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A FAVARD TYPE PROBLEM FOR 3-D CONVEX BODIES

STEFANO CAMPI AND PAOLO GRONCHI

ABSTRACT

A theorem due to Favard states that among all plane sets of given area and perimeter, the symmetric lens has maximum circumradius. This paper deals with the higher dimensional problem of finding the convex body in $\mathbb{R}^3$ of given volume and mean width with the largest possible diameter. It is shown that the solution is the convex hull of a surface of revolution with constant Gauss curvature and a segment lying on the axis of revolution. Such a body is conjectured to maximize also the circumradius in the same class.

1. *Introduction*

A well-known result in convex geometry states that the *symmetric lens* is the unique solution of the following problem:

Find the compact convex set in $\mathbb{R}^2$ *of given perimeter and area with the largest possible circumradius.*

We recall that a symmetric lens is the intersection of two discs with the same radius and that the circumradius of a set is the radius of the smallest disc containing it.

This extremal property of the symmetric lens was proved by Favard [**6**] and occupies a significant place in the framework of geometric inequalities of isoperimetric type. Among these inequalities, the Bonnesen inequalities

$$\frac{L^2}{4\pi} - A \geq \pi \left(\frac{L}{2\pi} - r \right)^2 , \tag{1.1}$$

$$\frac{L^2}{4\pi} - A \geq \pi \left(R - \frac{L}{2\pi} \right)^2 , \tag{1.2}$$

proved in [**2**], give lower bounds of the *isoperimetric deficit* of a set, the left-hand side, in terms of the inradius and circumradius, respectively. Here, L and A denote the perimeter and the area of the set and R and r its circumradius and inradius, the latter being the radius of the largest disc contained in the set.

In spite of the similarity, (1.1) and (1.2) differ in one important respect. Inequality (1.1) is sharp, since for every value of the isoperimetric deficit the convex hull of two suitable discs with the same radius gives equality. On the other hand, (1.2) is not sharp: If the isoperimetric deficit is strictly positive, then (1.2) is strict also. For more details on Bonnesen's inequalities and their variants, we refer to Bonnesen and Fenchel [**3**], Osserman [**7**] and Schneider [**8**].

A natural question that we shall deal with in the present paper is whether it is possible to find results in higher dimensions analogous to Favard's theorem. It is worth saying at once that the literature contains only a partial result of this

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type, due to Zalgaller [**9**]. To describe it, let K be a convex body in $\mathbb{R}^3$, i.e., a three-dimensional compact convex set, and let V and S denote its volume and surface area, respectively. The classical isoperimetric inequality (see, for instance, [**4**], p.145) states that

$$S^3 \geq 36\pi V^2 \,.$$

THEOREM 1.1 Zalgaller [**9**]. *Let S_0 and V_0 be two numbers such that $S_0^3 \geq 36\pi V_0^2$. Among all three-dimensional convex bodies such $S \leq S_0$ and $V \geq V_0$, the unique body having maximum diameter is a mean curvature spindle-shaped body of surface area S_0 and volume V_0.*

A mean curvature spindle-shaped body is a centrally symmetric convex body of revolution which can be seen as the convex hull of a surface of revolution with constant mean curvature and a segment lying on the axis of revolution.

Note that Zalgaller's result refers to the diameter of the set instead of the circumradius. Therefore, it answers a three-dimensional Favard-type question in the class of bodies of revolution. The same paper [**9**] contains the conjecture that the solution does not change in the full class of convex bodies.

While it is natural to replace the area of a plane set with the volume of a three-dimensional one, there are two possible substitutes for the perimeter, namely, the surface area and the mean width. Indeed, the perimeter of a planar convex body is an average of all its widths. The width along a direction v is the distance between the support lines (hyperplanes, in $\mathbb{R}^d$) orthogonal to v. These two possible choices are also suggested by the Steiner formula for parallel sets of a given convex body in $\mathbb{R}^d$ (see, for instance, [**8**], p.197). The d-dimensional measure of $K + tB$, where B is the unit d-ball and $+$ stands for the Minkowski sum (vector addition), is a polynomial of degree d in t. For $d = 2$, the perimeter is the coefficient of the linear term. For $d = 3$, a multiple of the mean width and the surface area are the coefficients of t and t^2, respectively.

An analytical expression for the mean width can be given in terms of the support function h_K of a convex body K, whose definition is:

$$h_K(z) = \max_{x \in K} \langle z, x \rangle \;, \quad \text{for every } z \in \mathbb{S}^2 \,,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product and $\mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$.

Thus, the mean width W of K can be expressed as

$$W = \frac{1}{2\pi} \int_{\mathbb{S}^2} h_K(z)\, dz \,. \tag{1.3}$$

The mean width W and the volume V of a three-dimensional convex body satisfy the isoperimetric inequality

$$\pi W^3 \geq 6V \,,$$

known as Urysohn inequality (see, for instance, [**4**], p.145).

This paper is devoted to the following problem.

PROBLEM 1. Let W_0 and V_0 be two numbers such that $\pi W_0^3 \geq 6V_0$. Among all three-dimensional convex bodies such $W \leq W_0$ and $V \geq V_0$, find the bodies of maximum diameter.

In Section 2 we shall prove that the solution exists and is unique, of course up to isometries. The description of the solution is given in Sections 3, 4 and 5 and is summarized in the following theorem.

THEOREM 1.2. *The unique solution of Problem 1 is a Gauss curvature spindle-shaped body of mean width W_0 and volume V_0.*

A Gauss curvature spindle-shaped body is a body analogous to the one introduced by Zalgaller, where mean curvature is replaced by Gauss curvature.

Actually, Theorem 1.2 is inspired by Theorem 1.1. Nevertheless, our proof makes use of different tools, such as the Brunn-Minkowski inequality, the Gauss map and the area measure of a convex body, all coming from the Brunn-Minkowski theory.

The constraints on surface area and mean width give rise to conditions on the mean curvature and the Gauss curvature of the extremal bodies, respectively. This fact can be explained by looking at the following expressions for S and W, when K is smooth:

$$S = \int_{\partial K} \left(\frac{1}{R_1} \frac{1}{R_2} \right) \, dp \,, \tag{1.4}$$

$$W = \frac{1}{4\pi} \int_{\partial K} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \, dp \,, \tag{1.5}$$

where R_1 and R_2 are the principal radii of curvature of ∂K at p and the integration is performed with respect to the $(n-1)$-dimensional Hausdorff measure (see [**3**], p. 72).

Obviously, the solution of Problem 1 also maximizes the circumradius in the class of bodies of revolution. As in the case of the problem solved by Zalgaller through Theorem 1.1, we may conjecture that the solution of Problem 1 has the largest possible circumradius in the whole class of three-dimensional convex bodies.

Authors other than Favard himself provided different proofs of the maximality of the symmetric lens, for example Besicovitch [**1**], Zalgaller [**9**] and Campi [**5**]. Unfortunately, none of the available proofs in the plane case seems to generalize to higher dimensions.

Finally, we recall that analogous problems concerning the minimum and the maximum of the *inradius* of three-dimensional convex body have been considered and solved by Zalgaller [**9**].

2. *Existence and uniqueness of the solution*

For convenience, let us denote by $\mathcal{K}(V_0, W_0)$ the class of all three-dimensional convex bodies K such that $V(K) \geq V_0$ and $W(K) \leq W_0$, and by $d(K)$ the diameter of K. The class $\mathcal{K}(V_0, W_0)$ is endowed with the Hausdorff metric and the functionals $V(K)$, $W(K)$, $d(K)$ are continuous this metric.

Let $K \in \mathcal{K}(V_0, W_0)$. Since K contains a segment whose length is $d(K)$, the monotonicity of the mean width implies that

$$W(K) \geq \frac{1}{2} \, d(K) \,.$$

Hence, if we identify translates of bodies, then $\mathcal{K}(V_0, W_0)$ is compact. Therefore, the maximum of $d(K)$ over $K \in \mathcal{K}(V_0, W_0)$ exists.

Let us show that *if K is a solution of Problem 1, then*

$$V(K) = V_0 \quad \textit{and} \quad W(K) = W_0\,. \tag{2.1}$$

Assume that $V(K) > V_0$ and let $\tilde K \in \mathcal{K}(V_0, W_0)$ be a proper subset of K such that $d(\tilde K) = d(K)$ and $W(\tilde K) < W(K)$. For suitable $\lambda > 1$, we have $\lambda\tilde K \in \mathcal{K}(V_0, W_0)$ and $d(\lambda\tilde K) = \lambda d(\tilde K) = \lambda d(K)$, which contradicts the assumption on K. Similarly, if $W(K) < W_0$, then, for suitable $\lambda > 1$, λK gives a contradiction.

We prove now that *the solution of Problem 1 is unique in $\mathcal{K}(V_0, W_0)$, up to isometries.*

Suppose, on the contrary, that K_1 and K_2 are two distinct solutions. Via a possible rigid motion, $K_1 \cap K_2$ still has maximum diameter. Consider the Minkowski sum $\tilde K = \frac{K_1+K_2}{2}$. By the Brunn-Minkowski inequality (see [**8**], p.309), we have

$$V(\tilde K) > \left(\frac{1}{2}V(K_1)^{\frac{1}{3}} + \frac{1}{2}V(K_2)^{\frac{1}{3}}\right)^3 = V_0\,, \tag{2.2}$$

where strict inequality holds since we are assuming that K_1 and K_2 are not homothetic. By the linearity of the mean width with respect to Minkowski addition, $W(\tilde K) = W_0$ and hence $\tilde K \in \mathcal{K}(V_0, W_0)$. Since $\tilde K$ has maximum diameter, the strict inequality (2.2) contradicts (2.1).

The same argument we just used can be repeated to show that *if K is a solution of Problem 1, then K is a centrally symmetric body of revolution about a diameter.*

3. *Regularity of the solution*

In order to describe the solution K, it is sufficient to characterize its section with a plane through the axis of revolution. Fix an orthonormal system $(O; x, y)$ in the plane and assume that the diameter of K lies on the x-axis symmetrically with respect to O. The boundary of K is then obtained by rotating the meridian curve γ, contained in the half-plane $y \geq 0$, around the x-axis. Let us denote by C the section of K with the xy-plane and by h_C its support function. For simplicity, we write $h_C(\theta)$ instead of $h_C(\cos\theta, \sin\theta)$.

The curve γ can be parametrized by

$$\begin{cases} x(\theta) = h_C(\theta)\cos\theta - h_C'(\theta)\sin\theta \\ y(\theta) = h_C(\theta)\sin\theta + h_C'(\theta)\cos\theta\,. \end{cases}$$

Hence

$$V(K) = \pi\int_{-D/2}^{D/2} y^2\,dx = \pi\int_0^\pi \big(h_C(\theta)\sin\theta + h_C'(\theta)\cos\theta\big)^2\big(h_C(\theta) + h_C''(\theta)\big)\sin\theta\,d\theta\,.$$

Integrating by parts gives

$$V(K) = \pi\int_0^\pi \big(\frac{2}{3}h_C(\theta)^3\sin\theta - \frac{1}{3}h_C'(\theta)^3\cos\theta - h_C(\theta)h_C'(\theta)^2\sin\theta\big)\,d\theta\,.$$

Note that the last equality requires no regularity assumptions on ∂C.

Furthermore,

$$W(K) = \int_0^\pi h_C(\theta)\sin(\theta)\,d\theta\,.$$

We now show that *every interior point of γ is regular, i.e., the normal cone has dimension one.*

Suppose, on the contrary, that there exists a point $p = (p_1, p_2)$ in the interior of γ whose normal cone contains the arc $[\theta_1, \theta_2]$. Cut the curve γ with a line orthogonal to the axis of symmetry of the arc at a distance ε from p and let q_1 and q_2 be the intersection points. Replace the arc of γ with the chord q_1q_2, and denote by $\tilde{\gamma}$ the curve so obtained. Let $\tilde{K}$ be the convex body enclosed by the surface of revolution of $\tilde{\gamma}$ about the x-axis and let $\tilde{C}$ be its section with the xy-plane. Clearly, for every $\varepsilon > 0$, $V(\tilde{K}) < V(K)$ and $W(\tilde{K}) < W(K)$. Precisely,

$$\begin{aligned} W(K) - W(\tilde{K}) &= \int_0^{\pi} (h_C(\theta) - h_{\tilde{C}}(\theta)) \sin\theta \, d\theta \\ &= \int_{\theta_1}^{(\theta_1+\theta_2)/2} \frac{\varepsilon \sin(\theta-\theta_1)}{\sin\frac{\theta_2-\theta_1}{2}} \sin\theta \, d\theta + \int_{(\theta_1+\theta_2)/2}^{\theta_2} \frac{\varepsilon \sin(\theta_2-\theta)}{\sin\frac{\theta_2-\theta_1}{2}} \sin\theta \, d\theta \\ &= \frac{\varepsilon}{2\sin\frac{\theta_2-\theta_1}{2}} (\cos\theta_1 - \cos\theta_2) \frac{\theta_2-\theta_1}{2} \, . \end{aligned}$$

If we denote by $\tilde{y}$ the barycenter of the triangle q_1pq_2 and by $A(\cdot)$ the area, then

$$V(K) - V(\tilde{K}) = 2\pi\tilde{y}A(q_1pq_2) = \frac{2\pi\varepsilon^2}{\tan\frac{\theta_2-\theta_1}{2}} \left(p_2 - \frac{\varepsilon(\cos\theta_1 - \cos\theta_2)}{3\sin\frac{\theta_2-\theta_1}{2}} \right) .$$

Consider the Minkowski sum $\hat{K}$ of $\tilde{K}$ and $\lambda\bar{x}$, where $\bar{x}$ is a unit segment parallel to the x-axis. Since $W(\lambda\bar{x}) = \lambda/2$, if we take

$$\lambda = \frac{\varepsilon}{\sin\frac{\theta_2-\theta_1}{2}} (\cos\theta_1 - \cos\theta_2) \frac{\theta_2-\theta_1}{2} \, ,$$

then $W(\hat{K}) = W(K) = W_0$. Moreover, denoting by $\tilde{K}|x^{\perp}$ the orthogonal projection of $\tilde{K}$ onto a plane orthogonal to the x-axis,

$$\begin{aligned} V(\hat{K}) &= V(\tilde{K}) + \lambda A(\tilde{K}|x^{\perp}) \\ &= V(K) - \frac{2\pi\varepsilon^2}{\tan\frac{\theta_2-\theta_1}{2}} \left(p_2 - \frac{\varepsilon(\cos\theta_1-\cos\theta_2)}{3\sin\frac{\theta_2-\theta_1}{2}} \right) + \frac{\varepsilon(\cos\theta_1-\cos\theta_2)}{\sin\frac{\theta_2-\theta_1}{2}} \frac{\theta_2-\theta_1}{2} A(\tilde{K}|x^{\perp}) \, . \end{aligned}$$

Hence, for ε sufficiently small, $V(\hat{K}) > V(K) = V_0$, which is a contradiction.

4. *Gauss curvature of the solution*

We show now that *the solution is the convex hull of a surface of revolution with positive constant Gauss curvature (except at the points on the axis) and a segment lying on the axis of revolution.*

Note that the above property does not exhaust the description of the solution. Indeed, two parameters need still to be determined, namely the value of the Gauss curvature and the length of the segment. Section 5 deals with this aspect.

We are going to prove the statement by means of local variations of the solution. To do this, it is convenient to introduce the Gauss map, which we adapt here to bodies of revolution about the x-axis. Let H be such a body. For every point p in ∂H, let $f(p)$ be the corresponding point on the meridian curve γ_H. We denote by $g_H(p)$ the set of outward unit normals to ∂H at $f(p)$, and by $G_H(p)$ the set of outward unit normals to ∂H at p. Note that G_H maps ∂H into $\mathbb{S}^2$, while we can regard g_H as a map from ∂H to $[0, \pi]$, by identifying each outer unit normal with its angle with the x-axis. For simplicity, if ω is a subset of $[0, \pi]$, then we define $s(\omega) = G_H(g_H^{-1}(\omega))$.

Let $\omega = [\theta_1, \theta_2]$ with $0 < \theta_1 \leq \theta_2 < \pi$ and let $\varepsilon > 0$. For every p in the subset $g_K^{-1}(\omega)$ of ∂K, consider the segment of length ε issuing from p along the outer normal to ∂K. Define $K(\omega, \varepsilon)$ as the convex hull of K and all these segments.

It is easy to check that

$$h_{K(\omega,\varepsilon)}(z) \leq h_K(z) + \varepsilon$$

and that equality occurs when $z \in s(\omega)$.

Moreover, let $K(\omega, -\varepsilon)$ be the largest convex body of revolution such that

$$h_{K(\omega,-\varepsilon)}(z) \leq h_K(z) \quad , \text{ for every } z \in \mathbb{S}^2$$

and

$$h_{K(\omega,-\varepsilon)}(z) \leq h_K(z) - \varepsilon \quad , \text{ for every } z \in s(\omega)\,.$$

Thus,

$$\begin{aligned} W(K(\omega,\varepsilon)) - W(K) &= \tfrac{1}{2\pi} \textstyle\int_{\mathbb{S}^2} \left[h_{K(\omega,\varepsilon)}(z) - h_K(z)\right] dz \\ &\leq \tfrac{\varepsilon}{2\pi} A(G_{K(\omega,\varepsilon)}(\partial K(\omega,\varepsilon) \setminus \partial K))\,. \end{aligned} \tag{4.1}$$

Analogously,

$$W(K) - W(K(\omega,-\varepsilon)) \geq \varepsilon A(s(\omega))\,. \tag{4.2}$$

From the definitions of $K(\omega, \varepsilon)$ and $K(\omega, -\varepsilon)$ it follows that

$$V(K(\omega,\varepsilon)) - V(K) \geq \varepsilon A(g_K^{-1}(\omega)) \tag{4.3}$$

and

$$V(K) - V(K(\omega,-\varepsilon)) \leq \varepsilon A(\partial K(\omega,-\varepsilon) \setminus \partial K)\,. \tag{4.4}$$

Let $\tilde{K} = [K(\omega_1, \varepsilon_1)](\omega_2, -\varepsilon_2)$, where the intervals ω_1, ω_2 are disjoint and have positive distance from the x-axis. It is easy to check that if ε_1, ε_2 are sufficiently small, then $d(\tilde{K}) = d(K)$.

From (4.1), (4.2), (4.3) and (4.4) we deduce that $\tilde{K}$ belongs to $\mathcal{K}(V_0, W_0)$ if ε_1 and ε_2 satisfy

$$\varepsilon_1 A(G_{K(\omega_1,\varepsilon_1)}(\partial K(\omega_1,\varepsilon_1) \setminus \partial K)) \leq \varepsilon_2 A(s(\omega_2)) \tag{4.5}$$

and

$$\varepsilon_1 A(g_K^{-1}(\omega_1)) \geq \varepsilon_2 A(\partial K(\omega_2,-\varepsilon_2) \setminus \partial K)\,. \tag{4.6}$$

If (4.5) or (4.6) is strict, then $\tilde{K}$ does not satisfy (2.1), which contradicts the fact that $\tilde{K}$ maximizes the diameter in $\mathcal{K}(V_0, W_0)$. This happens if

$$\frac{A(\partial K(\omega_2,-\varepsilon_2) \setminus \partial K)}{A(s(\omega_2))} < \frac{A(g_K^{-1}(\omega_1))}{A(G_{K(\omega_1,\varepsilon_1)}(\partial K(\omega_1,\varepsilon_1) \setminus \partial K))}\,. \tag{4.7}$$

Letting ε_1 and ε_2 tend to zero, inequality (4.7) holds if

$$\frac{A(g_K^{-1}(\omega_2))}{A(s(\omega_2))} < \frac{A(g_K^{-1}(\omega_1))}{A(s(\omega_1))}\,. \tag{4.8}$$

Therefore, if K is the solution, then

$$\frac{A(g_K^{-1}(\omega))}{A(s(\omega))} = \textit{constant}\,, \tag{4.9}$$

for every closed subinterval ω of $[0, \pi]$ with $g_K^{-1}(\omega)$ at positive distance from the x-axis. Since $g_K^{-1}(\omega) = G_K^{-1}(s(\omega))$, identity (4.9) states that the *area measure* σ_K

of K is proportional to the Hausdorff measure on $\mathbb{S}^2$ (see [**8**], Th. 4.2.5), except possibly two conical parts with vertex on the axis of revolution. We recall that for a smooth convex body K, the area measure of a Borel subset β of $\mathbb{S}^2$ is the integral over $G_K^{-1}(\beta)$ of the Gauss curvature of ∂K. Since a body is uniquely determined, up to translation, by its area measure, the proof is concluded.

5. *Characterization of the solution*

The final step consists in understanding whether the meridian curve of the solution has two segments at the endpoints or not. Of course, the answer might depend on the values V_0 and W_0. Actually, we show that the solution always contains two conical parts, except when $\pi W_0^3 = 6V_0$. In such a case the class $\mathcal{K}(V_0, W_0)$ contains only the ball with those parameters.

Let $K \in \mathcal{K}(V_0, W_0)$ and $y = f(x)$, $-d/2 \le x \le d/2$, a representation of its meridian curve γ_K, where $d = d(K)$. Denote by α the angle such that $\tan\alpha = f'(-d/2)$, $0 < \alpha \le \pi/2$ and by ℓ the common length of the two segments on the graph of $f(x)$. If k denotes the Gauss curvature of the central part of K, then the function $f(x)$ satisfies in $[\ell\cos\alpha - d/2, d/2 - \ell\cos\alpha]$ the differential equation

$$kf(x)\left(1 + f'(x)^2\right)^2 + f''(x) = 0\,.$$

A first integration yields

$$f'(x) = -sign(x)\sqrt{\frac{1}{kf(x)^2 + 1 - (1 + k\ell^2)\sin^2\alpha} - 1}\,, \tag{5.1}$$

where we used the condition

$$f'(\ell\cos\alpha - d/2) = \tan\alpha\,.$$

From $f'(0) = 0$, we deduce that

$$kf(0)^2 = (1 + k\ell^2)\sin^2\alpha\,. \tag{5.2}$$

Integrating (5.1) gives

$$\int_{\ell\sin\alpha}^{f(x)} \frac{dt}{\sqrt{\frac{1}{kt^2+1-(1+k\ell^2)\sin^2\alpha} - 1}} = -|x| + \frac{d}{2} - \ell\cos\alpha\,. \tag{5.3}$$

Note that for fixed W_0^3/V_0, different choices of V_0 and W_0 correspond to homothetic solutions. Since we are interested in the shape of the solution, we may assume $f(0) = 1$ as a normalization. Setting $\rho = \ell\sin\alpha$, from (5.2) and (5.3), the diameter is given by

$$d = 2\rho\frac{\cos\alpha}{\sin\alpha} + \frac{2}{\sin\alpha}\int_{\arcsin\rho}^{\pi/2} \Gamma(\alpha, \rho, t)\,dt\,, \tag{5.4}$$

where

$$\Gamma(\alpha, \rho, t) = \sqrt{1 - \rho^2 - \sin^2\alpha\cos^2 t}\,. \tag{5.5}$$

The volume V of K is the sum of the volume of the two cones and of the central part. Therefore, we obtain

$$V = \frac{2}{3}\pi\rho^3\frac{\cos\alpha}{\sin\alpha} + \frac{2\pi}{\sin\alpha}\int_{\arcsin\rho}^{\pi/2} \Gamma(\alpha, \rho, t)\sin^2 t\,dt\,. \tag{5.6}$$

Finally, let us show that the mean width W of K can be expressed, in terms of α, ρ, V and d, as

$$W = \frac{3V\sin^2\alpha + \pi d(\cos^2\alpha - \rho^2)}{2\pi(1-\rho^2)}\,. \tag{5.7}$$

Indeed, by definition (1.3),

$$W = \frac{1}{2\pi}\int_{\Sigma_1} h_K(z)\,dz + \frac{1}{2\pi}\int_{\Sigma_2} h_K(z)\,dz\,,$$

where $\Sigma_1 = s((\frac{\pi}{2}-\alpha, \frac{\pi}{2}+\alpha))$ and $\Sigma_2 = \mathbb{S}^2 \setminus \Sigma_1$.

It is convenient to use here the following expression for the volume of K (see [**3**], p. 64):

$$V = \frac{1}{3}\int_{\mathbb{S}^2} h_K(z)\,d\sigma_K(z)\,.$$

Consequently,

$$V = \frac{1}{3}\int_{\Sigma_1} \frac{h_K(z)}{k}\,dz + \frac{1}{3}\int_{\Sigma_2} h_K(z)\,d\sigma_K(z)\,.$$

Hence,

$$2\pi W - 3kV = \int_{\Sigma_2} h_K(z)\,dz - k\int_{\Sigma_2} h_K(z)\,d\sigma_K(z)\,.$$

Note that in Σ_2 the area measure is concentrated on the boundary and the support function can be easily evaluated. Thus, (5.7) follows.

Now, looking at the expressions (5.4), (5.6) and (5.7), we note that K has minimal mean width among all bodies with the same volume and diameter. Indeed, if L is a body such that

$$V(L) = V(K)\,, \quad d(L) = d(K) \quad \text{and} \quad W(L) < W(K)\,,$$

then L belongs to $\mathcal{K}(V_0, W_0)$ and solves Problem 1. This contradicts (2.1).

Thus, let us consider the function W in (5.7) under the constraint

$$\frac{V^{1/3}}{d} = \mu\,, \tag{5.8}$$

where V and d are defined by (5.6) and (5.4). The isodiametric inequality (see [**4**], p. 145), states that $\mu \le (\frac{\pi}{6})^{1/3}$, where equality holds only for balls.

In order to show the existence of the conical parts, for every fixed $\mu < (\frac{\pi}{6})^{1/3}$, it is sufficient to prove that the minimum of W is not attained when $\rho = 0$.

To this end, let us remark that $\frac{\partial}{\partial\alpha}\frac{V^{1/3}}{d}$ vanishes, for $\rho = 0$, only for $\alpha = \frac{\pi}{2}$. Hence, for every point $(0,\alpha)$, with $\alpha < \frac{\pi}{2}$, the level set (5.8) issuing from it defines a function $\alpha = \alpha(\rho)$ in a one-sided neighborhood of $\rho = 0$. Consequently, W can be thought of as a function of ρ, that, for simplicity, we denote by $W(\rho)$. We claim that $W'(0) = W''(0) = 0$ and $W'''(0) < 0$.

Indeed, calculations lead to the following asymptotic expansion:

$$W'(\rho) = \frac{3\Gamma_2\sin^2\alpha(6\Gamma_2 - 3\Gamma_4 - 2\Gamma_1) - \Gamma_1\Gamma_4\cos^2\alpha}{\sin\alpha\cos\alpha(3\Gamma_2\Gamma_3 - \Gamma_1\Gamma_4)}\,\rho^2 + O(\rho^3)\,, \tag{5.9}$$

where

$$\Gamma_1 = \int_0^{\pi/2} \Gamma(\alpha,0,t)\,dt\,, \qquad \Gamma_2 = \int_0^{\pi/2} \Gamma(\alpha,0,t)\sin^2 t\,dt\,,$$

$$\Gamma_3 = \int_0^{\pi/2} \frac{dt}{\Gamma(\alpha,0,t)} \quad \text{and} \quad \Gamma_4 = \int_0^{\pi/2} \frac{\sin^2 t}{\Gamma(\alpha,0,t)}\, dt\ .$$

First, $3\Gamma_2\Gamma_3 - \Gamma_1\Gamma_4 > 0$. Indeed, by Hölder's inequality,

$$1 = \int_0^{\pi/2} \sin t\, dt \le (\Gamma_2\Gamma_3)^{1/2}\,,$$

while $\Gamma_1 \le \frac{\pi}{2}$ and $\Gamma_4 \le 1$.

Let us check that the numerator in (5.9) is negative. By Hölder's inequality, $\Gamma_1\Gamma_4 \ge 1$ and

$$\frac{\pi}{4} = \int_0^{\pi/2} \sin^2 t\, dt \le (\Gamma_2\Gamma_4)^{1/2}\,,$$

$$\frac{1}{2}\left(\frac{\alpha}{\sin\alpha} + \cos\alpha\right) = \int_0^{\pi/2} \Gamma(\alpha,0,t)\sin t\, dt \le (\Gamma_1\Gamma_2)^{1/2}$$

and

$$\Gamma_2^2 \le \left(\int_0^{\pi/2} \sin^2 t\, dt\right)\left(\int_0^{\pi/2} \Gamma^2(\alpha,0,t)\sin^2 t\, dt\right) = \frac{\pi^2}{16}\,(1 - \frac{1}{4}\sin^2\alpha)\,.$$

Therefore, the numerator is bounded from above by

$$(\frac{3}{16}\pi^2 + 1)\sin^2\alpha - \frac{9}{32}\pi^2\sin^4\alpha - 1\,,$$

which is negative for every α.

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Stefano Campi
Dipartimento di Ingegneria dell'Informazione
Università degli Studi di Siena
Via Roma 56, 53100 Siena,
Italy

campi@dii.unisi.it

Paolo Gronchi
Dipartimento di Matematica e Applicazioni per l'Architettura
Università degli Studi di Firenze
Piazza Ghiberti 27, 50122 Firenze
Italy

paolo@fi.iac.cnr.it