TRIPLE-POINT DEFECTIVE SURFACES

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ABSTRACT. In this paper we study the linear series $|L - 3p|$ of hyperplane sections with a triple point p on a surface S embedded via a very ample line bundle L for a *general* point p. If this linear series does not have the expected dimension we call (S, L) *triple-point defective*. We show that on a triple-point defective *regular* surface through a general point every hyperplane section has either a triple component or the surface is rationally ruled and the hyperplane section contains twice a fibre of the ruling.

1. INTRODUCTION

Throughout this note, S will be a smooth projective surface, $K = K_S$ will denote the canonical class and L will be a divisor class on S such that L is *very ample* and $L - K$ is *ample and base-point-free*.

The classical *interpolation problem* for the pair (S, L) is devoted to the study of the varieties:

$$V^{gen}_{m_1,\ldots,m_n} = \big\{ C \in |L| \;\big|\; p_1, \ldots, p_n \in S \text{ general}, \; \mathrm{mult}_{p_i}(C) \geq m_i \big\}.$$

In a more precise formulation, we start from the incidence variety:

$$\mathcal{L}_{m_1,\ldots,m_n} = \{ (C, (p_1, \ldots, p_n)) \in |L| \times S^n \mid \mathrm{mult}_{p_i}(C) \geq m_i \}$$

together with the canonical projections:

$$
\begin{array}{ccc}
\mathcal{L}_{m_1,\ldots,m_n} & \xrightarrow{\;\alpha\;} & S^n \\
{\scriptstyle\beta}\big\downarrow & & \\
|L| = \mathbb{P}(H^0(L)^*) & &
\end{array}
\tag{1}
$$

As for the map α, the fibre over a fixed point $(p_1, \ldots, p_n) \in S^n$ is just the linear series $|L - m_1 p_1 - \cdots - m_n p_n|$ of effective divisors in $|L|$ having a point of multiplicity at least m_i at p_i. These fibres being irreducible, we deduce that if α is *dominant* then $\mathcal{L}_{m_1,\ldots,m_n}$ has a unique irreducible component, say $\mathcal{L}^{gen}_{m_1,\ldots,m_n}$, which dominates S^n. The closure of its image

$$V_{m_1,\ldots,m_n} := V_{m_1,\ldots,m_n}(S, L) := \overline{\beta(\mathcal{L}^{gen}_{m_1,\ldots,m_n})}$$

under β is an irreducible closed subvariety of $|L|$, a *Severi variety* of (S, L).

Imposing a point of multiplicity m_i corresponds to killing $\binom{m_i+1}{2}$ partial derivatives, so that

$$\dim |L - m_1 p_1 - \cdots - m_n p_n| \geq \max\left\{-1, \dim |L| - \sum_{i=1}^{n} \binom{m_i + 1}{2}\right\},$$

and one expects that the previous inequality is in fact an equality, for the choice of general points $p_1, \ldots, p_n \in S$.

When this is not the case, then the pair (S, L), is called *defective* and is endowed with some special structure. By abuse of notation we sometimes call the surface S defective, if L is understood.

The case when $m_i = 2$ for all i has been classically considered (and solved) by Terracini, who classified in [18] double–point defective surfaces. If S is double-point defective, then a general curve $C \in |L - 2p_1 - \cdots - 2p_n|$ has a double component passing through each point p_i, and the map β in Diagram (1) has positive dimensional fibres.

When the multiplicities grow, the situation becomes much more complicated. Even in the case $S = \mathbb{P}^2$, the situation is not understood and there are several, still unproved conjectures on the structure of defective embeddings (see [7] for an introductory survey).

Let us point out a first difference between the case of multiplicity two and the case of higher multiplicity. It is easy to show that imposing on $|L|$ multiplicity two at *one* general point always yields three independent conditions, so that $\dim |L - 2p| = \dim |L| - 3$, the expected dimension. Contrary to this, on some surfaces, it turns out that even imposing *just one* point of multiplicity 3, one may obtain a defective behaviour.

Example 1

Let $S = \mathbb{F}_e \xrightarrow{\pi} \mathbb{P}^1$ be a Hirzebruch surface, $e \geq 0$. We denote by F a fibre of π and by C_0 the section of π of minimal self intersection $C_0^2 = -e$ – both of which are smooth rational curves. The general element C_1 in the linear system $|C_0 + eF|$ will be a section of π which does not meet C_0 (see e.g. [12], Theorem 2.17).

Consider now the divisor $L = (2+b)\cdot F + C_1 = (2+b+e)\cdot F + C_0$ for some fixed $b \geq 0$. Then for a general $p \in S$ there are curves $D_p \in |bF + C_1 - p|$ and there is a unique curve $F_p \in |F - p|$, in particular $p \in F_p \cap D_p$. For each choice of D_p we have

$$2F_p + D_p \in |L - 3p|.$$

Since $F \cdot L = 1 = F \cdot (L - F)$ we see that every curve in $|L - 3p|$ must contain F_p as a double component, i.e.

$$|L - 3p| = 2F_p + |bF + C_1 - p|.$$

Moreover, since $bF + C_1$ is base-point free (see [12], Theorem 2.17) we have (see [10], Lemma 35)

$$\begin{aligned}
\dim|bF + C_1 - p| &= \dim|bF + C_1| - 1 \\
&= h^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b+e)\big) + h^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b)\big) - 2 = 2b + e
\end{aligned}$$

and, using the notation from above,

$$\dim(V_3) \geq \dim|bF + C_1 - p| + 2 = 2b + e + 2.$$

However,

$$\dim|L| = h^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2+b+e)\big) + h^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2+b)\big) - 1 = 2b + e + 5,$$

and thus

$$\mathrm{expdim}(V_3) = \dim|L| - 4 = 2b + e + 1 < 2b + e + 2 = \dim(V_3).$$

We say, $(\mathbb{F}_e, L)$ is *triple-point defective*, see Definition 1.2.

Note, moreover, that L is very ample, as is $L - K_S$ for $b \geq \max\{0, e-3\}$ (see [12], Corollary 2.18), and that

$$(L - K)^2 = \big((4 + 2e + b) \cdot F + 3 \cdot C_0\big)^2 2 = 24 + 6b + 3e > 16.$$

$\square$

It is interesting to observe that, even though, in the previous example, the general element of $|L - 3p|$ is non-reduced, still the map β of Diagram (1) has finite general fibres, since the general element of $|L - 3p|$ has no triple components.

The aim of this note is to investigate the structure of pairs (S, L) for which the linear system $|L - 3p|$ for $p \in S$ general has dimension bigger that the expected value $\dim |L| - 6$. We will show that, under some assumptions, ruled surfaces are the only case of triple point defective surfaces.

Definition 2

We say that the pair (S, L) is *triple-point defective* or, in classical notation, that (S, L) *satisfies one Laplace equation* if

$$\dim |L - 3p| > \max\{-1, \dim |L| - 6\} = \operatorname{expdim} |L - 3p|$$

for $p \in S$ general.

Remark 3

Going back to Diagram (1), one sees that (S, L) is triple-point defective if and only if either:

- $\dim |L| \leq 5$ and the projection $\alpha : \mathcal{L}_3 \to S$ dominates, or
- $\dim |L| > 5$ and the general fibre of the map α has dimension at least $\dim |L| - 5$.

In particular, (S, L) is triple-point defective if and only if the map α is *dominant* and

$$\dim(\mathcal{L}_3^{gen}) > \dim |L| - 4.$$

The particular case in which the general fibre of the map β in Diagram (1) is positive-dimensional, (i.e. the general member of V_3 contains a triple component through p) has been studied in [4], [8], and [3]. We will recall the classification of such surfaces in Theorem 2.2 below. Notice that these surfaces are almost always singular (i.e. L is not very ample), so that they do not appear in the statement of our main theorem, *where, indeed, we make no assumptions on the dimension of the fibres of β.*

One of the major subjects in algebraic interpolation theory, namely Segre's conjecture on defective linear systems *in the plane*, suggests in

our situation that, when (S, L) is triple-point defective, then the general element of $|L - 3p|$ must be non-reduced, with a double component through p.

We will show here, under some assumptions, that this extension of Segre's conjecture for triple point defectivity holds for a single triple point.

Our main result is:

Theorem 4

Let L be a very ample line bundle on S, such that $L - K$ is ample and base-point-free. Assume $(L - K)^2 > 16$ and (S, L) is triple-point defective.

Then S is ruled in the embedding defined by L. Moreover a general curve $C \in |L - 3p|$ contains the fibre of the ruling through p as fixed component with multiplicity at least two.

Remark 5

In the paper [6] we classify all triple-point defective linear systems L on ruled surfaces satisfying the assumptions of Theorem 1.4, and it follows from this classification that the linear system $|L - 3p|$ will contain the fibre of the ruling through p precisely with multiplicity two as a fixed component. In particular, the map β will automatically be generically finite.

Our method is an application of Reider's analysis of rank 2 bundles arising from triple points which do not impose independent conditions. Under the assumption that $(L - K)^2 > 16$, the bundle is Bogomolov unstable, and we show that from the destabilising divisors A and $B = L - K - A$ one gets the multiple fibre. We point out that we obtain in this way a natural geometric construction for the non–reduced divisor which must be part of any defective linear system.

This application of Reider's construction for the investigation of defective surfaces was introduced by Beltrametti, Francia and Sommese in [2]. We will refer to [2] for the first main properties of the destabilising divisors A and B (see Section 4 below).

Then, we will use the assumption "$L - K$ ample and base-point-free" to control curves of low degree on S. The freeness of $L - K$ seems

unavoidable in the argument of the crucial Lemma 14, which in turn implies that B is fixed part free and determines a ruling on S.

Let us finish by pointing out in the following corollary what happens if we apply our result to $\mathbb{P}^2$ and its blow ups, and notice that, combining results in [19] and [16] Corollary 2.6, one can give purely numerical conditions on r and the m_i such that $L - K$ there is ample and base-point-free.

Corollary 6

Fix multiplicities $m_1 \leq m_2 \leq \cdots \leq m_n$. Let H denote the class of a line in $\mathbb{P}^2$ and assume that, for $p_1, \ldots, p_n$ general in $\mathbb{P}^2$, the linear system $M = rH - m_1p_1 - \cdots - m_np_n$ is defective, i.e.

$$\dim |M| > \max \left\{ -1, \binom{r+2}{2} - \sum_{i=1}^{n} \binom{m_i+1}{2} \right\}.$$

Let $\pi : S \longrightarrow \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ at the points $p_2, \ldots, p_n$ and set $L := r\pi^*H - m_2E_2 - \cdots - m_nE_n$, where $E_i = \pi^*(p_i)$ is the i-th exceptional divisor. Assume that L is very ample on S, of the expected dimension $\binom{r+2}{2} - \sum_{i=2}^{n} \binom{m_i+1}{2}$, and that $L - K$ is ample and base-point-free on S, with $(L - K)^2 > 16$. Assume, finally, $m_1 \leq 3$.

Then $m_1 = 3$ and the general element of M is non-reduced. Moreover L embeds S as a ruled surface.

Proof. Just apply the Main Theorem 1.4 to the pair (S, L). $\square$

The reader can easily check that the previous result is exactly the translation of Segre's and Harbourne–Hirschowitz's conjectures on defective linear systems in the plane, for the case of a *minimally* defective system with lower multiplicity 3. The (-1)–curve predicted by the Harbourne–Hirschowitz conjecture, in this situation, is just the pull-back of a line of the ruling. Thus, although in a partial situation, we get new evidence for the conjecture, at least when the *minimal* multiplicity imposed at the points is 3.

The paper is organised as follows.

The case where β is not generically finite is pointed out in Theorem 2.2 in Section 2. In Section 3 we reformulate the problem as an h^1-vanishing problem. The Sections 4 to 7 are devoted to the proof of the

main result: in Section 4 we use Serre's construction and Bogomolov instability in order to show that triple-point defectiveness leads to the existence of very special divisors A and B on our surface; in Section 5 we show that $|B|$ has no fixed component; in Section 6 we then list properties of B and we use these in Section 7 to classify the triple-point defective surfaces.

The authors wish to thank the referee, who pointed out the possibility of weakening one assumption in a preliminary version of the main theorem.

2. TRIPLE COMPONENTS

In this section, we consider what happens when, in Diagram (1), the general fibre of β is positive-dimensional, in other words, when the general member of V_3 contains a triple component through p.

This case has been investigated (and essentially solved) in [4], and then rephrased in modern language in [8] and [3].

Although not strictly necessary for the sequel, as our arguments do not make any use of the generic finiteness of β, (and so we will not assume this), for the sake of completeness we recall in this section some example and the classification of pairs (S, L) which are triple-point defective, and such that a general curve $L_p \in |L - 3p|$ has a triple component through p.

The family $\mathcal{L}_3$ of pairs $(L, p) \in |L| \times S$ where $L \in |L - 3p|$ has dimension bounded below by $\dim |L| - 4$, and in Remark 1.3 it has been pointed out that (S, L) is triple-point defective exactly when α is dominant and the bound is not attained.

Notice however that $\dim |L| - 4$ is *not* necessarily a bound for the dimension of the subvariety $V_3 \subset |L|$, the image of $\mathcal{L}_3$ under β. The following example (exploited in [15]) shows that one may have $\dim(V_3) < \dim |L| - 4$ even when (S, L) is *not* triple-point defective.

Example 7 ((see [15]))
Let S be the blowing up of $\mathbb{P}^2$ at 8 general points $q_1, \ldots, q_8$ and L corresponds to the system of curves of degree nine in $\mathbb{P}^2$, with a triple point at each q_i.

$\dim |L| = 6$, but for $p \in S$ general, the unique divisor in $|L - 3p|$ coincides with the cubic plane curve through $q_1, \ldots, q_8, p$, counted three times. As there exists only a (non-linear) one-dimensional family of such divisors in $|L|$, then $\dim(V_3) = 1 < \dim |L| - 4$. On the other hand, these divisors have a triple component, so that the general fibre of β has dimension one, hence $\dim(\mathcal{L}_3) = 2 = \dim |L| - 4$.

The classification of triple-point defective pairs (S, L) for which the map β is not generically finite is the following.

Theorem 8

Suppose that (S, L) is triple-point defective. Then for $p \in S$ general, the general member of $|L - 3p|$ contains a triple component through p if and only if S lies in a three dimensional scroll W containing a one dimensional family of planes, and moreover W is developable, i.e. the tangent space to W is constant along the planes.

Proof. First, since we assume that S is triple-point defective and embedded in $\mathbb{P}^r$ via L, then the hyperplanes π that meet S in a divisor $H = S \cap \pi$ with a triple point at a general $p \in S$, intersect in a $\mathbb{P}^4$. Thus we may project down S to $\mathbb{P}^5$ and work with the corresponding surface.

In this setting, through a general $p \in S$ one has only one hyperplane π with a triple contact, and π has a triple contact with S along the fibre C of β. Thus V_3 is a curve.

If H', H'' are two consecutive infinitesimally near points to H on V_3, then C also belongs to $H \cap H' \cap H''$. Thus C is a plane curve and S is fibred by a 1-dimensional family of plane curves. This determines the three dimensional scroll W.

The tangent line to V_3 determines in $(\mathbb{P}^5)^*$ a pencil of hyperplanes which are tangent to S at any point of C, since this is the infinitesimal deformation of a family of hyperplanes with a triple contact along any point of C. Thus there is a $\mathbb{P}^4 = H_C$ which is tangent to S along C.

Assume that C is not a line. Then C spans a $\mathbb{P}^2 = \pi_C$ fibre of W, moreover the tangent space to W at a general point of C is spanned by π_C and $T_{S,P}$, hence it is constantly equal to H_C. Since C spans π_C,

then it turns out that the tangent space to W is constant at any point of π_C, i.e. W is developable.

When C is a line, then arguing as above one finds that all the tangent planes to S along C belong to the same $\mathbb{P}^3$. This is enough to conclude that S sits in some developable 3-dimensional scroll.

Conversely, if S is contained in the developable scroll W, then at a general point p, with local coordinates x, y, the tangent space t to W at p contains the derivatives $p, p_x, p_y, p_{xx}, p_{xy}$ (here x is the direction of the tangent line to C). Thus the $\mathbb{P}^4$ spanned by t, p_{yy} intersects S in a triple curve along C. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. The Equimultiplicity Ideal

If L_p is a curve in $|L - 3p|$ we denote by $f_p \in \mathbb{C}\{x_p, y_p\}$ an equation of L_p in local coordinates x_p and y_p at p. If $\mathrm{mult}_p(L_p) = 3$, the ideal sheaf $\mathcal{J}_{Z_p}$ whose stalk at p is the equimultiplicity ideal

$$\mathcal{J}_{Z_p, p} = \left\langle \frac{\partial f_p}{\partial x_p}, \frac{\partial f_p}{\partial y_p} \right\rangle + \langle x_p, y_p \rangle^3$$

of f_p defines a zero-dimensional scheme $Z_p = Z_p(L_p)$ concentrated at p, and the tangent space $T_{(L_p, p)}(\mathcal{L}_3)$ of $\mathcal{L}_3$ at (L_p, p) satisfies (see [17] Example 10)

$$T_{(L_p, p)}(\mathcal{L}_3) \cong \left(H^0\big(S, \mathcal{J}_{Z_p}(L_p)\big)/H^0(S, \mathcal{O}_S) \right) \oplus \mathcal{K},$$

where $\mathcal{K}$ is zero unless L_p is unitangential at p, in which case $\mathcal{K}$ is a one-dimensional vector space.

In particular, $\mathcal{L}_3$ is smooth at (L_p, p) of the expected dimension (see [17] Proposition 11)

$$\mathrm{expdim}(\mathcal{L}_3) = \dim |L| - 4$$

as soon as

$$h^1\big(S, \mathcal{J}_{Z_p}(L)\big) = 0.$$

We thus have the following proposition.

Proposition 9

Suppose that α is surjective, then (S, L) is not triple-point defective if

$$h^1\big(S, \mathcal{J}_{Z_p}(L)\big) = 0$$

for general $p \in S$ and $L_p \in |L|$ with $\mathrm{mult}_p(L_p) = 3$.
Moreover, if L is non-special, i.e. if $h^1(S, L) = 0$, the above h^1-vanishing
is also necessary for the non-triple-point-defectiveness of (S, L).

Note that by Kodaira vanishing L is non-special whenever $L - K$ is ample.

4. THE BASIC CONSTRUCTION

From now on we assume that for $p \in S$ general $\exists \, L_p \in |L|$ s.t.

$$h^1\bigl(S, \mathcal{J}_{Z_p}(L)\bigr) \neq 0.$$

Then by Serre's construction for a subscheme $Z'_p \subseteq Z_p$ with ideal sheaf $\mathcal{J}_p = \mathcal{J}_{Z'_p}$ of minimal length such that $h^1\bigl(S, \mathcal{J}_p(L)\bigr) \neq 0$ there is a rank two bundle $\mathcal{E}_p$ on S and a section $s \in H^0(S, \mathcal{E}_p)$ whose 0-locus is Z'_p, giving the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{E}_p \to \mathcal{J}_p(L - K) \to 0. \tag{2}$$

The Chern classes of $\mathcal{E}_p$ are

$$c_1(\mathcal{E}_p) = L - K \quad \text{and} \quad c_2(\mathcal{E}_p) = \mathrm{length}(Z'_p).$$

Moreover, Z'_p is automatically a complete intersection.
We would now like to understand what $\mathcal{J}_p$ is depending on $\mathrm{jet}_3(f_p)$, which in suitable local coordinates will be one of those in Table (3). For this we first of all note that the very ample divisor L separates all subschemes of Z_p of length at most two. Thus Z'_p has length at least 3, and due to Lemma 4.1 below we are in one of the following situations:

$\mathrm{jet}_3(f_p)$	$\mathcal{J}_{Z_p, p}$	$\mathrm{length}(Z_p)$	$\mathcal{J}_p = \mathcal{J}_{Z'_p, p}$	$c_2(\mathcal{E}_p)$
$x_p^3 - y_p^3$	$\langle x_p^2, y_p^2 \rangle$	4	$\langle x_p^2, y_p^2 \rangle$	4
$x_p^2 y_p$	$\langle x_p^2, x_p y_p, y_p^3 \rangle$	4	$\langle x_p, y_p^3 \rangle$	3
x_p^3	$\langle x_p^2, x_p y_p^2, y_p^3 \rangle$	5	$\langle x_p^2, y_p^2 \rangle$	4
x_p^3	$\langle x_p^2, x_p y_p^2, y_p^3 \rangle$	5	$\langle x_p, y_p^3 \rangle$	3

$$(3)$$

Lemma 10

If $f \in R = \mathbb{C}\{x, y\}$ with $\mathrm{jet}_3(f) \in \{x^3 - y^3, x^2 y, x^3\}$, and if $I = \langle g, h \rangle \lhd R$ such that $\dim_\mathbb{C}(R/I) \geq 3$ and $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle + \langle x, y \rangle^3 \subseteq I$, then we may assume that we are in one of the following cases:

  (a) $I = \langle x^2, y^2 \rangle$ *and* $\mathrm{jet}_3(f) \in \{x^3 - y^3, x^3\}$, *or*
  (b) $I = \langle x, y^3 \rangle$ *and* $\mathrm{jet}_3(f) \in \{x^2 y, x^3\}$.

Proof. If $>$ is any *local degree* ordering on R, then the Hilbert-Samuel functions of R/I and of $R/L_>(I)$ coincide, where $L_>(I)$ denotes the leading ideal of I (see e.g. [11] Proposition 5.5.7). In particular, $\dim_\mathbb{C}(R/I) = \dim_\mathbb{C}(R/L_>(I))$ and thus

$$L_>(I) \in \big\{ \langle x^2, xy^2, y^3 \rangle, \langle x^2, xy, y^2 \rangle, \langle x^2, xy, y^3 \rangle, \langle x^2, y^2 \rangle, \langle x, y^3 \rangle \big\},$$

since $\langle x^2, xy^2, y^3 \rangle \subset I$.

Taking $>$, for a moment, to be the local degree ordering on R with $y > x$ we deduce at once that I does not contain any power series with a linear term in y. For the remaining part of the proof $>$ will be the local degree ordering on R with $x > y$.

<u>1st Case:</u> $L_>(I) = \langle x^2, xy^2, y^3 \rangle$ or $L_>(I) = \langle x^2, xy, y^2 \rangle$. Thus the graph of the slope $H^0_{R/I}$ of the Hilbert-Samuel function of R/I would be as shown in Figure 1, which contradicts the fact that I is a complete intersection due to [13] Theorem 4.3.
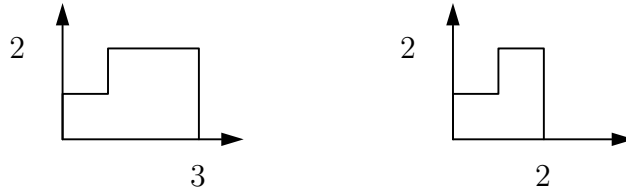

FIGURE 1. The graphs of $H^0_{R/\langle x^2, xy^2, y^3 \rangle}$ respectively of $H^0_{R/\langle x^2, xy, y^2 \rangle}$.

<u>2nd Case:</u> $L_>(I) = \langle x^2, xy, y^3 \rangle$. Then we may assume

$$g = x^2 + \alpha \cdot y^2 + h.o.t. \quad \text{and} \quad h = xy + \beta \cdot y^2 + h.o.t..$$

Since $x^2 \in I$ there are power series $a, b \in R$ such that

$$x^2 = a \cdot g + b \cdot h.$$

Thus the leading monomial of a is one, a is a unit and $g \in \langle x^2, h \rangle$. We may therefore assume that $g = x^2$. Moreover, since the intersection multiplicity of g and h is $\dim_{\mathbb{C}}(R/I) = 4$, g and h cannot have a common tangent line in the origin, i. e. $\beta \neq 0$. Thus, since $g = x^2$, we may assume that $h = xy + y^2 \cdot u$ with $u = \beta + h.o.t$ a unit.

In new coordinates $\widetilde{x} = x \cdot \sqrt{u}$ and $\widetilde{y} = y \cdot \frac{1}{\sqrt{u}}$ we have

$$I = \langle \widetilde{x}^2, \widetilde{x}\widetilde{y} + \widetilde{y}^2 \rangle.$$

Note that by the coordinate change $\mathrm{jet}_3(f)$ only changes by a constant, that $\frac{\partial f}{\partial \widetilde{x}}, \frac{\partial f}{\partial \widetilde{y}} \in I$ and that $\langle \widetilde{x}, \widetilde{y} \rangle^3 \subset I$, but $\widetilde{x}\widetilde{y}, \widetilde{y}^2 \notin I$. Thus $\mathrm{jet}_3(f) = x^3$. Setting now $\bar{x} = \widetilde{x}$ and $\bar{y} = \widetilde{x} + 2\widetilde{y}$, then $\bar{y}^2 = \widetilde{x}^2 + 4 \cdot (\widetilde{x}\widetilde{y} + \widetilde{y}^2) \in I$ and thus, considering colengths,

$$I = \langle \bar{x}^2, \bar{y}^2 \rangle.$$

Moreover, the 3-jet of f does not change with respect to the new coordinates, so that we may assume we worked with these from the beginning.

<u>3rd Case:</u> $L_>(I) = \langle x^2, y^2 \rangle$. Then we may assume

$$g = x^2 + \alpha \cdot xy + h.o.t. \quad \text{and} \quad h = y^2 + h.o.t.$$

As in the second case we deduce that w.l.o.g. $g = x^2$ and thus $h = y^2 \cdot u$, where u is a unit. But then $I = \langle x^2, y^2 \rangle$.

<u>4th Case:</u> $L_>(I) = \langle x, y^3 \rangle$. Then we may assume

$$g = x + h.o.t. \quad \text{and} \quad h = y^3 + h.o.t.$$

since there is no power series in I involving a linear term in y. In new coordinates $\widetilde{x} = g$ and $\widetilde{y} = y$ we have

$$I = \langle \widetilde{x}, \widetilde{h} \rangle,$$

and we may assume that $\widetilde{h} = \widetilde{y}^3 \cdot u$, where u is a unit only depending on $\widetilde{y}$. Hence, $I = \langle \widetilde{x}, \widetilde{y}^3 \rangle$. Moreover, the 3-jet of f does not change with respect to the new coordinates, so that we may assume we worked with these from the beginning. $\qquad \square$

From now on we assume that $(L - K)^2 > 16$.

Thus

$$c_1(\mathcal{E}_p)^2 - 4 \cdot c_2(\mathcal{E}_p) > 0,$$

and hence $\mathcal{E}_p$ is Bogomolov unstable. The Bogomolov instability implies the existence of a unique divisor A_p which destabilises $\mathcal{E}_p$. (See e. g. [9] Section 9, Corollary 2.) In other words, setting $B_p = L - K - A_p$, i. e.

$$A_p + B_p = L - K, \qquad (4)$$

there is an immersion

$$0 \to \mathcal{O}_S(A_p) \to \mathcal{E}_p \qquad (5)$$

where $(A_p - B_p)^2 \geq c_1(\mathcal{E}_p)^2 - 4 \cdot c_2(\mathcal{E}_p) > 0$ and $(A_p - B_p) \cdot H > 0$ for every ample H. The same construction was considered in [2] and with their Proposition 1.4 it follows:

(a) $\mathcal{E}_p(-A_p)$ has a global section that vanishes along a subscheme $\widetilde{Z}_p$ of codimension 2 and which gives rise to a short exact sequence:

$$0 \to \mathcal{O}_S(A_p) \to \mathcal{E}_p \to \mathcal{J}_{\widetilde{Z}_p}(B_p) \to 0. \qquad (6)$$

(b) The divisor B_p is effective and we may assume that $Z'_p \subset B_p$.

(c) The divisors A_p and B_p satisfy the following numerical condition:

$$\text{length}(Z'_p) \geq A_p \cdot B_p \geq B_p^2 + 1. \qquad (7)$$

(d) $A_p - B_p$ and A_p are big.

Now let p move freely in S. Accordingly the scheme Z'_p moves, hence the effective divisor B_p containing Z'_p moves in an algebraic family $\mathcal{B} \subseteq |B|_a$ which is the closure of $\{B_p \mid p \in S, L_p \in |L - 3p|, \text{ both general}\}$ and which covers S. A priori this family $\mathcal{B}$ might have a *fixed part* C, so that for general $p \in S$ there is an effective divisor D_p moving in a fixed-part free algebraic family $\mathcal{D} \subseteq |D|_a$ such that

$$B_p = C + D_p.$$

Whenever we only refer to the algebraic class of A_p respectively B_p respectively D_p we will write A respectively B respectively D for short.

For these considerations we assume, of course, that $\operatorname{length}(Z_p')$ is constant for $p \in S$ general, so either $\operatorname{length}(Z_p') = 3$ or $\operatorname{length}(Z_p') = 4$.

5. $C = 0$.

Our first aim is to show that actually $C = 0$ (see Lemma 5.5). But in order to do so we first have to consider the boundary case that $A_p \cdot B_p = \operatorname{length}(Z_p')$.

Proposition 11

If $A_p \cdot B_p = \operatorname{length}(Z_p')$, then there exists a non-trivial global section $0 \neq s \in H^0\big(B_p, \mathcal{J}_{Z_p'/B_p}(A_p)\big)$ whose zero-locus is Z_p'.

In particular, $A_p \cdot D_p = A_p \cdot B_p = \operatorname{length}(Z_p')$ and $A_p \cdot C = 0$.

Proof. By Sequence (6) we have

$$A_p \cdot B_p = \operatorname{length}(Z_p') = c_2(\mathcal{E}_p) = A_p \cdot B_p + \operatorname{length}\big(\widetilde{Z}_p\big).$$

Thus $\widetilde{Z}_p = \emptyset$.

If we merge the sequences (2), (6), and the structure sequence of B twisted by B we obtain the exact commutative diagram in Figure 2, where $\mathcal{O}_{B_p}(B_p) = \mathcal{J}_{Z_p'/B_p}(A_p + B_p)$, or equivalently $\mathcal{O}_{B_p} = \mathcal{J}_{Z_p'/B_p}(A_p)$. Thus from the rightmost column we get a non-trivial global section, say s, of this bundle which vanishes precisely at Z_p', since Z_p' is the zero-locus of the monomorphism of vector bundles $\mathcal{O}_S \hookrightarrow \mathcal{E}_p$. However, since p is general we have that $p \notin C$ and thus the restriction $0 \neq s_{|D_p} \in H^0\big(D_p, \mathcal{J}_{Z_p'/D_p}(A_p)\big)$ and it still vanishes precisely at Z_p'. Thus $A_p \cdot D_p = \operatorname{length}(Z_p') = A_p \cdot B_p$, and $A_p \cdot C = A_p \cdot B_p - A_p \cdot D_p = 0$. $\quad\square$

We next want to show that positive self-intersection of B imposes hard restrictions.

Lemma 12

$B^2 \leq 2$ and if $B^2 \in \{1, 2\}$ then $A \cdot B = \operatorname{length}(Z_p') = 4$.

Proof. We may suppose that $B^2 > 0$. By (7) we know that $4 \geq A \cdot B > B^2$ and by assumption $(A + B)^2 \geq 17$, so that

$$A^2 = (A + B)^2 - 2 \cdot A \cdot B - B^2 \geq 17 - 8 - 3 > 0$$

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 \longrightarrow & \mathcal{O}_S(A_p) & \longrightarrow & \mathcal{E}_p & \longrightarrow & \mathcal{O}_S(B_p) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathcal{O}_S(A_p) & \longrightarrow & \mathcal{J}_{Z_p'/S}(A_p + B_p) & \longrightarrow & \mathcal{O}_{B_p}(B_p) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}
$$

FIGURE 2. The diagram showing $\mathcal{O}_{B_p} = \mathcal{J}_{Z_p'/B_p}(A_p)$.

and the Hodge Index Theorem gives

$$
(A{\cdot}B)^2 \geq A^2 \cdot B^2 \geq (17 - 2 \cdot A{\cdot}B - B^2) \cdot B^2.
$$

But then $B^2 \geq 3$ leads to the contradiction $16 \geq 18$. Similarly, $A{\cdot}B \leq 3$ leads to $9 \geq (11 - B^2) \cdot B^2$ which is neither for $B^2 = 1$ nor for $B^2 = 2$ fulfilled. This shows that $A{\cdot}B = 4$, and thus by (7) also $\operatorname{length}(Z_p') = 4$. $\qquad\square$

Even though we do not know whether $\mathcal{B}$ has a fixed part or not, we can get some information about the moving part $\mathcal{D}$.

Lemma 13

Let $p \in S$ be general and suppose $\operatorname{length}(Z_p') = 4$.

(a) *If D_p is irreducible, then $\dim(\mathcal{D}) \geq 2$ and $D_p^2 \geq 3$.*

(b) *If D_p is reducible but the part containing p is reduced, then either D_p has a component singular in p and $D_p^2 \geq 5$ or at least two components of D_p pass through p and $D_p^2 \geq 2$.*

(c) *If $D_p^2 \leq 1$, then $D_p = k \cdot E_p$ where $k \geq 2$, E_p is irreducible and $E_p^2 = 0$. In particular, $D_p^2 = 0$.*

Proof.   (a) If D_p is irreducible, then $\dim(\mathcal{D}) \geq 2$, since D_p, containing Z_p', is singular in p by Table (3) and since $p \in S$ is general.

If through $p \in S$ general and a general $q \in D_p$ there is another $D' \in \mathcal{D}$, then due to the irreducibility of D_p

$$D_p^2 = D_p \cdot D' \geq \operatorname{mult}_p(D_p) + \operatorname{mult}_q(D_p) \geq 3.$$

Otherwise, $\mathcal{D}$ is a two-dimensional involution whose general element is irreducible, so that by [5] Theorem 5.10 $\mathcal{D}$ must be a linear system. This, however, contradicts the Theorem of Bertini, since the general element of $\mathcal{D}$ would be singular.

(b) Suppose $D_p = \sum_{i=1}^{k} E_{i,p}$ is reducible but the part containing p is reduced. Since D_p has no fixed component and p is general, each $E_{i,p}$ moves in an at least one-dimensional family. In particular $E_{i,p}^2 \geq 0$.

   If some $E_{i,p}$, say $i = 1$, would be singular in p for $p \in S$ general we could argue as above that $E_{1,p}^2 \geq 3$. Moreover, either $E_{2,p}$ is algebraically equivalent to $E_{1,p}$ and $E_{2,p}^2 \geq 3$, or $E_{1,p}$ and $E_{2,p}$ intersect properly, since both vary in different, at least one-dimensional families. In any case we have

$$D_p^2 \geq (E_{1,p} + E_{2,p})^2 \geq 5.$$

Otherwise, at least two components, say $E_{1,p}$ and $E_{2,p}$ pass through p, since D_p is singular in p and no component passes through p with higher multiplicity. Hence, $E_{1,p} \cdot E_{2,p} \geq 1$ and therefore

$$D_p^2 \geq 2 \cdot E_{1,p} \cdot E_{2,p} \geq 2.$$

(c) From the above we see that D_p is not reduced in p. Let therefore $D_p \equiv_a kE_p + E'$ where $k \geq 2$, E_p passes through p and E' does not contain any component algebraically equivalent to E_p.

   Suppose $E' \neq 0$. Since D_p has no fixed component both, E_p and E' vary in an at least one dimensional family covering S and must therefore intersect properly. In particular, $E_p \cdot E' \geq 1$ and $1 \geq D_p^2 \geq 2k \cdot E_p \cdot E' \geq 4$. Thus, $E' = 0$.

   We therefore may assume that $D_p = kE_p$ with $k \geq 2$. Then $0 \leq E_p^2 = \frac{1}{k^2} \cdot D_p^2 \leq \frac{1}{4}$, which leaves only the possibility $E_p^2 = 0$, implying also $D_p^2 = 0$.

$\square$

The following observations on the self intersection number of irreducible curves embedded via $L - K$ in our situation is an important tool in the proof that the fixed part C does not exist.

Lemma 14

Suppose that $R \subset S$ is an irreducible curve, L is very ample, and $L - K$ is base-point-free on S.

(a) *If $(L - K) \cdot R = 1$, then R is smooth, rational and $R^2 \leq -2$.*
(b) *If $(L - K) \cdot R = 2$, then one of the following two cases occurs:*
    (1) *R is smooth and rational with $R^2 \leq -1$, or*
    (2) *$|L - K|$ induces a $\mathfrak{g}_2^1$ on R and $L + R$ does not separate the points of this $\mathfrak{g}_2^1$.*
  In any case, if R moves in a one dimensional algebraic family, then $R^2 \neq 0$.

Proof. Since $|L - K|$ is base-point-free it defines a morphism

$$\varphi_{|L-K|} : S \longrightarrow \mathbb{P}^n$$

and if $C = \varphi_{|L-K|}(R)$ and $\varphi : R \longrightarrow C$ denotes the restriction of $\varphi_{|L-K|}$ then

$$\deg(\varphi) \cdot \deg(C) = (L - K) \cdot R.$$

Moreover, by the adjunction formula we know that

$$p_a(R) = \frac{R^2 + R \cdot K}{2} + 1,$$

and since L is very ample we thus get

$$1 \leq L \cdot R = (L - K) \cdot R + R \cdot K = (L - K) \cdot R + 2 \cdot \big(p_a(R) - 1\big) - R^2. \quad (8)$$

(a) If $(L - K) \cdot R = 1$, then C is a line in $\mathbb{P}^n$ and φ is a birational morphism from R to C. It thus is an isomorphism, and R must be a smooth, rational curve. We deduce from (8)

$$R^2 \leq (L - K) \cdot R - 3 = -2.$$

(b) If $(L - K) \cdot R = 2$, then either the degree of φ is one or two.
    Suppose first that $\deg(\varphi) = 1$. Then as above φ is a birational morphism and hence an isomorphism. C being an irreducible

conic it is smooth and rational, and so is R. We deduce from (8)

$$R^2 \leq (L - K){\cdot}R - 3 = -1.$$

Consider now the case $\deg(\varphi) = 2$. $|L - K|$ cuts out a $\mathfrak{g}_2^1$ on R which induces the morphism φ. Even if R is singular the dualizing sheaf on R is given by the restriction of $K + R$, and it satisfies the Riemann-Roch formula (see e.g. [12, Ex. IV.I.9]), i.e. if $\mathfrak{d}$ is any divisor on C we have

$$h^0(\mathfrak{d}) - h^0\big((K + R)_{|R} - \mathfrak{d}\big) = \deg(\mathfrak{d}) + 1 - p_a(R). \tag{9}$$

Suppose now that $P + Q \in \mathfrak{g}_2^1$ with P and Q in the smooth part of C. Then

$$h^0\big((K + R)_{|R} - (L + R)_{|R} + P\big) = h^0\big(P - (L - K)_{|R}\big) = h^0(-Q) = 0$$

and

$$h^0\big((K+R)_{|R}-(L+R)_{|R}+P+Q\big) = h^0\big(P+Q-(L-K)_{|R}\big) = h^0(\mathcal{O}_R) = 1.$$

The Theorem of Riemann-Roch (9) thus gives

$$h^0\big((L + R)_{|R} - P\big) = (L + R).R - 1 + 1 - p_a(R)$$

and

$$h^0\big((L + R)_{|R} - P - Q\big) - 1 = (L + R).R - 2 + 1 - p_a(R).$$

Hence

$$h^0\big((L + R)_{|R} - P\big) = h^0\big((L + R)_{|R} - P - Q\big),$$

i.e. each divisor in the linear series induced by $L + R$ on R which contains P contains automatically also Q. The divisors in $|L + R|$ thus do not seperate the points P and Q.

Suppose now that $\dim |R|_a \geq 1$ and $R^2 = 0$. Then $|R|_a$ is pencil and induces a fibration of S whose fibres are the elements of $|R|_a$ (see [14] App. B.1). But then $\mathcal{O}_R(R)$ is trivial (see e.g. [1, Lem. 8.1]) and thus $\mathcal{O}_R(L + R) = \mathcal{O}_R(L)$ is very ample, which contradicts the fact that it does not separate the points of the $\mathfrak{g}_2^1$.

$\square$

Lemma 15

The family $\mathcal{B}$ introduced on page 13 has no fixed part. I.e. under the assumptions of Section 4 and with the notation there, we have $C = 0$.

Proof. Suppose $C \neq 0$ and r is the number of irreducible components of C. Since $\mathcal{D}$ has no fixed component and $A - B$ is big we know that $(A - B) \cdot D > 0$, so that

$$A \cdot D \geq B \cdot D + 1 = D \cdot C + D^2 + 1 \tag{10}$$

or equivalently

$$D \cdot C \leq A \cdot D - D^2 - 1. \tag{11}$$

Moreover, since $A + B$ is ample we have $r \leq (A+B) \cdot C = A \cdot C + D \cdot C + C^2$ and thus

$$A \cdot C + D \cdot C = (A + B) \cdot C - C^2 \geq r - C^2. \tag{12}$$

1st Case: $C^2 \leq 0$. Then (12) together with (10) gives

$$A \cdot B = A \cdot C + A \cdot D \geq A \cdot C + D \cdot C + D^2 + 1 \geq r + (-C^2) + D^2 + 1 \geq 2, \tag{13}$$

or the slightly stronger inequality

$$A \cdot B \geq (A + B) \cdot C + (-C^2) + D^2 + 1. \tag{14}$$

2nd Case: $C^2 > 0$. Then necessarily $B^2 > 0$ and by Lemma 5.2 we have $A \cdot B = \text{length}(Z_p') = 4$ and

$$2 \geq B^2 = D^2 + 2 \cdot C \cdot D + C^2 \geq 1. \tag{15}$$

Since all the summands involved in the right hand side of (13) and all summands in (15) are non-negative, and since by Lemma 5.3 the case $D^2 = 1$ cannot occur when $\text{length}(Z_p') = 4$, and since by Lemma 5.2 $B^2 > 0$ is impossible when $\text{length}(Z_p') = 3$, we are left considering the cases shown in Figure 3, where for the additional information (the last four columns) we take Proposition 5.1, Lemma 5.2 and Lemma 5.3 into account.

Let us first and for a while consider the situation $\text{length}(Z_p') = 4$ and $D^2 = 0$, so that by Lemma 5.3 $D = kE$ for some irreducible curve E with $k \geq 2$ and $E^2 = 0$. Applying Lemma 5.4 to E we see that $(A + B) \cdot E \geq 3$, and thus

$$6 \leq 3k \leq (A + B) \cdot D = A \cdot D + C \cdot D. \tag{16}$$

	length(Z_p')	D^2	C^2	$C{\cdot}D$	r	$A{\cdot}B$	$A{\cdot}D$	$A{\cdot}C$	D
1)	4	0	-2		1	4	4	0	$kE, k \geq 2$
2)	4	0	-1		2	4	4	0	$kE, k \geq 2$
3)	4	0	0		3	4	4	0	$kE, k \geq 2$
4)	4	0	-1		1	3, 4			$kE, k \geq 2$
5)	4	2	0		1	4	4	0	
6)	4	0	0		2	3, 4			$kE, k \geq 2$
7)	4	0	0		1	2, 3, 4			$kE, k \geq 2$
8)	3	0	-1		1	3	3	0	
9)	3	0	0		2	3	3	0	
10)	3	0	0		1	2, 3			
11)	4	0	1	0		4	4	0	$kE, k \geq 2$
12)	4	0	2	0		4	4	0	$kE, k \geq 2$

FIGURE 3. The cases to be considered.

If in addition $A{\cdot}D \leq 4$, then (11) leads to

$$6 \leq 3k \leq A{\cdot}D + C{\cdot}D \leq 4 + C{\cdot}D \leq 7, \tag{17}$$

which is only possible for $k = 2$, $C{\cdot}E = 1$ and

$$C{\cdot}D = k \cdot C{\cdot}E = 2. \tag{18}$$

In Cases 1, 2 and 3 we have $A{\cdot}D = 4$, and we can apply (18), which by (12) then gives the contradiction

$$2 = A{\cdot}C + C{\cdot}D \geq r - C^2 = 3.$$

If, still under the assumption length(Z_p') = 4 and $D^2 = 0$, we moreover assume $2 \geq C^2 \geq 0$ then by Lemma 5.2

$$2 \geq B^2 = 2 \cdot C{\cdot}D + C^2 \geq 2 \cdot C{\cdot}D \geq 0,$$

and thus $C{\cdot}D \leq 1$ and $C{\cdot}D + C^2 \leq 2$, which due to (16) implies $A{\cdot}D \geq 5$. But then by Proposition 5.1 we have $A{\cdot}B \leq 3$ and hence $A{\cdot}C = A{\cdot}B - A{\cdot}D \leq -2$, which leads to the contradiction

$$(A + B){\cdot}C = A{\cdot}C + D{\cdot}C + C^2 \leq 0, \tag{19}$$

since $A + B$ is ample. This rules out the Cases 6, 7, 11 and 12.

In Case 4 Lemma 5.4 applied to C shows

$$2 \leq (A + B){\cdot}C = A{\cdot}C + D{\cdot}C + C^2. \tag{20}$$

Lemma 5.2 implies

$$2 \geq B^2 = 2 \cdot C{\cdot}D + C^2 = 2k \cdot C{\cdot}E - 1 \geq 4 \cdot C{\cdot}E - 1 \geq -1,$$

which is only possible for $C{\cdot}E = C{\cdot}D = 0$. But then (20) implies $A{\cdot}C \geq 3$, and since A is big and E is irreducible with non-negative self intersection we get the contradiction

$$2 \leq k \cdot A{\cdot}E \leq A{\cdot}D = A{\cdot}B - A{\cdot}C \leq 1.$$

This finishes the cases where $\mathrm{length}(Z'_p) = 4$ and $D^2 = 0$.

In Cases 5 and 10 we apply Lemma 5.4 to the irreducible curve C with $C^2 = 0$ and find

$$(A + B){\cdot}C \geq 2.$$

In Case 5 Equation (14) then gives the contradiction

$$4 = A{\cdot}B \geq 2 - C^2 + D^2 + 1 = 5.$$

In Case 10 we get

$$3 \geq A{\cdot}B \geq (A + B).C - C^2 + D^2 + 1 = (A + B).C + 1,$$

which shows that

$$2 = (A + B).C = A.C + D.C + C^2 \tag{21}$$

and that $A.B = 3 = \mathrm{length}(Z'_p)$. Then by Proposition 5.1 we get $A.C = 0$ and by (21)

$$D.C = 2 - A.C - C^2 = 2,$$

which due to Lemma 5.2 leads to the contradiction

$$2 \geq B^2 = D^2 + 2 \cdot D.C + C^2 = 4.$$

In very much the same way we get in Case 8 by Lemma 5.4

$$(A + B){\cdot}C \geq 2$$

and the contradiction

$$3 = A{\cdot}B \geq 2 - C^2 + D^2 + 1 = 4.$$

It remains to consider Case 9. Here we deduce from (14) that

$$2 \geq (A + B){\cdot}C \geq r = 2,$$

and hence

$$2 = (A + B){\cdot}C = A{\cdot}C + D{\cdot}C + C^2 = D{\cdot}C.$$

But then Lemma 5.2 leads to the final contradiction

$$2 \geq B^2 = D^2 + 2 \cdot D{\cdot}C + C^2 = 4.$$

$\square$

It follows that $B_p = D_p$, $\mathcal{B} = \mathcal{D}$, and that B_p is nef.

6. The General Case

Let us review the situation and recall some notation. We are considering a divisor L such that L is very ample and $L - K$ is ample and base-point-free with $(L - K)^2 > 16$, and such that for a general point $p \in S$ the general element $L_p \in |L - 3p|$ has no triple component through p and that the equimultiplicity ideal of L_p in p in suitable local coordinates is one of the ideals in Table (3) – and for all p the ideals have the same length. Moreover, we know that there is an algebraic family $\mathcal{B} = \overline{\{B_p \mid p \in S\}} \subset |B|_a$ without fixed component such that for a general point $p \in S$

$$B_p \in |\mathcal{J}_{Z'_p/S}(L - K - A_p)|,$$

where Z'_p is contained in the equimultiplicity scheme Z_p of L_p and A_p is the unique divisor linearly equivalent to $L - K - B_p$ such that B_p and A_p destabilise the vector bundle $\mathcal{E}_p$ in (2). Keeping these notations in mind we can now consider the two cases that either $\text{length}(Z'_p) = 4$ or $\text{length}(Z'_p) = 3$.

Proposition 16

With the above notation and assumptions, it is impossible that for a general point $p \in S$ the length of Z_p' is length$(Z_p') = 4$.

Proof. In Section 5 we have shown that $B = D$ is nef, and thus Lemma 5.2 shows

$$0 \le B^2 \le 2. \tag{22}$$

Then, however, Lemma 5.3 implies that B_p must be reducible.

Let us first consider the case that the part of B_p through p is reduced. Then by Lemma 5.3 and Equation (22) we know that $B_p = E_p + F_p + R$, where E_p and F_p are irreducible and smooth in p. In particular, $E_p \cdot F_p \ge 1$, and thus

$$2 = B^2 = E_p^2 + 2 \cdot E_p \cdot F_p + F_p^2 + 2 \cdot (E_p + F_p) \cdot R + R^2$$
$$\ge 2 + 2 \cdot (E_p + F_p) \cdot R.$$

Since $E_p \cdot F_p \ge 1$ and since the components E_p and F_p vary in at least one-dimensional families and R has no fixed component, $(E_p + F_p) \cdot R \ge 1$, unless $R = 0$. This would however give a contradiction, so $R = 0$. Therefore necessarily, $B_p = E_p + F_p$, $E_p \cdot F_p = 1$, and $E_p^2 = F_p^2 = 0$. Then by Lemma 5.4 $(A + B) \cdot E_p \ge 3$ and $(A + B) \cdot F_p \ge 3$, so that

$$4 \ge A \cdot B \ge (A + B) \cdot E_p + (A + B) \cdot F_p - B^2 \ge 4$$

implies $E_p \cdot A_p = 2 = F_p \cdot A_p$ and $(A + B) \cdot E_p = 3 = (A + B) \cdot F_p$.

Since $E_p^2 = 0$ the family $|E|_a$ is a pencil and induces a fibration on S (see [14] App. B.1). In particular, the generic element E_p in $|E|_a$ must be smooth (see e.g. [1] p. 110).

We claim that in p the curve L_p can share at most with one of E_p or F_p a common tangent, and it can do so at most with multiplicity one. For this consider local coordinates (x_p, y_p) as in the Table (3). Since length$(Z_p') = 4$ we know that $\mathcal{J}_{Z_p', p} = \langle x_p^2, y_p^2 \rangle$ does not contain $x_p y_p$, and since $B_p = E_p + F_p \in |\mathcal{J}_{Z_p'}(L - K - A)|$, where E_p and F_p are smooth in p, we deduce that in local coordinates their equations are

$$x_p + a \cdot y_p + h.o.t. \quad \text{respectively} \quad x_p - a \cdot y_p + h.o.t.,$$

where $a \ne 0$. By Table (3) the local equation f_p of L_p has either $\text{jet}_3(f_p) = x_p^3$ and has thus no common tangent with either E_p or F_p,

or $\mathrm{jet}_3(f_p) = x_p^3 - y_p^3$ and it is divisible at most once by one of $x_p - ay_p$ or $x_p + ay_p$.

In particular, E_p can at most once be a component of L_p, and we deduce

$$E_p \cdot K_S = E_p \cdot L_p - E_p \cdot A_p - E_p \cdot B_p = E_p \cdot L_p - 3 \geq \begin{cases} 0, & \text{if } E_p \not\subset L_p, \\ -1, & \text{if } E_p \subset L_p. \end{cases}$$

But then, since the genus is an integer,

$$p_a(E_p) = \frac{E_p^2 + E_p \cdot K_S}{2} + 1 = \frac{E_p \cdot K_S}{2} + 1 \geq 1. \tag{23}$$

Fix a general point p in S and a general point q on E_p, then $E_p = E_q$ since $|E|_a$ is a pencil. Hence,

$$A_p + F_p \sim_l L - K - E_p = L - K - E_q \sim_l A_q + F_q.$$

Since $A_q.B_q = 4 = \mathrm{length}(Z_q')$ by Proposition 5.1 there is a global section $s_q \in H^0\big(B_q, \mathcal{J}_{Z_q'/B_q}(A_q)\big)$ whose zero locus is Z_q'. Restricting s_q to E_q we get a global section of $\mathcal{O}_{E_p}(A_q) = \mathcal{O}_{E_q}(A_q)$ which cuts out $2q$ on $E_p = E_q$. Moreover, $\mathcal{O}_{E_p}(F_q) = \mathcal{O}_{E_q}(F_q)$ has a global section which cuts out q. Thus $\mathcal{O}_{E_p}(A_p + F_p) = \mathcal{O}_{E_p}(A_q + F_q)$ has for infinitely many points q on E_p a global section which cuts out $3q$. The linear system $|\mathcal{O}_{E_p}(A_p + F_p)|$ thus has degree three and contains the divisor $3q$ for infinitely many points q, and it hence has no base point. So it defines a morphism to $\mathbb{P}^k$, where k is the dimension of the linear system. k cannot be one, since otherwise the morphism would have infinitely many ramification points. If the dimension k is two, the morphism maps the curve E_p to the plane. Then either the morphism has degree three and the image is a line, which leads to the same contradiction, or the morphism is an isomorphism and the image is a cubic which has infinitely many reflection points, which is also impossible. It remains the case that the dimension k is three, but then E_p has a $\mathbf{g}_3^3$ and is rational, in contradiction to (23).

This finishes the case that the part of B_p through p is reduced.

It remains to consider the case that B_p is not reduced in p. Using the notation of the proof of Lemma 5.3 we write $B_p \equiv k \cdot E_p + E'$ with $k \geq 2$, E_p irreducible passing through p and E' not containing any component algebraically equivalent to E_p. We have seen there (see p.

16) that $E' \neq 0$ implies $B_p^2 \geq 4$ in contradiction to (22). We may therefore assume $B_p = k \cdot E_p$ with $E_p^2 \geq 0$. If $E_p^2 \geq 1$, then again $B_p^2 \geq 4$. Thus $E_p^2 = 0$. Applying Lemma 5.4 to E_p we get

$$3 \leq (A + B) \cdot E_p = A \cdot E_p,$$

and hence the contradiction

$$4 \geq A \cdot B = k \cdot A \cdot E_p \geq 6.$$

This finishes the case that B_p is not reduced in p, and shows thus that the case $\mathrm{length}(Z_p') = 4$ cannot occur. $\qquad\square$

Proposition 17

Let $p \in S$ be general and suppose that $\mathrm{length}(Z_p') = 3$. Then B_p is an irreducible, smooth, rational curve in the pencil $|B|_a$ with $B^2 = 0$, $A \cdot B = 3$ and $\exists \, s \in H^0\big(B_p, \mathcal{O}_{B_p}(A_p)\big)$ such that Z_p' is the zero-locus of s.

In particular, $S \to |B|_a$ is a ruled surface and $2B_p$ is a fixed component of $|L - 3p|$.

Proof. Since in Section 5 we have shown that B is nef, Lemma 5.2 implies

$$B^2 = 0. \tag{24}$$

Once we have shown that B_p is irreducible and reduced, we then know that $|B|_a$ is a pencil and induces a fibration on S whose fibres are the elements of $|B|_a$ (see [14] App. B.1). In particular, the general element of $|B|_a$, which is B_p, is smooth (see [1] p. 110).

Let us therefore first show that B_p is irreducible and reduced. Since $\mathcal{B}$ has no fixed component we know for each irreducible component B_i of $B_p = \sum_{i=1}^r B_i$ that $B_i^2 \geq 0$, and hence by Lemma 5.4 that $(A + B) \cdot B_i \geq 2$. Thus by (7) and (24)

$$2 \cdot r \leq (A + B) \cdot B = A \cdot B + B^2 = A \cdot B \leq 3,$$

which shows that B_p is irreducible and reduced and that $A \cdot B = 3$.

Since $A \cdot B = 3 = \mathrm{length}(Z_p')$ Proposition 5.1 implies that there is a section $s_p \in H^0\big(B_p, \mathcal{O}_{B_p}(A_p)\big)$ such that Z_p' is the zero-locus of s_p, which is just $3p$. Note that for $p \in S$ general and $q \in B_p$ general we

have $B_p = B_q$ since $|B|_a$ is a pencil, and thus by the construction of B_p and B_q we also have

$$A_p \sim_l L - K - B_p = L - K - B_q \sim_l A_q.$$

But if A_p and A_q are linearly equivalent, then so are the divisors s_p and s_q induced on the curve $B_p = B_q$. The curve B_p therefore contains a linear series $|\mathcal{O}_{B_p}(A_p)|$ of degree three which contains $3q$ for a general point $q \in B_p$. In particular, the linear series has no base point and induces a morphism $\varphi : B_p \longrightarrow \mathbb{P}^k$ where k is the dimension of the linear series.

Suppose that k was one, then φ would be a morphism of degree three from the curve B_p to a line and it would have infinitely many ramification points q, which is clearly not possible. If k is two, then either φ has degree three and its image is a line, which leads to the same contradiction, or φ has degree one and the image is a plane cubic. In that case φ is a birational morphism and either B_p is rational (if $\mathrm{Im}(\varphi)$ is singular) or B_p is elliptic (if $\mathrm{Im}(\varphi)$ is smooth). If B_p was an elliptic curve, then the general point q of the cubic $\mathrm{Im}(\varphi)$ embedded via the $\mathfrak{g}_3^2 = |\mathcal{O}_{B_p}(A_p)|$ would be an inflexion point. But that is clearly not possible. Finally, if k is three, then B_p has a $\mathfrak{g}_3^3$ and is thus rational. Alltogether we have shown that

$$p_a(B_p) = 0,$$

and by the adjunction formula we get

$$K{\cdot}B = 2 \cdot p_a(B) - 2 - B^2 = -2. \tag{25}$$

Note also, that $Z'_p \subset B_p$ in view of Table (3) implies that B_p and L_p have a common tangent in p. Suppose that B_p and L_p have no common component, i. e. $B_p \not\subset L_p$, then

$$3 \leq \mathrm{mult}_p(B_p) \cdot \mathrm{mult}_p(L_p) < L{\cdot}B = A{\cdot}B + B^2 + K{\cdot}B = 3 + K{\cdot}B = 1,$$

which contradicts (25). Thus, B_p is at least once contained in L_p. Moreover, if $2B_p \not\subset L_p$ then by Table (3) $L'_p := L_p - B_p$ has multiplicity two in p, and it still has a common tangent with B_p in p, so that

$$3 \leq L'_p{\cdot}B_p = L{\cdot}B - B^2 = A{\cdot}B + K{\cdot}B = 3 + K{\cdot}B = 1 \tag{26}$$

again is impossible. We conclude finally, that B_p is at least twice contained in L_p

Note finally, since $\dim |B|_a = 1$ there is a unique curve B_p in $|B|_a$ which passes through p, i. e. it does not depend on the choice of L_p, so that in these cases B_p respectively $2B_p$ is actually a fixed component of $|L - 3p|$. $\square$

7. Triple-Point Defective Surfaces are Ruled

The considerations of the previous sections prove the following theorem.

Theorem 18 ("S is a ruled surface.")
More precisely, let L be a line bundle on S such that L is very ample and $L - K$ is ample and base-point-free. Suppose that $(L - K)^2 > 16$ and that for a general $p \in S$ the linear system $|L - 3p|$ contains a curve L_p which has no triple component through p, but such that $h^1\big(S, \mathcal{J}_{Z_p}(L)\big) \neq 0$ where Z_p is the equimultiplicity scheme of L_p at p.
Then there is a ruling $\pi : S \to C$ of S such that L_p contains the fibre over $\pi(p)$ with multiplicity two.

In view of Propositon 3.1 this proves Theorem 1.4.

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