THE EFFECT OF POINTS FATTENING ON POSTULATION

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Abstract. ***We classify sets Z of points in the projective plane, for which the difference between the minimal degrees of curves containing $2Z$ and Z respectively, is small.***

MSC. 14N05

1. Introduction

Given a set of points $Z \subset \mathbb{P}^2$, an interesting problem concerns the study of the geometric properties of the fattening $2Z$ and their relations with the properties of Z itself. When Z is general, the postulation of $2Z$ is well understood, but as soon as the points come to some special position, our knowledge on the geometry of $2Z$ becomes less effective.

In [GMS], Geramita, Migliore and Sabourin study the possible Hilbert functions of $2Z$, when the Hilbert function of Z is given. In [GHM], Geramita, Harbourne and Migliore determine precisely what functions occur as Hilbert functions of ideals $I(Z)$ of fat point subschemes Z of $\mathbb{P}^2$ in case either $I(Z)$ contains some power of a form of degree 2 or in case the support of Z consists of 8 or fewer points.

It should be observed that the Hilbert function of Z does not determine uniquely the Hilbert function of $2Z$, as many easy examples show.

In this paper, we look at a specific point of this analysis, by considering the first interesting step of the Hilbert function, namely the smallest degree of a curve containing the set of points or its fattening. In more detail, let Z be a configuration of points in $\mathbb{P}^2$ and let $I = I(Z)$ be the homogeneous ideal. Put $I^{(2)} = I(2Z)$. We say that Z has type $(d-t, d)$ if the generators of I and $I^{(2)}$ have minimal degrees $d-t$ and d respectively.

The number t which measures the difference between the first degrees of curves containing Z and $2Z$, is directly related to some important invariants of sets of points, such as Seshadri constants or the resurgence. The knowledge of the type $(d-t, d)$ can give some hint for these two invariants. We refer to the papers of Harbourne and Roé [HR2] and Harbourne and the first author [BH1], [BH2], for an account on these links.

It is easy to show that t is at least 1. When the characteristic of the ground field is zero, this can be seen by taking any derivative of a form f of minimal degree d singular at Z. In characteristic $p > 0$, it may happen that all the derivatives of f vanish, but in this case f is the p-th power of a form g passing through Z. For a set Z of general points, it is classically known that, except for a few cases, both Z and $2Z$ have maximal rank, so their postulation follows just from arithmetic and we see that t is not far from $(\sqrt{3}-1)d$.

Here we consider the cases where t is small, giving a complete description of sets of points for which $t = 1$ or $t = 2$, regardless of the characteristic of the ground field.

The case $t = 1$ is relatively simple. In section 3 we obtain the classification result:

Theorem 1.1. *Let Z be a set of points in $\mathbb{P}^2$. Then Z has type $(d-1, d)$ if and only if*

(i) *Z is the set of $\binom{d}{2}$ distinct points given by the pair-wise intersection of d lines or*
(ii) *Z is a set of collinear points and $d = 2$.*

In particular, every set Z of cardinality smaller than four fits in one of the previous cases.

The case $t = 2$ is more complicated, and yields more examples. Here the minimal curve containing $2Z$ can be irreducible, e.g. when Z is the set of nodes of a rational curve. Hence, the classification requires a deeper understanding of curves of degree d singular at Z. A first investigation concerns whether this curve is reduced or not. In the reduced case, in order to obtain a complete description, we need to refer to some graph attached to a subset of the singular set of reducible curves.

The classification theorem is:

Theorem 1.2. *Let Z be a set of at least four points in $\mathbb{P}^2$, not in the list of Theorem 1.1. Then Z has type $(d-2, d)$ if and only if either:*

- i) *Z is contained in a conic and $d = 4$; or*
- ii) *there exists a line L such that the set $Z' := Z - L$ is non empty, of type $(d-3, d-2)$ and there are no curves of degree $d-2$ singular at Z' and passing through $Y := Z \cap L$, neither there are curves of degree $d-3$ passing through Z' and Y; or*
- iii) *Z is contained in the singular locus of a tame curve C of degree d, Z contains the singular locus of each component of C and the adapted graph of Z is a forest.*

The definitions of *tame curve* and *adapted graph* are given respectively in sections 4 and 5. In any case, from some point of view, the theorem shows that sets of points with $t = 2$ and $d > 4$ are contained in the set of nodes of some union of rational curves.
Notice that cases i) and ii) of theorem 1.2 correspond to the situations in which there exists a non-reduced curve of degree d singular at Z.
The proof of the theorem is divided in several sections. In section 4 we give some preliminary results. In particular we show that a set of points defined by the curves satisfying the conditions of the theorem is contained in a curve of degree $d-2$. Finally, in section 5 we use the notion of adapted graph to study the proper subsets of the singular locus of curves C which have type $(d-2, d)$ and we complete the proof.

2. Postulation under specific conditions

Through the paper, we work on projective space defined over an algebraically closed field of any characteristic.
Given a homogeneous ideal $I \subset R = k[\mathbf{P}^N]$, let $\alpha(I)$ be the least degree t such that the homogeneous component I_t in degree t is not zero. Thus α is, so to speak, the degree in which the ideal begins.
We will consider in this note ideals I defining configurations Z of points, i.e. *sets of distinct points*. In other words, I is the intersection $m_1 \cap \cdots \cap m_n$ where the m_i's are distinct maximal homogeneous ideals. Notice that $I^{(2)}$ is then equal to $m_1^2 \cap \cdots \cap m_n^2$.

Definition 2.1. Let Z be a configuration of points in $\mathbb{P}^2$ and let $I = I(Z)$. We say that Z has type $(d-t, d)$ if $\alpha(I) = d-t$ and $\alpha(I^{(2)}) = d$.

Remark 2.2. One has $1 \le t < d$. Indeed if f is any element of minimal degree in $I^{(2)}$, then any partial derivative of f belongs to I. If the characteristic p is positive, it could happen that all the derivatives of f vanish, but in this case f is the p-th power of another form g, which necessarily vanishes at Z.

We will use, in the sequel, the following, well known Remark (see e.g. lemma 7.5 of [GMS]).

Remark 2.3. Let $C = C_1 + \cdots + C_n$ be a reduced plane curve. Call $d_i = \deg(C_i)$ and let $d = d_1 + \cdots + d_n$ be the degree of C. Then C has at most $\delta = n - 1 + (d-1)(d-2)/2$ singular points.
The number of singular points is exactly δ when C is a set of lines, no three of them meeting at a point. Indeed it is just a matter of computation to show that:

$$n - 1 + (d-1)(d-2)/2 = \sum_i \frac{(d_i - 1)(d_i - 2)}{2} + \sum_{i<j} d_i d_j.$$

When $n < d$ or when more than two components meet at the same point, it follows that:

$$n - 1 + \frac{(d-1)(d-2)}{2} < \binom{d}{2}.$$

Hence the singular locus of a reduced curve C of degree d is always contained in some curve of degree $d-2$, unless all the components of C are lines, no three of them meeting at one point.
In any event, if we forget one singular point of Z, the remaining singular points are always contained in a curve of degree $d-2$.

3. The case $(d-1,d)$

In this section we describe sets of points of type $(d-1,d)$, which correspond to the smallest value of t. We will see that these sets fit in two precise categories.

Example 3.1. Let Z be a set of points on a line L. Thus $\alpha(I)=1$ and, since $L^2 \in I^{(2)}$, one has $\alpha(I^{(2)})=2$. Then Z has type $(1,2)$.

A more interesting example of sets of points of type $(d-1,d)$ is given in the following Proposition (see lemma 7.8 and theorem 7.10 of [GMS]):

Proposition 3.2. *Let Z be the set of $\binom{d}{2}$ points ($d \geq 2$) given by the pair-wise intersection of d lines. Then Z has type $(d-1,d)$.*

Proof. Obviously the union of the d lines is in $I^{(2)}$. Thus $\alpha(I^{(2)}) \leq d$. Let us suppose there exists a form f of degree $u \leq d-1$ vanishing at Z with multiplicity 2. Call F the curve defined by f. It is enough to show that, in this case, each of the d lines must be a fixed component of F. Let L be one of the d lines. Since $F \cdot L = 2(d-1) > u$, L is a fixed component of F and the same is true for each of the d lines, a contradiction. In a similar way we can show that $\alpha(I)=d-1$. If there exists a form F of degree $u' \leq d-2$ passing through Z, then again $F \cdot L = d-1 > u'$ and any line L is a fixed component of F. □

We show that the two previous examples exhaust all configurations of type $(d-1,d)$.

Theorem 3.3. *Let Z be a set of points of type $(d-1,d)$. Then*

(i) *Z is the set of $\binom{d}{2}$ distinct points given by a pair-wise intersection of d lines or*
(ii) *Z is a set of collinear points and $d=2$.*

Proof. Let q be the cardinality of Z. First of all, one has $q \geq \binom{d}{2}$: if not, by an obvious parameter count, there would be a curve of degree $d-2$ passing through the points.
Let C be a curve of degree d singular at Z. Assume that C is not reduced, and $d>2$. C_{red} cannot have degree $\leq d-2$, for it contains Z. Thus C is equal to the union of a double line $2L$ with a reduced curve C'. The set $Y = Z - L$ is not empty, for $d>2$, and C' is singular at Y. Taking derivatives of an equation f of C', or taking the p-th root of f, if the characteristic p is positive and all the partial derivatives of f vanish, one sees that there exist curves D of degree $\leq d-3$ passing through Y. Then $D \cup L$ has degree at most $d-2$ and passes through Z, a contradiction. Thus, if C is non reduced, $\deg(C)=2=d$ and C is a double line: we are in case (ii) of the statement.
Assume now C reduced. By assumption, Z cannot be contained in a curve of degree $d-2$. Hence, by remark 2.3, C is a union of lines, no three of them meeting at a point, and moreover Z is the whole set of the pair-wise intersections of the lines in C. □

Remark 3.4. Notice that, unless Z is one point, in the previous setting the curve of degree d singular at Z is unique.

Remark 3.5. Notice that every set Z with three or less points, fits in one of the cases of theorem 3.3.

4. The case $(d-2,d)$: preliminaries

We consider now in this section sets of points of type $(d-2,d)$. By the previous Remark 3.5, we know that the cardinality of Z is at least four. Let us start with some examples.

Example 4.1. Let Z be a non aligned set of at least four points on a (possibly reducible) conic Γ. Then Z has type $(2,4)$.
Indeed, let us see that no cubic curve is singular at Z. If Γ is smooth, then by Bezout any cubic singular at Z must contain Γ twice, a contradiction. Assume that Γ is reducible, $\Gamma = L_1 \cup L_2$. If one line, say L_1, contains at least three points of Z, then any cubic C singular at Z contains L_1 twice. Since by assumption there exists at least one point $P \in Z - L_1$ and C is singular at P, we get a contradiction. It remains to exclude the case in which each line contains exactly two points of Z (so that $Z \cap L_1 \cap L_2 = \emptyset$). Here, a cubic C singular at Z must contain both lines, so that $C = L_1 \cup L_2 \cup L$, where L is a third line which must contain Z. Again, this is absurd.

Example 4.2. Let Z' be a set of points of type $(d-3, d-2)$, and let L be a line, not intersecting Z'. Let Y be a non-empty set of points on L and write $Z = Z' \cup Y$. Assume that Z is not contained in a conic (which implies that Z' is not contained in a line). Assume also that Y is not contained in a curve of degree $d-2$ singular at Z', nor in a curve of degree $d-3$ passing through Z'. Then Z has type $(d-2, d)$.
Indeed, observe that we must have $d \geq 5$, for otherwise Z' lies in a line, so Z lies in a conic. It follows, by Theorem 3.3, that Z' is the set of pair-wise intersections of $d-2$ lines L_i.
If D is any curve of degree $d-3$ through Z', then $D \cup L$ is a curve of minimal degree $d-2$ through Z, by our assumptions. If C is a curve of degree $d-2$ singular at Z', then $C = \bigcup L_i$ and $C \cup 2L$ is a (non-reduced) curve of degree d singular at Z. So we just need to prove that we cannot find curves of degree $d-1$ singular at Z. If C' is such a curve, then it has $d-3$ singular points on each line L_i. Assume $d > 5$. Then $2(d-3) > d-1$, thus C' contains any line L_i. It follows that $C' = \bigcup L_i \cup M$ where M has degree 1 and is singular at Y, a contradiction. If $d = 5$, then Z' is a set of three non collinear points and C' is a quartic singular at Z' and Y, moreover Y has cardinality at least three, for Z is not contained in a conic. Thus C' contains L. Since the unique cubic singular at Z' is the union $C = L_1 \cup L_2 \cup L_3$, it follows that Y is the intersection of C with L, which contradicts our last assumption.

Remark 4.3. The two cases excluded in the previous example, namely the existence of a curve C of degree $d-2$ singular at Z' and passing through Y, or the existence of a curve D of degree $d-3$ passing through Z' and Y, yield particular configurations.
If C exists and D does not, then Z is of type $(d-2, d-1)$. Notice that, in this case, by Remark 2.3, the non existence of D implies that Y is equal to the intersection of C with L, so Z is the set of pair-wise intersections of $d-1$ lines. If D exists then Z is of type $(d-3, x)$, where x is either $d-1$ or d. In this case, as Z' is the set of pair-wise intersections of a set C' of $d-2$ lines, then C' is the unique curve of degree $d-2$ singular at Z' (recall that Z' is not aligned, for Z does not lie in a conic). So we cannot have a curve of degree $d-2$ singular at Z' and Y. If $Y \subset C'$, then Z has type $(d-3, d-1)$ (it enters in the classification below), otherwise it has type $(d-3, d)$.

The two previous examples exhaust all possible cases of sets Z, with cardinality at least four, of type $(d-2, d)$, such that there are curves of degree d singular at Z and non reduced.

Proposition 4.4. *Let Z be a set of points of type $(d-2, d)$. Assume that there is a curve C of degree d singular at Z and non reduced. Then either C is a double conic and $d = 4$, or C contains a double line $2L$ and the set $Z - L$ is non empty, of type $(d-3, d-2)$.*

Proof. Write $C = m_1C_1 + m_2C_2 + \cdots + m_nC_n$. If there is an index i such that $\deg(C_i) \geq 3$ and $m_i > 1$ then the curve $\Gamma = m_1C_1 + \cdots + C_i + \cdots m_nC_n$ passes through Z and $\deg(\Gamma) \leq d-3$ which is a contradiction. A similar fact happens if, for some i, either $\deg(C_i) \geq 2$ and $m_i > 2$, or $m_i > 3$. Thus we conclude that $m_i \leq 3$ for all $i = 1, \ldots, n$.
Assume that there are two indices i, j such that $m_i, m_j > 1$. Then C_i, C_j are both lines and $m_i = m_j = 2$, otherwise, arguing as before, we get a curve of degree smaller than $d-2$ through Z. Thus we are left with four possible cases:

(a) $m_1 = 3$, $d_1 = 1$, i.e. C_1 is a line, $m_2 = \cdots = m_n = 1$;
(b) $m_1 = 2$, $d_1 = 2$, $m_2 = \cdots = m_n = 1$;
(c) $m_1 = 2$, $d_1 = 1$, $m_2 = \cdots = m_n = 1$;
(d) $m_1 = m_2 = 2$, $d_1 = d_2 = 1$, $m_3 = \cdots = m_n = 1$.

Case (a) can be easily excluded. Namely if $C = 3C_1 + C_2 + \cdots + C_n$, then $2C_1 + C_2 + \cdots + C_n$ is a curve of degree $d-1$ singular at Z, a contradiction.
In case (b) we have $C = 2C_1 + C_2 + \cdots + C_n$ where C_1 is a conic. Write $Z' = Z - C_1$. Assume Z' is non empty. Let D be the curve defined either by a derivative of an equation f of $C_2 + \cdots + C_n$, or, if all these derivatives vanish, so that the characteristic p is positive, let D be defined by a p-th root of f . $C_1 + D$ has degree $\leq d-3$ and passes through Z, a contradiction. Hence $Z \subset C_1$ and we have points on a conic, moreover $d = \deg(2C_1) = 4$.
Case (d) is similar to the previous one, for $C_1 + C_2$ is a degenerate conic and again one proves in this case that $C = 2C_1 + 2C_2$, i.e. the points lie on a (reducible) conic and $d = 4$.
Finally in case (c), call $Z' = Z - C_1$. Notice that Z' cannot be empty, otherwise Z has type $(1, 2)$. $C_2 + \cdots + C_n$ is a curve of degree $d-2$, singular at Z'. Clearly there are no curves C' of degree $d' < d-2$ singular at Z', otherwise $2C_1 + C'$ is a curve of degree smaller than d and singular

at Z. Similarly, there are no curves of degree smaller than $d-3$ through Z'. By remark 2.2, we conclude that Z' has type $(d-3, d-2)$. □

Let us now turn to the case where there are curves of degree d singular at Z and reduced. We need the following

Definition 4.5. Let C be a plane reduced curve. C is called a *tame* curve if

a) all the components C_i of C are rational;
b) the singular points of each C_i are either nodes or ordinary cusps;
c) no singular point of C_i belongs to some other component C_j;
d) no components C_i, C_j are tangent at two points;
e) for $i \neq j$ the local intersection multiplicity of C_i and C_j at any point is at most two;
f) no components C_i, C_j of C are tangent anywhere if some point of $C_i \cap C_j$ belongs to a third component;
g) for $i < j < k$, the intersection $C_i \cap C_j \cap C_k$ consists of at most one point;
h) if some point of $C_i \cap C_j$ belongs to a third component, then no other components contain points of $C_i \cap C_j$; in particular no point belongs to four components of C.

Definition 4.6. In the sequel, we say that P is a *contact point* of two irreducible curves C_1, C_2 if $P \in C_1 \cap C_2$ and the two curves share a common tangent at P. In other words, P is a contact point of C_1, C_2 when the intersection multiplicity of C_1, C_2 at P is bigger than 1.

Next proposition shows why we introduced the notion of *tame* curve.

Proposition 4.7. *Let $C = C_1 + \cdots + C_n$ be a reduced plane curve, where all C_i are irreducible. Put $d_i = \deg(C_i)$ and let $d = d_1 + \cdots + d_n$ be the degree of C. Assume that C is not tame. Then there exists a curve D of degree $d-3$ containing all the singular points of C.*

Proof. We show that when at least one of the assumptions (a),..., (h) of definition 4.5 fails, then we can find the curve D.

(a) Assume C_1 is not rational. Then C_1 has an adjoint D_1 of degree $d_1 - 3$, which passes through all singular points of C_1. Then $D = D_1 + C_2 + \cdots + C_n$ is a curve of degree $d-3$ which contains all singular points of C.

(b) Assume that C_1 is rational and one singular point P of C_1 is not a node nor a cusp. Then P is responsible for a difference of at least two between the arithmetic and geometric genera of C_1, thus it turns out that C_1 has at most $(d_1-1)(d_1-2)/2 - 1$ singular points. So for dimensional reasons, one finds a curve D_1 of degree $d_1 - 3$ containing all singular points of C_1. As above, $D = D_1 + C_2 + \cdots + C_n$ is a curve of degree $d-3$ which contains all singular points of C.

(c) Consider the reducible curve $C_1 + C_2$. If C_2 passes through a singular point of C_1, then $C_1 \cap C_2$ consists of at most $d_1d_2 - 1$ points. Hence $C_1 + C_2$ has at most $(d_1-1)(d_1-2)/2 + (d_2-1)(d_2-2)/2 + d_1d_2 - 2$ singular points. Since

$$\frac{(d_1-1)(d_1-2)}{2} + \frac{(d_2-1)(d_2-2)}{2} + d_1d_2 - 2 = \binom{d_1+d_2-1}{2} - 1$$

one finds a curve D_1 of degree $d_1 + d_2 - 3$ passing through all the singular points of $C_1 + C_2$. Then $D = D_1 + C_3 + \cdots + C_n$ is a curve of degree $d-3$ which contains all the singular points of C.

(d) The proof is almost identical to the previous one, since in this case again $C_1 + C_2$ has at most $(d_1-1)(d_1-2)/2 + (d_2-1)(d_2-2)/2 + d_1d_2 - 2$ singular points.

(e) Again if C_1, C_2 have intersection multiplicity $m > 2$ at some P then $C_1 + C_2$ has at most $(d_1-1)(d_1-2)/2 + (d_2-1)(d_2-2)/2 + d_1d_2 - m$ singular points. Thus there exists a curve D_1 of degree $d_1 + d_2 - 3$ containing the singularities of $C_1 + C_2$. $D = D_1 + C_3 + \cdots + C_n$ yields the contradiction.

(f) If C_3 passes through a contact point of C_1, C_2, then the number of singular points of the curve $C_1 + C_2 + C_3$ is at most:

$$\frac{(d_1-1)(d_1-2)}{2} + \frac{(d_2-1)(d_2-2)}{2} + \frac{(d_3-1)(d_3-2)}{2} + d_1d_2 + d_1d_3 + d_2d_3 - 3.$$

One computes that this number is smaller than $\binom{d_1+d_2+d_3-1}{2}$, hence the singular locus of $C_1 + C_2 + C_3$ is contained in a curve D_1 of degree $d_1 + d_2 + d_3 - 3$. The curve $D = D_1 + C_4 + \cdots + C_n$ has degree $d-3$ and contains the singular locus of C, a contradiction.

(g) As above, one computes that the number of singular points of the curve $C_1+C_2+C_3$ is smaller than $\binom{d_1+d_2+d_3-1}{2}$, and concludes similarly.
(h) Assume that C_1, C_2, C_3 pass through a point P and C_1, C_2, C_4 pass through a point Q (we do not exclude the case $P = Q$). Then one computes that the number of singular points of the reducible curve $C_1 + C_2 + C_3 + C_4$ is at most:

$$\sum_{i=1,\dots,4} \frac{(d_i-1)(d_i-2)}{2} + \sum_{i<j} d_i d_j - 4.$$

As this last number is smaller than $\binom{d_1+d_2+d_3+d_4-1}{2}$, there exists a curve D_1 of degree $d_1 + d_2 + d_3 + d_4 - 3$ passing through all the singular points of of $C_1 + C_2 + C_3 + C_4$. Then $D = D_1 + C_5 + \cdots + C_n$ is a curve of degree $d - 3$ which contains all singular points of C. □

Thus we have immediately:

Proposition 4.8. *Let Z be a configuration of points of type $(d-2, d)$. Let C be a curve of minimal degree d, singular at Z, and assume that $C = C_1 + \cdots + C_n$ is reduced. Then C is tame.*

We briefly analyze now the case $d = 4$ for the set of points of type $(d - 2, d)$.

Remark 4.9. Notice that no configuration of points is of type $(1, 3)$. Namely such a configuration should be aligned, but any set of aligned points has type $(1, 2)$.

Proposition 4.10. *Z has type $(2, 4)$ if and only if it lies in a conic and not in a line, and has cardinality at least four.*

Proof. If Z lies in a line, then it has type $(1, 2)$. If the cardinality is smaller than four, Z cannot have type $(2, 4)$, by remark 3.5.
Then we conclude by example 4.1. □

Remark 4.11. If Z is a set of cardinality ≥ 3 on a line L_1 plus one point $P \notin L_1$, then Z has type $(2, 4)$ and any quartic C singular at Z is a union $C = 2L_1 + L_2 + L_3$ where L_2, L_3 are lines such that $P = L_2 \cap L_3$.

5. The case $(d - 2, d)$: the classification

We are ready now to give the classification of configurations of type $(d - 2, d)$. After remarks 3.5 and 4.9, we will assume $d \geq 4$.

As usual, we call C a curve of degree d which is singular at Z. Since the case in which C is non reduced has already been exploited in proposition 4.4, we will assume here that C is reduced.
Write $C = C_1 + \cdots + C_n$, where each C_i is irreducible. From proposition 4.8 we may assume that C is tame.

Lemma 5.1. *Let $C = C_1 + \cdots + C_n$ be a tame reduced curve of degree d. Call c the number of contact points of two components of C and call t the number of points at which three components meet. Then the cardinality of the singular locus of C is:*

$$\frac{(d-1)(d-2)}{2} + n - 1 - c - 2t.$$

Proof. We prove the claim by induction on n. If $n = 1$ the claim is obvious since $C = C_1$ is rational, with only nodes and ordinary cusps as singularities.
Assume that the claim is true for $C' = C_2 + \cdots + C_n$. Put $d_1 = \deg(C_1)$, so that the degree of C' is $d' = d - d_1$. Call t' the number of points where three components of C' meet and call c' the number of contact points between two components of C'. C_1 has $(d_1 - 1)(d_1 - 2)/2$ singular points and its intersection number with C' is $d_1 d'$. By conditions (f), (g), (h), C_1 misses all points in which three components of C' meet, as well as contact points of two components of C'. Moreover C' misses all singular points of C_1 and C_1 misses all singular points of any component of C', by condition (c). Call t'' the number of singular points of C' belonging to C_1 and call c'' the number of points where C_1 is tangent to C'. Then $c = c' + c''$, $t = t' + t''$. Moreover, the number of

singular points of C is the number of singular points of C' plus $(d_1-1)(d_1-2)/2$ plus dd' minus $2t''$ minus c''. Using induction, one computes that the number of singular points of C is:

$$\frac{(d'-1)(d'-2)}{2}+n-2-c'-2t'+\frac{(d_1-1)(d_1-2)}{2}+d'd_1-c''-2t''=$$
$$=\frac{(d-1)(d-2)}{2}+n-1-c-2t.$$

□

We define now the notion of *graph adapted to* C, which is crucial in our classification.

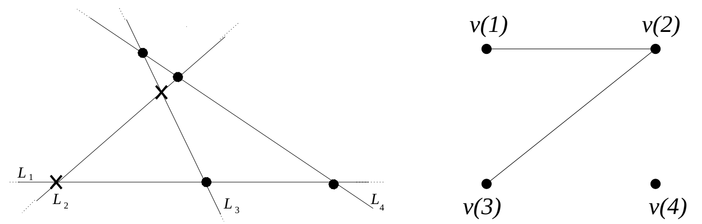
Definition 5.2. Let $C=C_1+\cdots+C_n$ be a reduced, reducible plane curve. C defines a (labelled) graph G_C as follows.
Any component C_i of C corresponds to a vertex $v(i)$ of G_C.
For any point P where C_i and C_j meet with multiplicity m, we draw m edges $e_1(i,j,P)$, …, $e_m(i,j,P)$ joining $v(i)$ and $v(j)$.
Notice that the edges of G_C are labelled by the points where two components meet, while any such point labels at least one (but maybe several) edge.

Definition 5.3. Let $C=C_1+\cdots+C_n$ be as above and let Z be a subset of the singular locus of C. A subgraph G of G_C is *adapted to* Z if it is obtained from G_C with the following procedure: for each point $P\in Z$ where two or more components meet, we erase *exactly one* edge labelled by P.

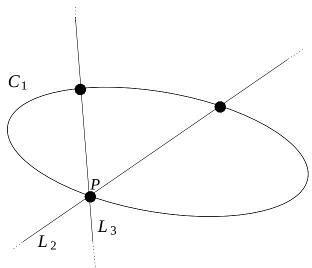
Remark 5.4. Notice that a subgraph adapted to Z is uniquely determined, unless there are at least three components of C meeting at some point $P\in Z$.

Remark 5.5. By definition, if $Z'\subset Z$ are subsets of the singular locus of C, then any subgraph of G_C adapted to Z is contained in a subgraph adapted to Z'.

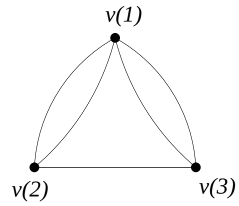
Example 5.6. Let $C=L_1+L_2+L_3+L_4$, where each L_i is a line and these lines are in general position. Then G_C is the complete graph with four vertices.
If Z is the subset of the singular locus of C obtained by forgetting the two points $L_1\cap L_2$ and $L_2\cap L_3$, then the unique subgraph adapted to Z has one edge joining $v(1)$ and $v(2)$ and one edge joining $v(2)$ and $v(3)$, while $v(4)$ is isolated.



Example 5.7. Let $C=C_1+L_2+L_3$, where each L_i is a line C_1 is a conic, and the components are in general position except that there is a point P where the three components meet.



G_C has three vertices $v(1),v(2),v(3)$, one edge joining $v(2),v(3)$, two edges joining $v(1),v(2)$ and two edges joining $v(1),v(3)$.



Let Z be the whole singular locus of C. Then any graph on the three vertices which contains only two consecutive edges (as $v(1), v(2)$ and $v(2), v(3)$, etc.) is adapted to Z.

Definition 5.8. Let $C = C_1 + \cdots + C_n$ be a reduced, tame plane curve.
We say that a subset Z of the singular locus of C is *admissible* when Z contains all singular points of any component C_i and any subgraph of G_C adapted to Z is a forest.

Remark 5.9. It is not hard to prove that, when $C = C_1 + \cdots + C_n$ is tame and one subgraph of G_C adapted to Z is a forest, then any other subgraph adapted to Z is.
Namely, as explained in remark 5.4, the subgraph is not uniquely determined when three components C_1, C_2, C_3 of C meet. In this case, we may erase either an edge $e(1,2)$ joining $v(1), v(2)$, or an edge $e(2,3)$ joining $v(2), v(3)$, or an edge $e(1,3)$ joining $v(1), v(3)$.
Call G the graph obtained by erasing $e(1,2)$ and G' the graph obtained by erasing $e(2,3)$. Assume that G has a loop. If the loop does not involve $e(2,3)$ then also G' has the same loop. If the loop ℓ involves $e(2,3)$, then G' has the loop obtained from ℓ by substituting $e(2,3)$ with the chain $e(1,2), e(1,3)$.

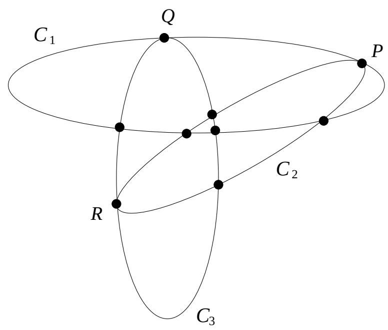
Remark 5.10. It is easy to see that if Z misses two or more points of intersection of two components C_i, C_j, then any subgraph adapted to Z cannot be a forest. Namely they contain two edges joining $v(i), v(j)$ and these two edges form a loop.
Similarly, we can prove that when the subgraphs adapted to Z are forests, then Z cannot miss a contact point of two components of C or a triple point of C.

Remark 5.11. Assume that the subset Z of the singular locus of the tame curve $C = C_1 + \cdots + C_n$ of degree d misses a contact point of two components. Then Z sits in a curve of degree $d-3$. Namely, if Z misses the contact point of C_1, C_2, then by lemma 5.1, it contains at most

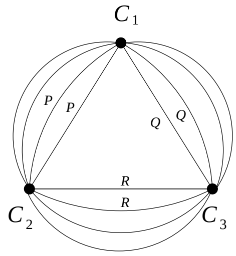
$$\binom{d_1 + d_2 - 1}{2} - 1$$

singular points of $C_1 + C_2$. Since this number is smaller than $\binom{d_1+d_2-1}{2}$, there exists a curve D' of degree $d_1 + d_2 - 3$ passing through the singular points of $C_1 + C_2$ belonging to Z. Then $D' + C_3 + \cdots + C_n$ is the required curve.
A similar conclusion holds when Z misses one point where three components of C meet.

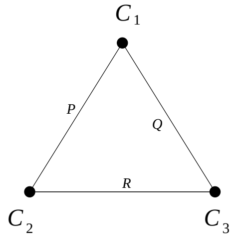
Example 5.12. Even if Z coincides with the whole singular locus of a tame reduced curve C, it may happen that Z is not admissible.
Consider $C = C_1 + C_2 + C_3$ where each C_i is a conic, and assume that the three conics have pairwise a common tangent (contact point) at three distinct points, while are transversal elsewhere.



The graph adapted to C is given by



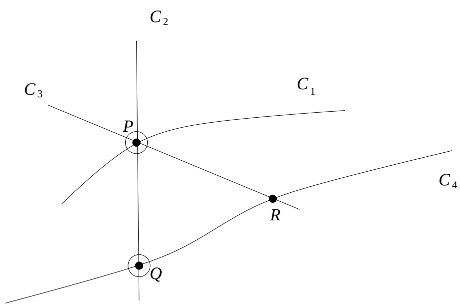
Then the unique graph adapted to the whole singular set Z contains a triangle since we can remove only one edge labelled by P, Q and R thus it is not a forest.

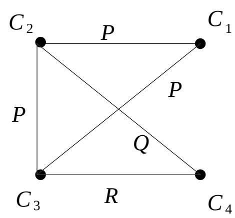


Lemma 5.13. *Let $C = C_1 + \cdots + C_n$ be a tame curve of degree d, and let Z be any admissible subset of the singular locus Z_0 of C. Fix $j \leq n$ and call $C' = C_1 + \cdots + C_j$, of degree d', and $C'' = C_{j+1} + \cdots + C_n$. Define $Z' = Z - C''$. Then Z' is an admissible subset of the singular locus of C'.*

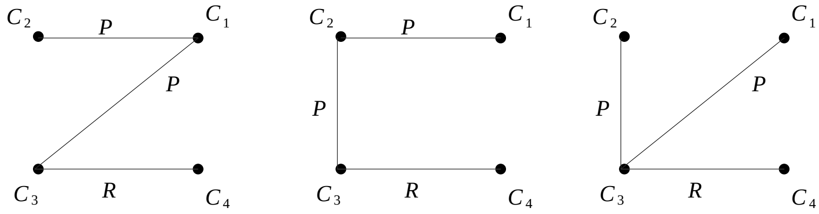
Proof. First of all, notice that C' satisfies condition (a), ..., (h). The graph $G_{C'}$ is a subgraph of G_C. We claim that any subgraph of $G_{C'}$ adapted to Z' is a subgraph of a graph adapted to Z. This is essentially obvious when C'' contains no singular points of C'. If, on the contrary, there are two components C_1, C_2 of C' and one component C_n of C'' meeting at P (this is the only case in which C'' can contain a singular point of C', in accordance with (a), ..., (h)), then Z' misses P and a graph adapted to Z' has one edge joining $v(1), v(2)$, but the same happens to *some* graphs adapted to Z. □

Example 5.14. Given a tame curve $C = C_1 + \cdots + C_n$ and a set $Z \subset$Sing(C), consider $C' = C_1 + \cdots + C_j$ and $Z' \subset Z \cap$Sing(C'). When Z contains a triple point, we can choose among several adapted graphs. In this case, it is not true that the adapted graph of Z' with respect to C' is a subgraph of each of the adapted graphs of Z with respect to C.
In fact, consider the following configuration $C = C_1 + C_2 + C_3 + C_4$ with a partial view of its adapted graph:



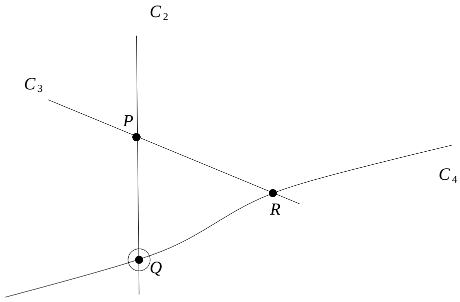


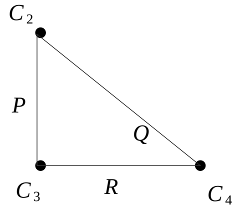
Let Z be the set $\{P, Q\}$ as denoted in the previous figure. Thus we have three different ways to erase an edge labelled by P and we obtain the following graphs:



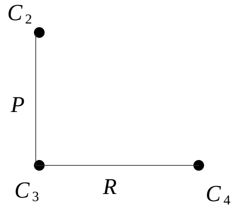
Case A Case B Case C

Consider now the set $Z' = Z - C_1$. Then $Z' = \{Q\}$. The residual curve $C_2 + C_3 + C_4$ and its adapted graph are respectively





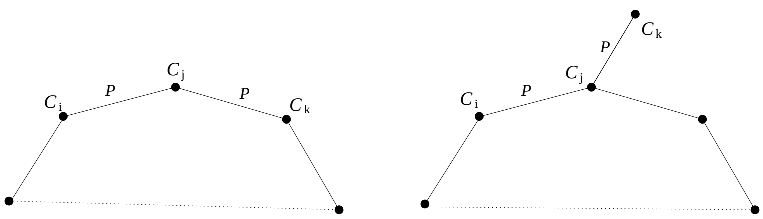
Thus the adapted graph of Z' is given



that is a subgraph of some adapted graphs of Z, but only for cases B and C.

Theorem 5.15. *Let $C = C_1 + \cdots + C_n$ be a tame reduced plane curve of degree d. Put $d_i = \deg(C_i)$. Let Z be a subset of the singular locus of C. If Z is not admissible, then Z belongs to some curve of degree $d-3$. In particular, Z cannot have type $(d-2, d)$.*

Proof. If some component, say C_1, has a singular point not in Z, then the intersection Z' of Z with the singular locus of C_1 has cardinality at most $(d_1-1)(d_1-2)/2-1$, so there exists a curve D of degree $d_1 - 3$ through Z'. Then $D + C_2 + \cdots + C_n$ is a curve of degree $d-3$ through Z.
Assume that a subgraph G of G_C, adapted to Z, is not a forest. Notice that, by remark 5.11, the conclusion holds when G contains either two edges labelled by the same contact point of two components, or three edges labelled by the same triple point. Thus we may exclude these cases.
Fix a minimal loop of G, with r edges. The loop is then formed by r distinct vertices, by minimality.
Assume that an edge of the loop, connecting the vertices corresponding to the components C_i, C_j, is labeled by a point P belonging to another component C_k. As G contains necessarily another edge labelled by P, then either the loop contains two edges labelled by P, or C_k cannot appear in the loop.



Case I Case II

After renumbering, we may assume that $C_1, \dots, C_r$ are the components corresponding to the vertices of the loop and $C_{r+1}, \dots, C_s$ are components passing through a point P which labels some edge of the loop, but not corresponding to vertices of the loop. Put $C' = C_1 + \dots + C_s$. As usual, set $d_i = \deg(C_i)$ and set $d' = \sum_{i=1}^s d_i$. Call Z' the intersection of Z with the singular locus of C'.
Assume there are c' contact points of two components of C' and that C' has t' triple points. As C' is tame by lemma 5.1, the number of singular points of C' is

$$\sum_{i=1}^{s} \frac{(d_i-1)(d_i-2)}{2} + \sum_{i<j\leq s} d_i d_j - c' - 2t' = \binom{d'-1}{2} + s - 1 - c' - 2t'.$$

By our construction of C', every edge of the loop corresponds either to a singular point of C' missed by Z, or to a contact point of two components of C', or to a triple point P of C'. In this last case, either P labels two edges of the loop, or it labels only one edge and lies exactly in one component among $C_{r+1}, \dots, C_s$. If we call x the number of singular points of C' missed by Z, it is thus easy to see that the number r of edges in the loop satisfies

$$r \leq x + c' + (s-r) + 2(t' - s + r)$$

i.e. $x \geq s - c' - 2t'$. Thus, the number of points in Z' is at most

$$\binom{d'-1}{2} + s - 1 - c' - 2t' - x \leq \binom{d'-1}{2} - 1.$$

It follows that there exists a curve D of degree $d' - 3$ passing through the points of Z'. Hence $D + C_{r+1} + \dots + C_n$ is a curve of degree $d - 3$ through Z. □

We pass now to study a sort of converse of the previous theorem.

Lemma 5.16. *Let $C = C_1 + \dots + C_n$ be a tame reduced curve of degree d. If Z is any admissible subset of the singular locus Z_0 of C, then Z has cardinality at least $(d-1)(d-2)/2$.*

Proof. Any graph adapted to Z has at least one edge for any point of $Z_0 - Z$, one edge for any contact point between two components of C and two edges for any point where three components of C meet. Call c the number of contact points of two components of C and call t the number of points at which three components meet. Since every graph with n vertices and at least n edges is not a forest (see [Diestel]), then the number of points in $Z_0 - Z$ is at most $n - 1 - c - 2t$. Using lemma 5.1, one sees that Z has at least:

$$\frac{(d-1)(d-2)}{2} + n - 1 - c - 2t - (n - 1 - c - 2t) = \frac{(d-1)(d-2)}{2}$$

points. □

Theorem 5.17. *Let $C = C_1 + \dots + C_n$ be a tame reduced plane curve of degree d. Let Z be a subset of the singular locus of C. If Z is admissible, then Z cannot lie in a curve of degree $d - 3$.*

Proof. The proof goes by induction on the number n of components of C.
If there is only one component, i.e. C is irreducible, then the admissible set Z is necessarily the full set of singular points of the rational curve C. In particular Z has cardinality $(d-1)(d-2)/2$. Since these points are just nodes or ordinary cusps, a curve D of degree $d-3$ through Z is an adjoint curve to C and it would cut effective canonical divisors on C, a contradiction since C is rational.
Assume now that the claim is true when the number of components is at most $n-1$. Let Z be an admissible subset of the singular locus Z_0 of C. We know from lemma 5.16 that Z consists of at least $(d-1)(d-2)/2$ points.
If Z belongs to a curve D of degree $d-3$, then computing the intersection number, one sees immediately that D must contain some component of C, for $d(d-3) < (d-1)(d-2)$. Assume that D contains $C_1 + \dots + C_j$, with $j < n$, and call D' the residue. Also write $C' = C_{j+1} + \dots + C_n$ and define $Z' = Z - (C_1 + \dots + C_j)$. Notice that C' still satisfies conditions (a), …, (h) and Z' is an admissible subset of the singular locus of C', by lemma 5.13. Put $d' = \deg(C')$. Then D' is a curve of degree $d' - 3$ containing Z'. As C' contains at least one component less than C, by induction we get a contradiction.

□

Corollary 5.18. *Let C be a tame curve of degree d and let Z be an admissible subset of the singular locus of C. Assume that either Z misses some singular point of C, or there is a component of C which is not a line. Then Z has either type $(d-2, d-1)$ or type $(d-2, d)$.*

Proof. Assume that C has some component whose degree is bigger than 1, or Z misses some singular point of C. Then, by remark 2.3, there exists a curve of degree $d-2$ through Z. By theorem 5.17, Z does not lie in a curve of degree $d-3$. Since C contains $2Z$, the conclusion follows. □

Proof of Theorem 1.2. The forward implication is due to Proposition 4.4 together with Theorem 5.15. The reverse implication is due to Examples 4.1, 4.2 and Theorem 5.17, corresponding to the three case of Theorem 1.2. □

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