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THREE-DIMENSIONAL BONNESEN TYPE INEQUALITIES

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A well-known result in convex geometry proved by Favard states that among all convex plane sets of given perimeter and area, the symmetric lens is the unique element of maximum circumradius. In this note a new proof of Favard's theorem is exhibited and possible extensions in higher dimensions are discussed.

1. Bonnesen type inequalities.

Let K denote a convex body in $\mathbb{R}^2$, i.e. a compact convex subset of the plane with non-empty interior. A Bonnesen type inequality is a geometric inequality that involves the perimeter L, the area A, the inradius r and/or the circumradius R of the body K. We recall that the inradius and the circumradius of K are the radius of the largest disc contained in K and the radius of the smallest disc containing K, respectively. The original Bonnesen inequalities, contained in [2], are the following:

$$\frac{L^2}{4\pi} - A \geq \pi \left(\frac{L}{2\pi} - r \right)^2 \tag{1}$$

$$\frac{L^2}{4\pi} - A \geq \pi \left(R - \frac{L}{2\pi} \right)^2 . \tag{2}$$

In (1) equality holds if and only if K is a "stadium" (or a baby biscuit, if one prefers), namely a set obtained from a rectangle by gluing two semidiscs of

radius t to opposite sides of length $2t$. In (2) equality holds if and only if K is a disc. The quantity on the left-hand side of both inequalities is the isoperimetric deficit of K. In fact, a common feature of Bonnesen style inequalities is estimating from below the isoperimetric deficit of K in terms of quantities involving the inradius and/or the circumradius. Thus all these inequalities are sharper versions of the classical isoperimetric inequality. Inequalities (1) and (2) imply

$$\frac{L^2}{4\pi} - A \geq \frac{\pi}{4}(R-r)^2 \,, \tag{3}$$

the well-known estimate of the isoperimetric deficit in terms of R and r. In the literature many variants of the original Bonnesen inequalities are known. The main sources are the books by Bonnesen [2], Bonnesen and Fenchel [3], Schneider [9] and the survey by Osserman [6], which is an excellent guide in the world of these inequalities. However, in this note, we shall focus our attention on the original Bonnesen inequalities only. Inequality (1) is sharp, since for every value of the isoperimetric deficit there exists a set for which equality holds. This fact can be rephrased as follows: Among all convex bodies of given L and A, the stadium is the one with minimum inradius. On the other hand, inequality (2) is not sharp in the following sense. If the isoperimetric deficit is strictly positive, then (2) is strict also and it does not provide the maximum possible circumradius, for fixed L and A. Favard showed in 1929 that under these constraints the symmetric lens is the only maximizing set. We recall that a symmetric lens is the intersection of two discs with the same radius. It is natural to ask which are in higher dimensions the convex sets corresponding to lenses in the plane case, that is the sets with maximum circumradius under suitable restrictions on the volume, the surface area and so on. Surprisingly, the problems which naturally correspond in higher dimensions to the one solved by Favard in the plane are unsolved. In the first part of this note (Sections 2-4) we deal with Favard's result and we give a new proof of it. In the second (Section 5) we discuss 3-dimensional extensions of Favard's theorem and we present a couple of results, by Zalgaller and by Campi and Gronchi respectively, concerning convex sets of maximum diameter under suitable restrictions.

2. Favard's problem.

Let L_0 and A_0 be two positive numbers such that

$$L_0^2 - 4\pi A_0 \geq 0$$

and denote by $\Lambda(L_0, A_0)$ the class of all plane convex bodies of perimeter L and A such that $L \leq L_0$ and $A \geq A_0$.

Problem 2.1. Which is the element from $\Lambda(L_0, A_0)$ with maximum circumradius?

In the literature this problem is called Favard's problem. Here is the solution.

Theorem 2.2. (Favard [5], 1929) *The symmetric lens of perimeter L_0 and area A_0 is the unique solution of Problem 2.1.*

Notice that, if D denotes the diameter of the convex body K, then

$$\frac{1}{2}D \leq R\,.$$

Since for the symmetric lens $D/2 = R$, Favard's theorem 2.2 implies also the following result:

The symmetric lens of perimeter L_0 and area A_0 is the unique element from $\Lambda(L_0, A_0)$ of maximum diameter.

Such a result can be also obtained directly. Indeed, it is easy to check that the area between a chord and an arc of given length is maximal for the arc of circle.

Favard's original proof consists in finding an upper bound for the circumradius and in showing that the circumradius of the lens attains just that value. Besicovitch [1] and, more recently, Zalgaller [10] provided two new and independent proofs of Theorem 2.2. Besicovitch's proof makes use of local variations of the set assumed to be the maximizer. In such a way it turns out that the candidates are reduced to the lens and the Reuleaux triangle. A direct computation leads to the conclusion. Zalgaller's proof is based on Pólya symmetrization, a circular version of Steiner symmetrization. A similar symmetrization to Pólya's was introduced earlier by Bonnesen. We shall describe both symmetrizations in the next section.

3. Pólya symmetrization and Bonnesen symmetrization.

Let K be a planar convex body and $\Gamma(\rho, \tau)$ a circular annulus of radii $\rho < \tau$ containing the boundary of K. We can assume that the annulus is centered at the origin o. Denote by (r, θ) the polar coordinates in the plane and let s be the half-line from the origin corresponding to $\theta = 0$. For $\rho < r < \tau$, let $4\theta^*$ be the linear measure of $K \cap \partial C_r$, where C_r is the disk with center at o and radius r. Moreover, let $n(r)$ be the number of components of $K \cap \partial C_r$ with positive linear measure.

3.1. Pólya symmetrization. (see [2], p.194; see also [8] and [3], p.77)

The Pólya symmetral K_P of K is defined as the set such that the points $(r, 2\theta^*)$, $(r, 2\pi - 2\theta^*)$ belong to its boundary, for every r.

3.1. Bonnesen symmetrization. (see [2], p.67)

The Bonnesen symmetral K_B of K is defined as the set such that the points (r, θ^*), $(r, 2\pi - \theta^*)$, $(r, \pi - \theta^*)$, $(r, \pi + \theta^*)$ belong to its boundary, for every r.

For Pólya and Bonnesen symmetrals the following properties hold:

(i) $A(K) = A(K_P) = A(K_B)$;
(ii) $L(K) \geq L(K_P)$;
(iii) if $n \geq 2$, when $\theta^*(r) < \pi/2$, then $L(K) \geq L(K_B)$.

The functions $A(K \cap C_r)$, $A(K_P \cap C_r)$, $A(K_B \cap C_r)$ have the same derivative with respect to r; hence property (i) follows. Properties (ii) and (iii) can be deduced from Jensen's inequality, applied to the function $\sqrt{1 + x^2}$.

In (i) equality holds if and only if $n(r) \equiv 1$ and K has an axis of symmetry (through o).

In (ii) equality holds if and only if $n(r) \equiv 2$ and each of the two components of $K \setminus C_\rho$ has an axis of symmetry (through o).

It is worth noticing that K_P and K_B need not be convex.

4. A new proof of Favard's theorem.

This proof consists of showing that a solution of Problem 2.1 has two antipodal points on the smallest circle containing it. Thus such a solution has the largest possible diameter; therefore it has to be the lens.

Assume that K is a solution of Favard's problem and let $\Gamma(\rho, \tau)$ be the minimal annulus of K, i.e. the unique annulus containing ∂K such that $\tau - \rho$ is minimum (see [2], p.45 and p.67). Such an annulus has the property that every circle with center at o and radius between ρ and τ intersects K at least in two arcs.

If K has two antipodal points on the largest circle of $\Gamma(\rho, \tau)$, then τ is just the circumradius of K. Assume that it is not so. Let K_B be the Bonnesen symmetral of K with respect to $\Gamma(\rho, \tau)$ and K^* the convex hull of K_B. Since K is assumed to be a solution, we have to exclude that $L(K^*) < L(K)$. The equality $L(K^*) = L(K)$ implies that in the process of symmetrization every circle with radius between ρ and τ has just two arcs in common with K. Therefore, on the circle of radius τ the set K has only two components, not containing two antipodal points and so contained in an arc strictly smaller than

half circle. Thus the circumradius of K would be less than τ, while the one of K^* is just τ. The conclusion is that a solution of Favard's problem must have two antipodal points on its circumcircle.

5. Favard type problems in three dimensions.

It is natural to ask whether in higher dimensions it is possible to find estimates of the circumradius of a convex set in terms of quantities like volume, surface area and so on. Let us focus our attention on possible extensions of Favard's theorem in three dimensions. Let K be a convex body of $\mathbb{R}^3$. How to state a three-dimensional version of Problem 2.1? How to replace the constraints on perimeter and area?

While the volume V is a natural substitute of the area, there are at least two possibilities for the other quantity. According to Kubota's integral formula (see [9], p. 295), the perimeter of a planar convex body is an average of the lengths of its orthogonal projections. Thus the same formula suggests that the role played by the perimeter L in the plane can be interpreted in the 3-space by the *surface area* S, that is the average of the areas of its two-dimensional projections, or by the *mean width* B, the average of the lengths of its one-dimensional projections. For a smooth K, the *total mean curvature* is defined by

$$M = \frac{1}{2}\int_{\partial K}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)d\sigma\,,$$

where R_1, R_2 are the principal radii of curvature of ∂K and σ is the $(n-1)$-dimensional Hausdorff measure. We have that

$$M = 2\pi B\,.$$

On the other hand, the surface area S of K is given by

$$S = \int_{\partial K}\left(\frac{1}{R_1}\frac{1}{R_2}\right)d\sigma\,.$$

The quantities V, S, B satisfy the following inequalities of isoperimetric type (see [4], p. 145):

$$S^3 \geq 36\pi V^2\,,$$
$$\pi B^3 \geq 6V\,.$$

Let $\Omega(S_0, V_0)$ be the class of all convex bodies in $\mathbb{R}^3$ such that $S \leq S_0$, $V \geq V_0$. Unfortunately a three-dimensional result analogous to Favard's theorem is not available. The following weaker result concerning the maximum of the diameter holds.

Theorem 5.1. (Zalgaller [10], 1994) *The unique body in* $\Omega(S_0, V_0)$ *having maximum diameter is a mean curvature spindle-shaped body of surface area* S_0 *and volume* V_0.

According to Definition 16 in [10], a mean curvature spindle-shaped body is a centrally symmetric convex body of revolution whose surface has constant mean curvature in the central part and consists of two cones in the parts adjacent to the axis of revolution.

Let $\Psi(B_0, V_0)$ be the class of all convex bodies in $\mathbb{R}^3$ such that $B \leq B_0$, $V \geq V_0$. Recently Campi and Gronchi obtained a result analogous to Zalgaller's theorem. Precisely they showed that *in* $\Psi(B_0, V_0)$ *the unique body of maximum diameter is a Gaussian curvature spindle-shaped body of mean width* B_0 *and volume* V_0.

The above result is the object of a forthcoming paper. The strategy of the proof is analogous to that used by Zalgaller for Theorem 5.1 and it is based mainly on local variations of the maximizer.

In conclusion, the problems of finding in $\Omega(S_0, V_0)$ or in $\Psi(B_0, V_0)$ the element of maximum circumradius remain open. It is reasonable to conjecture that the solutions are the same of the corresponding problems for the diameter. For solving such a conjecture it would be sufficient to show that the bodies with largest circumradius have two antipodal points on the circumscribed sphere.

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