0)$ and can be derived from $\mathcal{V}(0)$ and $E(0)$.

Several remarks can be made on the results depicted in Fig. 7. First, one may notice that if $E(0)$ is close to a vertex of $\mathcal{V}(0)$ then $\underline{\delta}$ approaches 1, and hence no improvement can be obtained by playing centralized pursuers' strategies. This fact is not surprising since these cases correspond to initial conditions in which two pursuers are close to the evader and to the corresponding vertex; in this situation, the longest path the evader may travel is towards the farthest vertex of $\mathcal{V}(0)$, where it will be captured, even by pursuers playing the $\mathcal{D}$-strategy.

If $\mathcal{V}(0)$ is an equilateral triangle (Fig. 7-a), the minimum value of $\underline{\delta}$ is $\sqrt{3}/2$, which is attained when $E(0)$ lies in the geometrical center of $\mathcal{V}(0)$. If $\mathcal{V}(0)$ is an isosceles triangle with all acute angles, the performance improvement in playing centralized strategies is remarkably small: for the games reported in Fig. 7-b, one has $\underline{\delta} \geq 0.968$ for any possible initial condition of the evader. This means that the maximum advantage of the pursuers playing in a centralized way with respect to the $\mathcal{D}$-strategy is about 3%. A different situation arises when dealing with obtuse isosceles triangles. In this case, for a wide obtuse angle and a suitable choice of $E(0)$, $\underline{\delta}$ may approach 0.5. Notice that this is consistent with the argument in the proof of Theorem 5. For the games illustrated in Fig. 7-c, where the obtuse angle is equal to 165°, one has $0.526 \leq \underline{\delta} \leq 1$. Fig. 7-d reports the case of a right triangle, where $\underline{\delta}$ ranges from 0.901 to 1. Finally, generic obtuse and acute scalene triangles are given in Fig. 7-e and 7-f, for which $\underline{\delta} \geq 0.63$ and $\underline{\delta} \geq 0.913$, respectively.

## 7. Conclusions

The benefits of adopting centralized pursuit strategies with respect to decentralized ones, in the three-pursuer single-evader game, have been quantified. The main result shows that the knowledge of the full game state allows a pursuer team to reduce the capture time by a factor two. Although the considered setting is admittedly simple and quite specific, the result is a first contribution towards the quantitative evaluation of the advantages provided by centralized algorithms in pursuit-evasion games. Clearly, the problem deserves deeper investigation in many directions. The extension to the case of $n$ pursuers, although non trivial, may reveal a different gap between the performance of centralized and decentralized strategies. Further studies should consider more complex game settings, such as 3D environments, non-holonomic motion models, different agent speeds, or the presence of obstacles. For all these scenarios, it is believed that there is room for assessing the actual improvement provided by centralized strategies over decentralized ones.


## Acknowledgment

This work has been supported by the Italian Ministry for Research in the framework of the 2017 Program for Research Projects of National Interest (PRIN), Grant No. 2017YKXYXJ.


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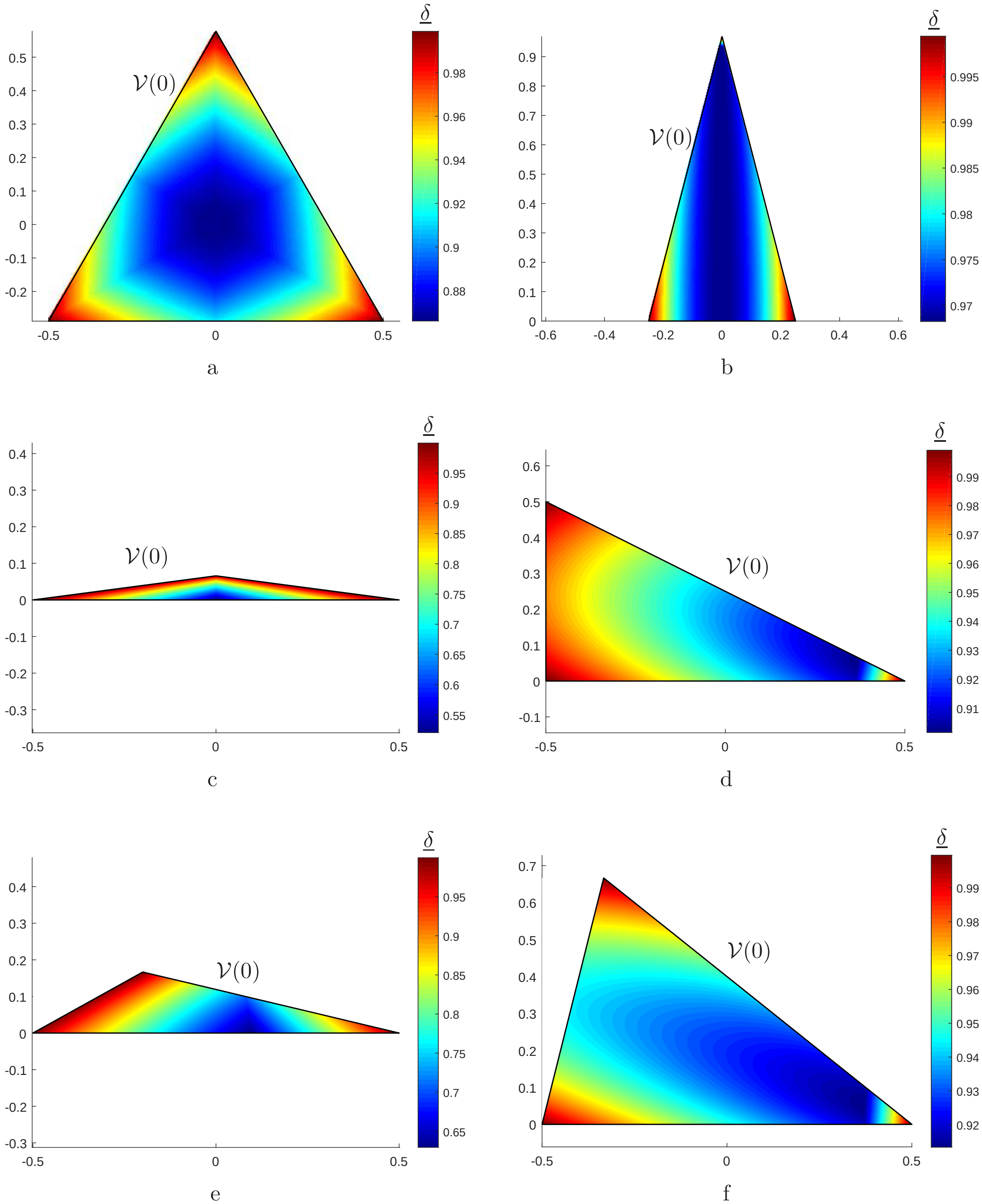


Figure 7: Lower bound $\underline{\delta}$ for different Voronoi cells $\mathcal{V}(0)$ and initial evader location $E(0)$. Equilateral triangle (a), acute isosceles triangle (b), obtuse isosceles triangle (c), right triangle (d), obtuse scalene triangle (e) and acute scalene triangle (f).

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## Appendix A. Proof of Theorem 1

In the first step of the $\mathcal{E}$-strategy, the evader goes from $E(0)$ to $Q$. So, step 1 takes a time $\tau_1 = \|E(0) - Q\|$. In step 1, according to (4a), $P_1$ and $P_3$ move in the same direction of the evader along $e(0) = v_{QE}$, while $P_2$ obeys to (4b) going towards $Q$. As a result, only the smallest edge of $\mathcal{V}$ moves (along $e(0)$), while the others remain the same. The Voronoi cell at time $\tau_1$ is depicted in red in Fig. 2.

Since $V_1(\tau_1) = V_1(0) \in \mathcal{V}(\tau_1)$, in the second step of the $\mathcal{E}$-strategy the evader moving along $v_{V_1Q}$ may safely approach $V_1(0)$ at time $\tau_{12} = \tau_1 + \tau_2$, with $\tau_2 = \|Q - V_1(0)\|$. So, $V_1(\tau_{12}) = V_1(0)$. Notice that, as in the previous step, only the smallest edge of $\mathcal{V}$ is moving in the second step. In Fig. 2, $\mathcal{V}(\tau_2)$ is colored in blue.

During the final step, the evader points towards the farthest vertex of $\mathcal{V}(\tau_2)$ moving along $v_{SE}$. Since the Voronoi cell at any time is a triangle similar to $\mathcal{V}(0)$, the farthest vertex from $V_1(\tau_2) = V_1(0)$ turns out to be $V_2(\tau_2)$. By defining $\tau_3 = \|V_1(0) - V_2(\tau_2)\|$, it is easy to see that $\tau_3 = \|S - E(0)\|$ and then the total traveled time is $\tau_1 + \tau_2 + \tau_3$, which coincides with (8). At such a time, the Voronoi cell collapses to one point and capture occurs.

It remains to prove that the $\mathcal{E}$-strategy is optimal when the pursuers play the $\mathcal{D}$-strategy, i.e., there exists no other evader's strategy guaranteeing a longer survival.

At a given time $\tau$, according to (8), let $M_{\mathcal{D}}(\tau)$ denote the residual game length if the pursuers and the evader play the $\mathcal{D}$-strategy and the $\mathcal{E}$-strategy, respectively, from time $\tau$ onwards. Let the evader move along a direction $\widehat{e}$: $\|\widehat{e}\| = 1$ for a time $\Delta\tau > 0$, i.e., $\dot{E}(t) = \widehat{e}$, $\tau \le t \le \tau + \Delta\tau$. Let $M_{\mathcal{D}}(\tau + \Delta\tau)$ be the corresponding residual game length at time $\tau + \Delta\tau$. Let us define

$$\Delta M(\Delta\tau) = M_{\mathcal{D}}(\tau + \Delta\tau) - M_{\mathcal{D}}(\tau)\,. \tag{A.1}$$

As it has been shown above, if $\widehat{e} \in \{v_{QE},\, v_{V_1Q},\, v_{SE}\}$, one has $\Delta M(\Delta\tau) = -\Delta\tau$. Hence, along the three directions the evader follows in the $\mathcal{E}$-strategy, it holds

$$\frac{dM_{\mathcal{D}}(t)}{dt} = \lim_{\Delta\tau \to 0} \frac{\Delta M(\Delta\tau)}{\Delta\tau} = -1\,.$$

In the following, we prove that for any direction $\widehat{e} \notin \{v_{QE},\, v_{V_1Q},\, v_{SE}\}$ one has $\frac{dM_{\mathcal{D}}(t)}{dt} < -1$. So, the evader will be captured in a shorter time and hence any evader's strategy involving a move $\widehat{e} \notin \{v_{QE},\, v_{V_1Q},\, v_{SE}\}$ cannot be optimal.

Let $\widehat{e} = [\cos(\theta), \sin(\theta)]'$ with $\theta \in [0, 2\pi]$. Assume the evader moves along direction $\widehat{e}$ for a time $\Delta\tau$. We want to compute $\frac{dM_{\mathcal{D}}(t)}{dt}$ as a function of $\theta$. Let $v_{ij}$ be defined as in (5). Let us consider the six directions $\pm v_{12}$, $\pm v_{13}$, $\pm v_{23}$, and the resulting six angular intervals in which they partition the interval $[0, 2\pi]$, as shown in Fig. A.8. Let $l = \|V_1 - V_2\|$, $m = \|V_1 - V_3\|$ and denote by $\varphi_1$ and $\varphi_2$ the angles associated to vertices $V_1$ and $V_2$, respectively.

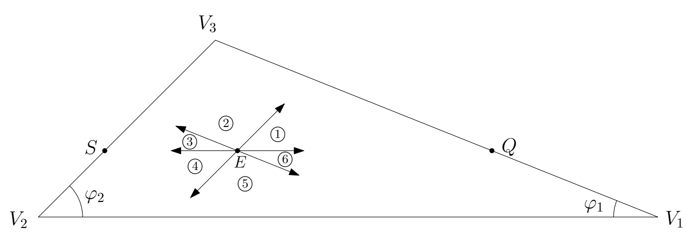


Figure A.8: Proof. of Theorem 1. The six directions given by $\pm v_{12}$, $\pm v_{V_1 3}$, $\pm v_{23}$ are depicted along with the related angular intervals. If the evader does not move along one of these directions, its move is surely not optimal.

Let us start by assuming $\theta \in [0, \varphi_2)$ and derive the expression of $\Delta M(\Delta\tau)$. Let us refer to Fig. A.9, where $\mathcal{V}(\tau)$ and $\mathcal{V}(\tau + \Delta\tau)$ are depicted in black and red, respectively. By (8), $M_{\mathcal{D}}(\tau) = \|S(\tau) - Q(\tau)\| + \|Q(\tau) - V_1(\tau)\|$. It is easy to see that $\|Q(\tau + \Delta\tau) - V_1(\tau + \Delta\tau)\| = \|Q(\tau) - V_1(\tau)\|$. Let $\widehat{Q} = Q(\tau) + \Delta\tau\widehat{e}$ and define $b = \|Q(\tau + \Delta\tau) - \widehat{Q}\|$, see Fig. A.9. One has

$$\|S(\tau + \Delta\tau) - Q(\tau + \Delta\tau)\| = \|S(\tau) - Q(\tau)\| - b\,,$$

and hence

$$\Delta M_{\mathcal{D}}(\Delta\tau) = \|S(\tau + \Delta\tau) - Q(\tau + \Delta\tau)\| - \|S(\tau) - Q(\tau)\| = -b\,.$$

By the law of sines, one has

$$\frac{b}{\sin(\pi - \theta - \varphi_1)} = \frac{\Delta\tau}{\sin(\varphi_1)}\,,$$

that is

$$b = \Delta\tau \frac{\sin(\pi - \theta - \varphi_1)}{\sin(\varphi_1)} = \Delta\tau \frac{\sin(\theta + \varphi_1)}{\sin(\varphi_1)}\,.$$

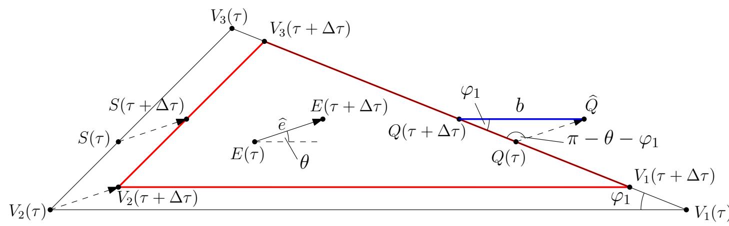


Figure A.9: Proof. of Theorem 1. The evader moves along a direction belonging to the angular interval ① of Fig. A.8. $\mathcal{V}(\tau)$ and $\mathcal{V}(\tau+\Delta\tau)$ are depicted in black and red, respectively. The length of the blue segment $b$ is equal to $M_{\mathcal{D}}(\tau)-M_{\mathcal{D}}(\tau+\Delta\tau)$. Since $b$ is greater than the path $\widehat{e}$ traveled by the evader, the depicted move cannot be optimal.

Thus, one has

$$\Delta M_{\mathcal{D}}(\Delta\tau) = -\Delta\tau \frac{\sin(\theta+\varphi_1)}{\sin(\varphi_1)},$$

and hence

$$\frac{dM_{\mathcal{D}}(t)}{dt} = \lim_{\Delta\tau\to 0} \frac{\Delta M(\Delta\tau)}{\Delta\tau} = -\frac{\sin(\theta+\varphi_1)}{\sin(\varphi_1)}.$$

By using a similar reasoning, one can compute $dM_{\mathcal{D}}(t)/dt$ for all the other cases. Table A.1 reports the expressions of $dM_{\mathcal{D}}(t)/dt$ for $\theta$ belonging to the six angular intervals.

By straightforward calculus arguments, it is possible to show that such a function has three maxima in $[0, 2\pi)$, all equal to $-1$. As expected, they are achieved when $\theta$ is equal to 0, $\pi$ and $2\pi - \varphi_1$, which correspond to the directions $v_{QE}$, $v_{SE}$, $v_{V_1Q}$ adopted in the $\mathcal{E}$-strategy. Therefore, any other direction leads to a greater reduction of $M_{\mathcal{D}}$ and thus it cannot be optimal. □

Table A.1: Expressions of $dM_{\mathcal{D}}(t)/dt$ as a function of $\theta$

| Case | $\theta$ interval | $\frac{dM_{\mathcal{D}}(t)}{dt}$ |
|---|---|---|
| 1 | $[0, \varphi_2)$ | $-\frac{\sin(\theta+\varphi_1)}{\sin(\varphi_1)}$ |
| 2 | $[\varphi_2, \pi-\varphi_1)$ | $-\frac{\sin(\theta+\varphi_1)}{\sin(\varphi_1)} - \frac{\sin(\theta-\varphi_2)}{\sin(\varphi_2)}$ |
| 3 | $[\pi-\varphi_1, \pi)$ | $-\frac{\sin(\theta-\varphi_2)}{\sin(\varphi_2)}$ |
| 4 | $[\pi, \pi+\varphi_2)$ | $-\frac{\sin(\theta-\varphi_2)}{\sin(\varphi_2)} + \frac{\sin(\theta)}{\sin(\varphi_1)}$ |
| 5 | $[\pi+\varphi_2, 2\pi-\varphi_1)$ | $\frac{\sin(\theta)}{\sin(\varphi_1)}$ |
| 6 | $[2\pi-\varphi_1, 2\pi)$ | $\frac{\sin(\theta)}{\sin(\varphi_1)} - \frac{\sin(\theta+\varphi_1)}{\sin(\varphi_1)}$ |