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INTERSECTIONS OF DILATATES OF CONVEX BODIES

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Abstract. We initiate a systematic investigation into the nature of the function $\alpha_K(L,\rho)$ that gives the volume of the intersection of one convex body K in $\mathbb{R}^n$ and a dilatate ρL of another convex body L in $\mathbb{R}^n$, as well as the function $\eta_K(L,\rho)$ that gives the $(n-1)$-dimensional Hausdorff measure of the intersection of K and the boundary $\partial(\rho L)$ of ρL. The focus is on the concavity properties of $\alpha_K(L,\rho)$. Of particular interest is the case when K and L are symmetric with respect to the origin. In this situation, there is an interesting change in the concavity properties of $\alpha_K(L,\rho)$ between dimension 2 and dimensions 3 or higher. When L is the unit ball, an important special case with connections to E. Lutwak's dual Brunn-Minkowski theory, we prove that this change occurs between dimension 2 and dimensions 4 or higher, and conjecture that it occurs between dimension 3 and dimension 4. We also establish an isoperimetric inequality with equality condition for subsets of equatorial zones in the sphere S^2, and apply this and the Brunn-Minkowski inequality in the sphere to obtain results related to this conjecture, as well as to the properties of a new type of symmetral of a convex body which we call the equatorial symmetral.

1. Introduction

Suppose that K is a convex body and B is the unit ball in $\mathbb{R}^n$, and $\rho \geq 0$ is such that $\partial K \cap \rho S^{n-1} \neq \emptyset$. Let $\alpha_K(\rho) = V(K \cap \rho B)$ be the volume of $K \cap \rho B$ and let $\eta_K(\rho) = \mathcal{H}^{n-1}(K \cap \rho S^{n-1})$ be the area ($(n-1)$-dimensional Hausdorff measure) of $K \cap \rho S^{n-1}$. (See Section 2 for other definitions and notation.) This study originates from an observation of Benguria, Levitin, and Parnovski [1, Lemma 5.1 and Remark 5.2] that when $n = 2$ and K is o-symmetric, $\eta_K(\rho)$ is decreasing and hence $\alpha_K(\rho)$ is concave, while this is not true in general when $n \geq 3$. (The result for $n = 2$ also follows from a more general one of Campi [5, Lemma 1].)

The authors of [1] were motivated by two conjectures concerning the set $N(K)$ of zeros of the Fourier transform $\widehat{1_K}$ of the characteristic function of an o-symmetric convex body K in $\mathbb{R}^n$. They conjecture that the distance from $N(K)$ to the origin is minimized when K is a ball, and is bounded above by $\sqrt{\lambda_2(K)}$, where $\lambda_2(K)$ is the second Dirichlet eigenvalue of K. Their observation mentioned above is used to prove some weaker versions of these conjectures in the plane.

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Motivated by this observation and its application, as well as by connections E. Lutwak's dual Brunn-Minkowski theory (see Section 2 for more details), we initiate a systematic investigation into the properties of the functions $\alpha_K(L,\rho)$ and $\eta_K(L,\rho)$ (always considered as functions of ρ) that give the volume $V(K\cap\rho L)$ and area $\mathcal{H}^{n-1}(K\cap\partial(\rho L))$ for the intersection of K and a dilatate ρL of L. (When $L=B$, we write $\alpha_K(\rho)$ and $\eta_K(\rho)$ instead of $\alpha_K(B,\rho)$ and $\eta_K(B,\rho)$.) As far as we are aware, such a study is new, somewhat surprising in view of the basic nature of the operations of intersection and dilatation.

A simple application (see Theorem 3.1) of the Brunn-Minkowski inequality yields that for arbitrary convex bodies K and L in $\mathbb{R}^n$ with $o\in L$, the function $\alpha_K(L,\rho)^{1/n}$ is concave, and in general the exponent cannot be improved, even when $L=B\subset K$. However, we show in Theorem 3.2 that if in addition $n=2$ and the origin belongs to the so-called kernel of K with respect to L, the exponent can be improved and in fact $\alpha_K(L,\rho)$ is concave. In particular, this holds when K and L are planar o-symmetric convex bodies (see Corollary 3.3). Specializing further by taking $L=B$, we see that Theorem 3.2 yields the observation in [1, Lemma 5.1], but is of course considerably more general. In this generality, it is not possible to approach this result via the function $\eta_K(L,\rho)$, and in any case, we construct after Corollary 3.3 planar o-symmetric convex polygons K and L such that $\eta_K(L,\rho)$ increases rather than decreases. The final result in Section 3, Theorem 3.4, shows that the exponent $1/n$ in Theorem 3.1 is optimal for $n\geq 3$ even when K and L are o-symmetric.

In Section 4, we focus on the special case when K is o-symmetric and $L=B$, the situation in [1], and consider the possibility that $\alpha_K(\rho)^{1/(n-1)}$ is concave. From the above we know that this holds when $n=2$. In Theorem 4.5, by making computations when K is an o-symmetric slab, we show that in general the exponent $1/(n-1)$ would be optimal, and moreover when $n\geq 4$, the exponent $1/n$ is best possible. However, for this slab K, $\alpha_K(\rho)^{1/2}$ is concave when $n=3$. The extremal nature of the slab led us to make the intriguing Conjecture 4.6, that if K is an o-symmetric convex body in $\mathbb{R}^3$, then $\alpha_K(\rho)^{1/2}$ is concave.

Thus we find that for o-symmetric convex bodies, the concavity properties of $\alpha_K(L,\rho)$ change between 2 and 3 dimensions, and we conjecture that the concavity properties of $\alpha_K(\rho)$ change between 3 and 4 dimensions. If the latter is true, this would be a remarkable and surprising fact. Our attempts to shed light on Conjecture 4.6 via the Brunn-Minkowski inequality in the sphere are described in Section 5. The method allows estimates on the derivative $\eta_K'(\rho)$ and hence $\alpha_K''(\rho)$, when they exist—see Theorem 5.2—but so far has not led to a proof of the conjecture. Instead we obtain in Theorem 5.3 an optimal strengthening of the result in [1, Lemma 5.1] and [5, Lemma 1] that when $n=2$ and K is o-symmetric, $\eta_K(\rho)$ is decreasing.

In Theorem 6.1 we establish an isoperimetric inequality with equality condition for subsets of equatorial zones in the sphere S^2. This is combined with the method described in the previous paragraph in Theorem 7.1, which states that $\eta_K(\rho)/\rho$ is decreasing when K is an o-symmetric convex body in $\mathbb{R}^3$. As we explain, this statement is equivalent to the quasiconvexity of a new type of symmetral of K which we call the equatorial symmetral.

In Section 8, we study a related pair of functions. For one of these, defined by $\overline{\alpha}_K(L,\rho) = \max_{x\in\mathbb{R}^n} V(K\cap(\rho L+x))$, the Brunn-Minkowski inequality is applied again to conclude in Theorem 8.1 that $\overline{\alpha}_K(L,\rho)^{1/n}$ is concave.

We thank Michael Levitin for drawing our attention to [1].

2. Definitions and preliminaries

As usual, S^{n-1} denotes the unit sphere and o the origin in Euclidean n-space $\mathbb{R}^n$. The unit ball in $\mathbb{R}^n$ will be denoted by B. The standard orthonormal basis for $\mathbb{R}^n$ will be $\{e_1,\dots,e_n\}$. We write $[x,y]$ for the line segment with endpoints x and y.

If X is a set, we denote by ∂X, $\operatorname{int} X$, and $\operatorname{conv} X$ the *boundary*, *interior*, and *convex hull* of X, respectively.

If X and Y are subsets of $\mathbb{R}^n$, their *vector* or *Minkowski sum* is

$$X+Y=\{x+y : x\in X, y\in Y\},$$

and if $t\in\mathbb{R}$, then

$$tX=\{tx : x\in X\}.$$

When $t>0$, tX is called a *dilatate* of X. The set $-X$ is the *reflection* of X in the origin.

We write $\mathcal{H}^k$ for k-dimensional Hausdorff measure in $\mathbb{R}^n$, where $k=1,\dots,n$. If K is a k-dimensional convex body in $\mathbb{R}^n$, then $V(K)$ is its *volume* $\mathcal{H}^k(K)$. The notation dz will always mean $d\mathcal{H}^k(z)$ for the appropriate $k=1,\dots,n$.

By Jensen's inequality for integrals [9, (B.8), p. 414], if $p\le q$ and a real-valued function f is q-concave (that is, f^q is concave) on its support, then f is p-concave on its support.

We follow Schneider [20] by writing κ_n for the volume $V(B)$ of the unit ball in $\mathbb{R}^n$, so that $\kappa_n=\pi^{n/2}/\,\Gamma(1+n/2)$.

A set is *o-symmetric* if it is centrally symmetric, with center at the origin.

A *convex body* is a compact convex set with nonempty interior.

Let K and L be convex bodies in $\mathbb{R}^n$ with $o\in L$. We write

$$r(K,L)=\min\{\rho : \partial K\cap\partial(\rho L)\neq\emptyset\}$$

and

$$R(K,L)=\max\{\rho : \partial K\cap\partial(\rho L)\neq\emptyset\}.$$

For $r(K,L)\le\rho\le R(K,L)$, we let

$$\alpha_K(L,\rho)=V(K\cap\rho L)$$

and

$$\eta_K(L,\rho)=\mathcal{H}^{n-1}\left(K\cap\partial(\rho L)\right).$$

Note that the domain of both functions is $[r(K,L),R(K,L)]$. In the special case when $L=B$, we simplify the notation by writing $r_K=r(K,B)$, $R_K=R(K,B)$, $\alpha_K(\rho)=\alpha_K(B,\rho)$, and $\eta_K(\rho)=\eta_K(B,\rho)$. Note that the domain of the latter functions is $[r_K,R_K]$.

It is of course possible to extend these definitions, though we shall only use the previous definitions in our results. For example, when $L = B$, we can define for any bounded Borel set C in $\mathbb{R}^n$,

$$\eta_C(\rho) = \mathcal{H}^{n-1}\left(C \cap \rho S^{n-1}\right), \tag{1}$$

for all $\rho \geq 0$. Similarly, we can also define $\alpha_C(\rho) = \mathcal{H}^n(C \cap \rho B)$, for all $\rho \geq 0$.

In 1975, Lutwak [14] initiated the dual Brunn-Minkowski theory (dBMt), in which the intersections of star bodies with subspaces replace the projections of convex bodies onto subspaces in the classical Brunn-Minkowski theory (BMt) expounded in [20]. He discovered that integrals over S^{n-1} of products of radial functions (giving the distance from the origin to the boundary) behave like the mixed volumes in the BMt, and called them dual mixed volumes. Special cases of dual mixed volumes, analogous to the intrinsic volumes in the BMt, are called dual volumes. A formula called the dual Kubota integral recursion (see [15] or [9, Theorem A.7.2]) allows dual volumes to be represented as averages of volumes of intersections with subspaces, just as intrinsic volumes can be represented as averages of projections onto subspaces.

For our purposes, it is useful to define the *ith dual volume* $\widetilde{V}_i(C)$ of any bounded Borel set C in $\mathbb{R}^n$, for real $i > 0$, by

$$\widetilde{V}_i(C) = \frac{i}{n}\int_C \|x\|^{i-n}\,dx, \tag{2}$$

where integration is with respect to $\mathcal{H}^n$. When C is a star body, (2) agrees with the original definition in [14] via a change to polar coordinates; see [11, Theorem 4.1]. Moreover the definition (2) allows dual volumes to be seen as moments of sets, imparting a clear and fundamental geometrical significance to these quantities. For a bounded Borel set C in $\mathbb{R}^n$, we have

$$\widetilde{V}_i(C) = \frac{i}{n}\int_C \|x\|^{i-n}\,dx = \frac{i}{n}\int_0^\infty \int_{\rho S^{n-1}\cap C} \|\rho u\|^{i-n}\,du\,d\rho = \frac{i}{n}\int_0^\infty \eta_C(\rho)\rho^{i-n}\,d\rho,$$

where η_C is defined by (1). If C is a star-shaped body with respect to the origin and r_C and R_C are the minimum and maximum values of the radial function of C, then integration by parts leads to the formula

$$\widetilde{V}_i(C) = \frac{i}{n}\mathcal{H}^n(C)R_C^{i-n} + \frac{\kappa_n(n-i)}{n}r_C^i + \frac{i(n-i)}{n}\int_{r_C}^{R_C} \alpha_C(\rho)\rho^{i-n-1}\,d\rho.$$

Thus the functions η_C and α_C are natural components of the dBMt.

In recent years the dBMt has been the subject of quite intense activity. It is now recognized that the dBMt has connections and applications to many areas, such as integral geometry, Minkowski geometry, the local theory of Banach spaces, stereology, and information theory; see [9] and the references given there. Even when restricted to o-symmetric convex bodies, the dBMt can count among its successes the solution of the Busemann-Petty problem in [7], [10], [16], [21], and [22].

3. Properties of $\alpha_K(L,\cdot)$ and $\eta_K(L,\cdot)$

Theorem 3.1. *If K and L are convex bodies in $\mathbb{R}^n$ with $o \in L$, then $\alpha_K(L,\rho)^{1/n}$ is concave.*

Proof. Let $H = K \times [r(K,L), R(K,L)] \subset \mathbb{R}^{n+1}$, let

$$C = \operatorname{conv}\left\{o, \{(x_1,\dots,x_{n+1}) : (x_1,\dots,x_n) \in R(K,L)L,\, x_{n+1} = R(K,L)\}\right\},$$

and let $M = H \cap C$. Then, identifying the plane $\{x_{n+1} = \rho\}$ with $\mathbb{R}^n$, we have

$$M \cap \{x_{n+1} = \rho\} = K \cap \rho L,$$

for $r(K,L) \le \rho \le R(K,L)$, so by the Brunn-Minkowski inequality (see [8], [9, Section B.2], and [20, Section 6.1]) the function

$$\alpha_K(L,\rho)^{1/n} = V\left(M \cap \{x_{n+1} = \rho\}\right)^{1/n}$$

is concave. □

The previous result is the best possible, even when $L = B \subset K$. To see this, let $M = B \cap \{x_n \ge 0\}$ be a half-ball containing the origin, so that $r_M = 0$, $R_M = 1$, and $\alpha_M(\rho) = \kappa_n \rho^n/2$, $0 \le \rho \le 1$. Let $q > 1/n$ and note that $\alpha_M(\rho)^q = (\kappa_n/2)^q \rho^{nq}$ is not concave. Let $0 < s < 1/2$ and let $K_s = M - se_n$. Then $sB \subset K_s$ and α_{K_s} converges uniformly to α_M as $s \to 0$. Therefore there is an $s_0 > 0$ such that for $s \le s_0$, $\alpha_{K_s}(\rho)^q$ is not concave. Letting $K = (1/s_0)K_{s_0}$, we have $B \subset K$ and $\alpha_K(\rho)^q$ is not concave.

Despite this example, there are situations in which Theorem 3.1 can be strengthened. To state one, let K and L be convex bodies in $\mathbb{R}^n$ such that $o \in L$. The *relative inradius* $\overline{r}(K,L)$ of K with respect to L is defined by

$$\overline{r}(K,L) = \sup\{r : rL + x \subset K \text{ for some } x \in \mathbb{R}^n\}.$$

The *kernel* of K with respect to L is denoted by

$$K_L = \{x : \overline{r}(K,L)L + x \subset K\}.$$

See, for example, [19]. Note that if $o \in K_L$, then $\overline{r}(K,L) = r(K,L)$ and $r(K,L)L \subset K$ but $sL + x \not\subset K$ for any $s > r(K,L)$ and $x \in \mathbb{R}^n$.

Theorem 3.2. *If K and L are convex bodies in $\mathbb{R}^2$ such that $o \in K_L$, then $\alpha_K(L,\rho)$ is concave.*

Proof. It suffices to show that

$$\alpha_K(L,\rho+h) - \alpha_K(L,\rho) \le \alpha_K(L,\rho) - \alpha_K(L,\rho-h),$$

for every $\rho, h > 0$ such that $[\rho-h, \rho+h] \subset [r(K,L), R(K,L)]$. If we let

$$C = (\rho+h)L \setminus \rho L \quad \text{and} \quad D = \rho L \setminus (\rho-h)L,$$

then we want to show that $\mathcal{H}^2(C \cap K) \le \mathcal{H}^2(D \cap K)$. The set $\operatorname{int} K \setminus r(K,L)L$ consists of countably many components. Let E be any such component, and let $C_1 = E \cap \operatorname{int} C$ and $D_1 = E \cap \operatorname{int} D$ be the corresponding (possibly empty) components of $\operatorname{int}(C \cap K)$ and $\operatorname{int}(D \cap K)$, respectively. Then it suffices to prove that $\mathcal{H}^2(C_1) \le \mathcal{H}^2(D_1)$.

The component D_1 is bounded by four arcs, each contained in ∂K, or $\partial\left((\rho-h)L\right)$, or $\partial(\rho L)$. Let x_1, y_1, y_2, and x_2 be the endpoints of these arcs, labeled clockwise and such that x_1 and x_2 belong to $(\rho-h)L$. For $i=1,2$, the line through x_i and y_i intersects $\partial\left((\rho+h)L\right)$ in a point z_i, say.

Let $C_1'\supset C_1$ be the region bounded by $[y_1,z_1]$, $[y_2,z_2]$, the arc in $\partial(\rho L)$ between y_1 and y_2, and the arc in $\partial\left((\rho+h)L\right)$ between z_1 and z_2. Similarly, let $D_1'\subset D_1$ be the region bounded by the line segments $[x_1,y_1]$ and $[x_2,y_2]$, the arc in $\partial\left((\rho-h)L\right)$ between x_1 and x_2, and the arc in $\partial(\rho L)$ between y_1 and y_2.

The lines l_i containing the segments $[x_i,y_i]$, $i=1,2$, are either parallel or meet at a point $w\neq o$. Let $u\in S^1$ be parallel to l_1 and l_2 in the first case, and let $u=-w/\|w\|$ in the second case. We claim that in the second case, $[o,w]\cap C_1'\neq\emptyset$. Indeed, if this is not true, then the ray emanating from the origin in the direction u meets C_1'. Note that the closed cone T with apex at w bounded by l_1 and l_2 is such that $r(K,L)L\subset K\cap T$. Let m_i, $i=1,2$ be the lines parallel to l_i and tangent to $r(K,L)L$ at p_i, say, and let m_i meet $\partial(\rho L)$ at q_i. Then for $i=1,2$, we have $p_i,q_i\in K$ and the relative interior of the line segment $[p_i,q_i]$ is contained in $\operatorname{int}K$. From this we see that for sufficiently small $\varepsilon>0$ we have $r(K,L)L+\varepsilon u\subset\operatorname{int}K$ and hence that there is an $r>r(K,L)$ such that $rL+\varepsilon u\subset K$, contradicting the definition of $r(K,L)$ and proving the claim.

In either of the two cases under consideration, let S be the closed strip bounded by the lines through y_1 and y_2 parallel to u, and let $C_1''=S\cap C$ and $D_1''=S\cap D$. Now $C_1\subset C_1'\subset C_1''$ and $D_1''\subset D_1'\subset D_1$, so it suffices to show that $\mathcal{H}^2(C_1'')\leq\mathcal{H}^2(D_1'')$.

To this end, choose a Cartesian coordinate system with o as the origin and the y-axis parallel to u, with the positive y-axis intersecting C_1'. Suppose that

$$L=\{(x,y)\,|\,f(x)\leq y\leq g(x)\},$$

for suitable functions f and g. Then for any $s>0$, we have

$$sL=\{(x,y)\,|\,sf(x/s)\leq y\leq sg(x/s)\}.$$

It follows that if the lines bounding S correspond to $x=a$ and $x=b$, where $a<0<b$, then

$$\mathcal{H}^2(C_1'')=\int_a^b\left((\rho+h)g\left(x/(\rho+h)\right)-\rho g(x/\rho)\right)\,dx$$

and

$$\mathcal{H}^2(D_1'')=\int_a^b\left(\rho g(x/\rho)-(\rho-h)g\left(x/(\rho-h)\right)\right)\,dx.$$

Thus the desired inequality $\mathcal{H}^2(C_1'')\leq\mathcal{H}^2(D_1'')$ would follow immediately from the inequality

$$2\rho g\left(\frac{x}{\rho}\right)\geq(\rho-h)g\left(\frac{x}{\rho-h}\right)+(\rho+h)g\left(\frac{x}{\rho+h}\right).$$

But the latter is a direct consequence of the concavity of g, since with $t=(\rho+h)/2\rho$, so that $0<t<1$, we have

$$\frac{x}{\rho}=(1-t)\frac{x}{\rho-h}+t\frac{x}{\rho+h}.$$

□

Corollary 3.3. *If K and L are o-symmetric convex bodies in $\mathbb{R}^2$, then $\alpha_K(L,\rho)$ is concave.*

In view of Theorem 3.2 and Corollary 3.3, it is natural to ask whether under the same hypotheses, the function $\eta_K(L,\rho)$ is decreasing. This is false, as the following example shows. Let $K=[-2,2]^2$ and $L=\operatorname{conv}\{(\pm 2,0),(\pm 1,\pm 1),(\pm 2,\pm 1)\}$, so that K and L are o-symmetric polygons in $\mathbb{R}^2$. For $1<\rho<2$, $K\cap\rho L$ is a hexagon and $K\cap\partial(\rho L)$ consists of the line segment $[(-2,2(\rho-1)/3),(\rho,\rho)]$, the line segment $[(\rho,\rho),(2,\rho)]$, and their reflections in the origin. Therefore

$$\eta_K(L,\rho)=2\left(\frac{\sqrt{10}(\rho+2)}{3}+2-\rho\right),$$

for $1<\rho<2$, which actually increases with ρ.

For $n\geq 3$, Theorem 3.1 implies that under the hypotheses of Theorem 3.2 or Corollary 3.3, $\alpha_K(L,\rho)^{1/n}$ is concave. Our next result shows that in contrast to the case $n=2$, the exponent is optimal.

Theorem 3.4. *If $n\geq 3$ and $q>1/n$, there are o-symmetric convex bodies K and L in $\mathbb{R}^n$ such that $\alpha_K(L,\rho)^q$ is not concave.*

Proof. Let D be an o-symmetric $(n-1)$-dimensional ball of radius 1 contained in the plane $\{x_n=0\}$ in $\mathbb{R}^n$, and let $L=\operatorname{conv}\{D,e_n,-e_n\}$, so that L is a double cone. Let $0<r<R$ and let $K=\{-r\leq x_n\leq r\}\cap RL$. Then $r(K,L)=r$ and $R(K,L)=R$. We claim that for $n\geq 3$ and $q>1/n$, $\alpha_K(L,\rho)^q$ is not concave. Indeed, for $\rho\geq r$ we have

$$\alpha_K(L,\rho)=\frac{2\kappa_{n-1}}{n}\int_0^r(\rho-t)^{n-1}\,dt=\frac{2\kappa_{n-1}}{n}\left(\rho^n-(\rho-r)^n\right).$$

Let $g(\rho)=\alpha_K(L,\rho)^q$. Then $g''(\rho)=q\alpha_K(L,\rho)^{q-2}I(\rho)$, where

$$I(\rho)=(q-1)\alpha_K'(L,\rho)^2+\alpha_K(L,\rho)\alpha_K''(L,\rho).$$

Let $r=1$. A direct calculation with $q=1/n+c$ and $c>0$ shows that

$$\lim_{\rho\to 1+}I(\rho)=4\kappa_{n-1}^2\left(c-\frac{n-1}{n}\right)+4\kappa_{n-1}^2\frac{n-1}{n}=4c\kappa_{n-1}^2>0,$$

for all $n\geq 3$. It follows that $g''(\rho)>0$ for ρ close to 1, proving the claim. □

4. Properties of α_K and η_K

The next result is just the special case $L=B$ of Theorem 3.1.

Corollary 4.1. *If K is a convex body in $\mathbb{R}^n$, then $\alpha_K(\rho)^{1/n}$ is concave.*

We now establish differentiability properties, some of which we require later. Note firstly that while α_K is clearly continuous (meaning, as always, on its domain $[r_K,R_K]$), η_K may not be. For example, let $K\subset\mathbb{R}^2$ be the convex hull of S^1, the arc $\{(r,\theta):r=2,\pi/4\leq\theta\leq\pi/2\}$ in $2S^1$, and the point $(2\sqrt{2},0)$. Then η_K has jump discontinuities at $\rho=1$ and $\rho=2$.

Observe that this example is easily modified to produce convex bodies with arbitrarily smooth boundaries for which η_K is not even continuous.

Theorem 4.2. *Let K be a convex body in $\mathbb{R}^n$.*

(i) *The left and right one-sided derivatives $(\alpha_K)'_-$ and $(\alpha_K)'_+$ of α_K exist and $(\alpha_K)'_+ \le (\alpha_K)'_-$. We have $(\alpha_K)'_- = \eta_K$ and η_K is continuous on the left and its one-sided limits exist. There is a countable subset M of $[r_K, R_K]$ such that $\alpha_K(\rho)$ is differentiable and $\alpha'_K(\rho) = \eta_K(\rho)$ for $\rho \in [r_K, R_K] \setminus M$. Consequently, the only discontinuities of η_K are jump discontinuities at points in M, and at such a discontinuity we have*

$$\lim_{s \to \rho+} \eta_K(s) < \eta_K(\rho).$$

Hence η_K is upper semicontinuous.

(ii) *There is a set N of measure zero such that $\alpha_K(\rho)$ is twice differentiable and hence $\eta_K(\rho)$ is differentiable for $\rho \in [r_K, R_K] \setminus (M \cup N)$.*

Proof. For (i), we use Corollary 4.1 and a standard result on concave or convex functions of one variable (see, for example, the statement and proof of [12, Theorem 1.4]) to conclude that if $g_K = \alpha_K^{1/n}$, then $(g_K)'_-$ and $(g_K)'_+$ exist and are continuous on the left and right, respectively, that $(g_K)'_+ \le (g_K)'_-$, and that there is a countable set M of $[r_K, R_K]$ such that $g_K(\rho)$ is differentiable for $\rho \in [r_K, R_K] \setminus M$. Since α_K is increasing, the same is true when g_K is replaced by α_K. Now using the convexity of K, we see that

$$\eta_K(\rho) = \lim_{s \to \rho-} \frac{\alpha_K(\rho) - \alpha_K(s)}{\rho - s} = (\alpha_K)'_-(\rho).$$

This proves (i).

To prove (ii), we similarly apply Corollary 4.1 and another result on concave or convex functions of one variable (see [12, Theorem 1.7]; this is the one-dimensional case of Aleksandrov's theorem on the almost-everywhere twice-differentiability of concave or convex functions, which according to Schneider [20, p. 32] was first observed by Jessen). □

As is explained in [12, p. 10], the previous result means that $\eta'_K(\rho)$ exists almost everywhere in $[r_K, R_K]$ in the sense that

$$\lim_{s \to \rho} \frac{\eta_K(\rho) - \eta_K(s)}{\rho - s}$$

exists for $\rho \in [r_K, R_K] \setminus (M \cup N)$, where in the limit $s \in [r_K, R_K] \setminus M$.

Theorem 4.3. *Let K be a convex body in $\mathbb{R}^n$ and let $\varepsilon > 0$. There is a convex body L in $\mathbb{R}^n$ such that $r_L = r_K$, $R_L = R_K$, the Hausdorff distance between K and L is less than ε, and η_L is continuous. Moreover, if K is o-symmetric, there is an o-symmetric L with these properties.*

Proof. If η_K is not continuous, then by Theorem 4.2(i) it has a jump discontinuity, which can only occur if for some $r_K \le \rho < R_K$, we have $\mathcal{H}^{n-1}(\partial K \cap \rho S^{n-1}) > 0$. Without loss of

generality, we may suppose that $K \subset \{-s \le x_n \le r_K\}$, where $s \ge r_K$ and $s = r_K$ if K is o-symmetric.

Let E be a non-spherical o-symmetric ellipsoid and choose $a > 0$ sufficiently small so that if $L' = K + aE$, the Hausdorff distance between K and L' is less than ε. Using the fact that the support functions of K, L', and E are related by $h_{L'} = h_K + ah_E$, it is easy to see that $\mathcal{H}^{n-1}(\partial L' \cap \rho S^{n-1}) = 0$ for each ρ and hence $\eta_{L'}$ is continuous. Let $L = L' \cap \{-s \le x_n \le r_K\} \cap R_K B$. Then L retains all the desired properties and is o-symmetric if K is. □

The following result is proved by a straightforward direct argument in [1, Lemma 5.1].

Proposition 4.4. *If K is an o-symmetric convex body in $\mathbb{R}^2$, then $\eta_K(\rho)$ is decreasing and hence $\alpha_K(\rho)$ is concave.*

The first statement in Proposition 4.4 is also implied by [5, Lemma 1], which when K is o-symmetric, states that $\rho \sin(\eta_K(\rho)/(4\rho))$ is decreasing. Indeed, if $\rho_1 < \rho_2$, then

$$\rho_2 \sin\left(\frac{\eta_K(\rho_2)}{4\rho_2}\right) \le \rho_1 \sin\left(\frac{\eta_K(\rho_1)}{4\rho_1}\right) \le \rho_2 \sin\left(\frac{\eta_K(\rho_1)}{4\rho_2}\right),$$

and $\eta_K(\rho_2) \le \eta_K(\rho_1)$ follows directly. A still stronger result is given in Theorem 5.3 below.

The second statement in Proposition 4.4 follows immediately from Corollary 3.3. However, as we have seen, the fact that $\eta_K(L, \rho)$ is not always decreasing means that the first statement cannot be obtained from a more general result.

Theorem 4.5. *If $n = 2$ or 3 and $q > 1/(n-1)$, or if $n \ge 4$ and $q > 1/n$, there is a o-symmetric convex body K in $\mathbb{R}^n$ such that $\alpha_K(\rho)^q$ is not concave.*

Proof. Let $n \ge 2$, let $R > 1$, and let $K = \{-1 \le x_n \le 1\} \cap RB$. Then $r_K = 1$, $R_K = R$, and

$$\alpha_K(\rho) = 2\kappa_{n-1} \int_0^1 (\rho^2 - t^2)^{(n-1)/2}\, dt, \tag{3}$$

for $1 \le \rho \le R$. As in the proof of Theorem 3.4, we let $g(\rho) = \alpha_K(\rho)^q$, so that $g''(\rho) = q\alpha_K(L, \rho)^{q-2} I(\rho)$, where

$$I(\rho) = (q-1)\alpha_K'(L, \rho)^2 + \alpha_K(L, \rho)\alpha_K''(L, \rho).$$

It suffices in each case to show that $I(\rho)$ may be positive.

Suppose that $n = 2$. Then

$$\alpha_K(\rho) = 2\left(\rho^2 \arcsin(1/\rho) + \sqrt{\rho^2 - 1}\right).$$

Therefore

$$I(\rho) = 8\left((2q-1)\rho^2 \arcsin^2(1/\rho) - \frac{\arcsin(1/\rho)}{\sqrt{\rho^2 - 1}} - 1\right).$$

Since $I(\rho) \to 16(q-1)$ as $\rho \to \infty$, we have $I(\rho) > 0$ when $q > 1$ and ρ (and hence R) is sufficiently large.

If $n = 3$, then $\alpha_K(\rho) = 2\pi(\rho^2 - 1/3)$. Therefore

$$I(\rho) = 8\pi^2 \left((2q-1)\rho^2 - 1/3\right), \tag{4}$$

so again,we have $I(\rho) > 0$ when $q > 1/2$ and ρ (and hence R) is sufficiently large.

Now suppose that $n \geq 4$. In view of (3), we have

$$\lim_{\rho\to 1+} \alpha_K(\rho) = \kappa_n = \kappa_{n-1}\frac{\pi^{1/2}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}, \tag{5}$$

$$\lim_{\rho\to 1+} \alpha_K'(\rho) = \lim_{\rho\to 1+} 2\kappa_{n-1}(n-1)\rho\int_0^1 (\rho^2 - t^2)^{\frac{n-3}{2}}\,dt = \kappa_{n-1}(n-1)\frac{\pi^{1/2}\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}, \tag{6}$$

and, using the relation $x\Gamma(x) = \Gamma(x+1)$,

$$\begin{aligned}
\lim_{\rho\to 1+} \alpha_K''(\rho) &= \lim_{\rho\to 1+} 2\kappa_{n-1}(n-1)(n-3)\rho^2\int_0^1 (\rho^2 - t^2)^{(n-5)/2}\,dt + \\
&\qquad + 2\kappa_{n-1}(n-1)\int_0^1 (\rho^2 - t^2)^{(n-3)/2}\,dt \\
&= \kappa_{n-1}(n-1)(n-3)\frac{\pi^{1/2}\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)} + \kappa_{n-1}(n-1)\frac{\pi^{1/2}\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \\
&= \pi^{1/2}\kappa_{n-1}(n-1)^2\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \qquad (7)
\end{aligned}$$

Therefore (5), (6), and (7) yield

$$\begin{aligned}
\lim_{\rho\to 1+} I(\rho) &= \pi(n-1)^2\kappa_{n-1}^2\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\left((q-1)\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} + \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\right) \\
&= \pi\left((n-1)\kappa_{n-1}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^2(q - 1/n).
\end{aligned}$$

It follows that if $q > 1/n$, then $I(\rho) > 0$ when ρ is sufficiently close to 1. This completes the proof. □

Our results so far leave open the following possibility.

Conjecture 4.6. *If K is an o-symmetric convex body in $\mathbb{R}^3$, then $\alpha_K(\rho)^{1/2}$ is concave.*

Note that when $n = 3$, it follows from (4) with $q = 1/2$ that $\alpha_K(\rho)^{1/2}$ is indeed concave for the convex body K in Theorem 4.5.

5. Consequences of the Brunn-Minkowski inequality in the sphere

For any set D in S^{n-1}, let $D_\varepsilon = \{x \in S^{n-1} : d(x, D) \leq \varepsilon\}$, where d denotes the angular metric in S^{n-1} (i.e., the induced metric in S^{n-1}; see [4, p. 10]). The following result is sometimes called the Brunn-Minkowski inequality in the sphere; see [8, p. 380] and the references given there, particularly [4, p. 77].

Proposition 5.1. *Let A be an $\mathcal{H}^{n-1}$-measurable subset of S^{n-1} and let $\varepsilon > 0$. Let C be a spherical cap in S^{n-1} such that $\mathcal{H}^{n-1}(C) = \mathcal{H}^{n-1}(A)$. Then*

$$\mathcal{H}^{n-1}(A_\varepsilon) \geq \mathcal{H}^{n-1}(C_\varepsilon).$$

Henceforth we denote by $m(s)$ the $\mathcal{H}^{n-1}$-measure of a cap in S^{n-1} with angular radius s.

Theorem 5.2. *If K is a convex body containing the origin in $\mathbb{R}^n$, then*

$$F_{n,K}(\rho) = m^{-1}(\eta_K(\rho)/\rho^{n-1}) + \arccos(r_K/\rho)$$

is decreasing.

Proof. For $\rho > 0$, let

$$f_K(\rho) = \mathcal{H}^{n-1}((1/\rho)K \cap S^{n-1}) = \eta_K(\rho)/\rho^{n-1} \tag{8}$$

be the normalized version of η_K.

Fix $\rho > r_K$ and let $0 < \varepsilon < \rho - r_K$. Let $x \in K \cap \rho S^{n-1}$, let $D_x = \operatorname{conv}\{r_K B, x\}$, and note that by convexity,

$$(1/(\rho-\varepsilon))K \cap S^{n-1} \supset (1/(\rho-\varepsilon))D_x \cap S^{n-1}, \tag{9}$$

a closed cap in S^{n-1}. By elementary geometry, we find that the angular radius of this cap is

$$\phi = \arccos(r_K/\rho) - \arccos(r_K/(\rho-\varepsilon)). \tag{10}$$

Let $A = (1/\rho)K \cap S^{n-1}$. Since (9) holds for any $x \in K \cap \rho S^{n-1}$, we obtain

$$A_\phi = ((1/\rho)K \cap S^{n-1})_\phi \subset (1/(\rho-\varepsilon))K \cap S^{n-1}.$$

This and Proposition 5.1 yield

$$f_K(\rho-\varepsilon) = \mathcal{H}^{n-1}\left((1/(\rho-\varepsilon))\,K \cap S^{n-1}\right) \geq \mathcal{H}^{n-1}(A_\phi) \geq \mathcal{H}^{n-1}(C_\phi), \tag{11}$$

where C is a cap in S^{n-1} such that $\mathcal{H}^{n-1}(C) = \mathcal{H}^{n-1}(A) = f_K(\rho)$. Choose ψ such that $m(\psi) = f_K(\rho)$, i.e. $\psi = m^{-1}(f_K(\rho))$. Then the angular radius of C is ψ, so the angular radius of C_ϕ is $\psi + \phi$. Therefore (11) implies that

$$f_K(\rho-\varepsilon) \geq m(\psi+\phi) = m(m^{-1}(f_K(\rho)) + \phi). \tag{12}$$

Using (10), we obtain

$$m^{-1}(f_K(\rho-\varepsilon)) + \arccos(r_K/(\rho-\varepsilon)) \geq m^{-1}(f_K(\rho)) + \arccos(r_K/\rho),$$

and the theorem follows immediately. □

The previous result cannot be improved, in the sense that there are convex bodies K containing the origin in $\mathbb{R}^n$ for which the function $F_{n,K}$ is constant. This is the case whenever $K = \operatorname{conv}\{A, rB\}$, where $r = r_K > 0$ and A is a (possibly degenerate) spherical cap in $RS^{n-1} = R_K S^{n-1}$ with angular radius in the interval $[0, \arcsin(r/R)]$.

We can use the argument of Theorem 5.2 to obtain a stronger result than Proposition 4.4. Indeed, it is not difficult to prove that it is also stronger than the result for planar o-symmetric convex bodies from [5, Lemma 1] that was mentioned after Proposition 4.4.

Theorem 5.3. *If K is an o-symmetric convex body in $\mathbb{R}^2$, then*

$$G_{2,K} = \eta_K(\rho)/(4\rho) + \arccos(r_K/\rho)$$

is decreasing, and this implies that $\eta_K(\rho)$ is decreasing.

Proof. When K is o-symmetric in $\mathbb{R}^2$, we can use the symmetry and the fact that if $r_K < \rho < R_K$ and E is a component of $K \cap \rho S^1$, then $-E$ is another such component disjoint from E, to obtain

$$f_K(\rho - \varepsilon) \geq 2m(m^{-1}(f_K(\rho)/2) + \phi) \tag{13}$$

instead of (12) in the proof of Theorem 5.2. With this and the fact that $m(s) = 2s$ when $n = 2$, it follows as in the proof of Theorem 5.2 that $G_{2,K}$ is decreasing.

Suppose that η_K is not decreasing, so that there exist $\rho_1 < \rho_2$ such that $\eta_K(\rho_1) < \eta_K(\rho_2)$. Let $M = \sup\{\eta_K(\rho) : \rho_1 \leq \rho \leq \rho_2\}$. By Theorem 4.2(i), η_K is continuous on the left and upper semicontinuous, so M is a maximum and the set $\{s \in [\rho_1, \rho_2] : \eta_K(s) = M\}$ is closed. Let ρ_0 be the minimum number in this set. Then for $0 < \varepsilon \leq \varepsilon_0 = \rho_0 - \rho_1$, we have

$$\eta_K(\rho_0 - \varepsilon) < \eta_K(\rho_0). \tag{14}$$

Next, we observe that the function $g(\rho) = \arccos(r_K/\rho)$ is strictly increasing and concave in ρ, so for $0 < \varepsilon \leq \varepsilon_0$,

$$\arccos\frac{r_K}{\rho_0} - \arccos\frac{r_K}{\rho_0 - \varepsilon} = g(\rho_0) - g(\rho_0 - \varepsilon) > \varepsilon g'(\rho_0) = \frac{\varepsilon r_K}{\rho_0\sqrt{\rho_0^2 - r_K^2}}. \tag{15}$$

Using the fact that $G_{2,K}$ is decreasing, (15), and (14), we obtain

$$\begin{aligned} \frac{\eta_K(\rho_0 - \varepsilon)}{4(\rho_0 - \varepsilon)} + \arccos\frac{r_K}{\rho_0 - \varepsilon} &> \frac{\eta_K(\rho_0)}{4\rho_0} + \arccos\frac{r_K}{\rho_0} \\ &> \frac{\eta_K(\rho_0 - \varepsilon)}{4\rho_0} + \arccos\frac{r_K}{\rho_0 - \varepsilon} + \frac{\varepsilon r_K}{\rho_0\sqrt{\rho_0^2 - r_K^2}}, \end{aligned}$$

and hence

$$\eta_K(\rho_0 - \varepsilon) > \frac{4(\rho_0 - \varepsilon)r_K}{\sqrt{\rho_0^2 - r_K^2}}.$$

Using (14), letting $\varepsilon \to 0$, and letting $\theta_0 = \arcsin(r_K/\rho_0)$, we have $0 < \theta_0 < \pi/2$ and

$$\eta_K(\rho_0) \geq 4\rho_0 \tan\theta_0 > 4\rho_0\theta_0.$$

However, since K is o-symmetric, it is contained in a closed o-symmetric strip S of width $2r_K$. Therefore

$$\eta_K(\rho_0) \leq V_1(S \cap \rho_0 S^1) = 4 \arcsin(r_K/\rho_0) = 4\rho_0\theta_0.$$

This contradiction proves that η_K is decreasing. □

The previous theorem cannot be improved, since when $K = \{-r_K \leq x_2 \leq r_K\} \cap RB$ where $R > r_K$, the function $G_{2,K}$ is constant.

6. An isoperimetric inequality for equatorial zones in S^2

By a *sector* of S^2 we mean the closed region between two lines of longitude, i.e. $\{(1, \theta, \phi) : \theta_1 \leq \theta \leq \theta_2\}$, where θ is the horizontal angle and ϕ the vertical angle in spherical polar coordinates. If $\theta_2 - \theta_1 = \gamma$, we denote such a sector by $S(\gamma)$, i.e., this is any sector of S^2 subtending a horizontal angle of γ at the origin.

An *equatorial zone* in S^2 is the intersection of S^2 with a closed slab of the form $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : -a \leq x_3 \leq a,\ 0 < a \leq 1\}$. We denote this equatorial zone by $E(a)$.

If A is an $\mathcal{H}^2$-measurable set in S^2, we denote by $P(A)$ its *perimeter*. We follow Burago and Zalgaller [4, p. 106] in defining

$$P(A) = \inf\{\liminf_{i\to\infty} \mathcal{H}^1(\partial M_i)\},$$

where the infimum is taken over all sequences of spherical polygons M_i in S^2 with boundaries ∂M_i such that $M_i \to A$ as $i \to \infty$ in the symmetric difference metric. (Note that the latter condition implies in particular that $\mathcal{H}^2(M_i) \to \mathcal{H}^2(A)$ as $i \to \infty$.) We also follow Burago and Zalgaller [4, p. 69] in defining

$$\mu_+(A) = \liminf_{\varepsilon\to 0+} \frac{\mathcal{H}^2(A_\varepsilon) - \mathcal{H}^2(A)}{\varepsilon}.$$

However, we shall refer to $\mu_+(A)$ as the *Minkowski perimeter* of A instead of the outer Minkowski content of ∂A.

Theorem 6.1. *If $0 < a < 1$ and A is an $\mathcal{H}^2$-measurable subset of $E(a)$, then*

$$\mathcal{H}^2(A) \leq \frac{a}{\sqrt{1-a^2}}\, P(A), \tag{16}$$

with equality if and only if $A = E(a)$.

Proof. Since $A \subset E(a)$ we may, by replacing the spherical polygons M_i in the definition of $P(A)$ by $M_i \cap E(a)$, if necessary, take these spherical polygons to be subsets of $E(a)$. From the definition of $P(A)$ it then follows that if (16) is false for A, there is a spherical polygon M_i in $E(a)$ such that (16) is also false for M_i. Thus we may assume that A itself is a spherical polygon with $\mathcal{H}^2(A) > 0$. We may further assume that $\operatorname{int} A$ is connected, since if (16) is false, there must be a component of $\operatorname{int} A$ such that (16) is also false when A is replaced by the closure of this component.

Choose $0 < b \le a$ and $0 < \gamma \le 2\pi$ to be minimal such that $\operatorname{int} A \subset C = E(b) \cap S(\gamma)$ for some sector $S(\gamma)$. Clearly, $\mathcal{H}^2(A) \le \mathcal{H}^2(C) = 2b\gamma$. We claim that

$$P(A) \ge 2\gamma\sqrt{1-b^2}, \tag{17}$$

with equality if and only if $A = E(b)$. This will suffice to prove the theorem, since it then follows that

$$\mathcal{H}^2(A) \le 2b\gamma \le \frac{b}{\sqrt{1-b^2}} P(A) \le \frac{a}{\sqrt{1-a^2}} P(A),$$

with equality if and only if $A = E(a)$.

To prove the claim, assume first that $\gamma < 2\pi$. The set $C \cap \partial S(\gamma)$ consists of two arcs, A_1 and A_2 say, and A, being closed, must contain a point in each arc. Therefore ∂A contains two paths, disjoint except at their endpoints, each joining A_1 and A_2.

The shortest paths in C joining A_1 and A_2 are the two arcs that comprise $C \cap \partial E(b)$, each of which has length $\gamma\sqrt{1-b^2}$. To see this, note first that a standard compactness argument guarantees the existence of a minimal path Q. We may assume that Q is contained in $\{x_3 \ge 0\}$, since otherwise we can replace $Q \cap \{x_3 \le 0\}$ by its reflection in $\{x_3 = 0\}$ to obtain a new path of the same length as Q that is contained in $\{x_3 \ge 0\}$. We may also assume that the endpoints of Q have the same height $h \ge 0$. Indeed, by the minimality of Q, the meridian of longitude (i.e., horizontal angle in spherical polar coordinates) $(\theta_1 + \theta_2)/2$ divides Q into two arcs. The union of the shorter of these arcs with its reflection in this meridian has length no greater than that of Q and endpoints at equal height.

Thus we have a path Q joining A_1 and A_2 of minimal length, contained in $\{x_3 \ge 0\}$ and with endpoints at the same height $h \ge 0$. Suppose that $p = (p_1, p_2, p_3) \in Q$ has minimal height in Q and $p_3 < h$. Let $0 < \varepsilon < h - p_3$ and let v and w be the points in $Q \cap \{x_3 = p_3 + \varepsilon\}$ that are the endpoints of the arc R in $Q \cap \{x_3 \le p_3 + \varepsilon\}$ containing p. If ε is sufficiently small, there is a geodesic G joining v and w contained in $C \cap \{x_3 \ge p_3 + \varepsilon\}$. If Q' is the path obtained from Q by replacing R with G, then the length of Q' is less than that of Q. This contradiction proves that $Q \subset \{x_3 \ge h\}$.

Suppose that $q = (q_1, q_2, q_3) \in Q$ has maximal height in Q, and that $q_3 > h$. Let θ_1, θ_2 and θ_q be the longitudes of the meridians containing the arcs A_1, A_2 and of the point q, respectively. The minimality of Q implies that the meridian of longitude θ_q divides Q into two arcs, B_1 and B_2, say, where B_1 meets A_1 and B_2 meets A_2. For $i = 1, 2$, let $D_i \subset S^2$ be the reflection of B_i in the meridian of longitude $(\theta_i + \theta_q)/2$, and let $Q'' = D_1 \cup D_2$. Then Q'' is a path in C joining A_1 and A_2, with the same length as Q, and with endpoints at height q_3 and containing a point at minimal height $h < q_3$. As in the previous paragraph, such a path cannot be minimal.

We conclude that a minimal path must lie in $\{x_3 = c\}$ for some $-b \le c \le b$. Clearly, the shortest such paths are the two that comprise $C \cap \partial E(b)$. This proves (17) when $\gamma < 2\pi$.

The above argument shows that the shortest path in $E(b)$ joining two meridians lies in $\partial E(b)$. With this in hand, (17) and its equality conditions follow easily in the remaining case when $\gamma = 2\pi$. □

Corollary 6.2. *If $0 < a < 1$ and A is an $\mathcal{H}^2$-measurable subset of $E(a)$, then*

$$\mathcal{H}^2(A) \leq \frac{a}{\sqrt{1-a^2}}\,\mu_+(A), \tag{18}$$

with equality if and only if $A = E(a)$.

Proof. The inequality and equality condition both follow from Theorem 6.1 and the general inequality $P(A) \leq \mu_+(A)$ (see [4, Theorem 14.2.1]). □

7. Equatorial symmetrization

We begin by applying the isoperimetric inequality obtained in the previous section to prove a property of η_K in $\mathbb{R}^3$.

Theorem 7.1. *If K is an o-symmetric convex body in $\mathbb{R}^3$, then $\eta_K(\rho)/\rho$ is decreasing.*

Proof. Let $F(\rho) = \eta_K(\rho)/\rho$ and assume to the contrary that there exist $\rho_1 < \rho_2$ such that $F(\rho_1) < F(\rho_2)$. Let $M = \sup\{F(\rho) : \rho_1 \leq \rho \leq \rho_2\}$. Clearly F inherits the properties of η_K established in Theorem 4.2(i) and in particular F is continuous on the left and upper semicontinuous. Therefore M is a maximum and the set $I = \{s \in [\rho_1, \rho_2] : F(s) = M\}$ is closed. Let ρ_0 be the minimum number in this set. Then for $0 < \varepsilon \leq \varepsilon_0 = \rho_0 - \rho_1$, we have

$$F(\rho_0 - \varepsilon) < F(\rho_0). \tag{19}$$

Now let $A = (1/\rho_0)K \cap S^2$, so that $\mathcal{H}^2(A) = F(\rho_0)/\rho_0$. Since K is o-symmetric, $r_K > 0$ and K is contained in a closed slab of width $2r_K$, bounded by two common supporting hyperplanes to K and $r_K B$. It follows that if we let $a = r_K/\rho_0$, then $A \subset E(a)$.

If $A = E(a)$, then (19) cannot hold. Indeed, in that case we would have $F(\rho) \geq 4\pi r_K$ for all $\rho \in [r_K, \rho_0]$ with equality for $\rho = \rho_0$. Therefore $A \neq E(a)$.

Suppose that the Minkowski perimeter $\mu_+(A)$ of A is finite. By (18), there is a $c > 0$ such that

$$\mathcal{H}^2(A) = \frac{a}{\sqrt{1-a^2}}\,(\mu_+(A) - c). \tag{20}$$

By the definition of $\mu_+(A)$, there is an $\varepsilon_1 > 0$ such that

$$\mathcal{H}^2(A_\varepsilon) \geq \mathcal{H}^2(A) + \varepsilon(\mu_+(A) - c/2), \tag{21}$$

for all $\varepsilon \in (0, \varepsilon_1)$.

Choose $\varepsilon > 0$ so that $\varepsilon < \varepsilon_0$,

$$\varepsilon < \frac{c\rho_0}{2\mu_+(A) - c}, \tag{22}$$

and

$$\phi = \arccos\frac{r_K}{\rho_0} - \arccos\frac{r_K}{\rho_0 - \varepsilon} < \varepsilon_1. \tag{23}$$

Using (19), (11), (21), and (23), we obtain

$$\frac{\rho_0}{\rho_0-\varepsilon}\mathcal{H}^2(A)=\frac{F(\rho_0)}{\rho_0-\varepsilon}>\frac{F(\rho_0-\varepsilon)}{(\rho_0-\varepsilon)}\geq\mathcal{H}^2(A_\phi)\geq\mathcal{H}^2(A)+\phi(\mu_+(A)-c/2), \tag{24}$$

which gives

$$\frac{\varepsilon}{\rho_0-\varepsilon}\mathcal{H}^2(A)>\left(\arccos\frac{r_K}{\rho_0}-\arccos\frac{r_K}{\rho_0-\varepsilon}\right)(\mu_+(A)-c/2).$$

By (15), we have

$$\arccos\frac{r_K}{\rho_0}-\arccos\frac{r_K}{\rho_0-\varepsilon}>\frac{\varepsilon r_K}{\rho_0\sqrt{\rho_0^2-r_K^2}}=\frac{\varepsilon a}{\rho_0\sqrt{1-a^2}}.$$

Consequently

$$\mathcal{H}^2(A)>\frac{(\rho_0-\varepsilon)a}{\rho_0\sqrt{1-a^2}}(\mu_+(A)-c/2). \tag{25}$$

This and (20) yield

$$\mu_+(A)-c>\frac{\rho_0-\varepsilon}{\rho_0}(\mu_+(A)-c/2),$$

which is a contradiction to (22).

It remains to consider the case when $\mu_+(A)=\infty$. Using (19) but not the assumption that $\mu_+(A)$ is finite, the argument above leading to (24) yields instead the conclusion that for each $M\in\mathbb{N}$, there is an $0<\varepsilon<\varepsilon_0$ such that

$$\frac{\rho_0}{\rho_0-\varepsilon}\mathcal{H}^2(A)=\frac{F(\rho_0)}{\rho_0-\varepsilon}>\frac{F(\rho_0-\varepsilon)}{(\rho_0-\varepsilon)}\geq\mathcal{H}^2(A_\phi)\geq\mathcal{H}^2(A)+\phi M.$$

Now the argument above leading to (25) gives

$$\mathcal{H}^2(A)>\frac{(\rho_0-\varepsilon)a}{\rho_0\sqrt{1-a^2}}M,$$

a contradiction to $\mathcal{H}^2(A)<\mathcal{H}^2(E(a))$ if M is sufficiently large. □

Theorem 7.1 has the following nice interpretation. Let K be a convex body in $\mathbb{R}^n$. For each $\rho\in[r_K,R_K]$, choose $h(\rho)$ such that

$$\mathcal{H}^{n-1}\left(\{x\in\mathbb{R}^n:-h(\rho)\leq x_n\leq h(\rho)\}\cap\rho S^{n-1}\right)=\eta_K(\rho),$$

and let

$$E_K=\bigcup\{\{x\in\mathbb{R}^n:-h(\rho)\leq x_n\leq h(\rho)\}\cap\rho S^{n-1}:r_K\leq\rho\leq R_K\}.$$

We call E_K the *equatorial symmetral* of K. Clearly E_K is a body of revolution about the x_n-axis. If K is o-symmetric, it is not even obvious a priori that E_K is a star body, but when $n=3$, Theorem 7.1 yields much more, namely, that E_K is "quasiconvex" in the sense that the function h is decreasing. Indeed, when $n=3$ we have

$$\mathcal{H}^2\left(\{x\in\mathbb{R}^3:-h(\rho)\leq x_n\leq h(\rho)\}\cap\rho S^2\right)=4\pi\rho h(\rho),$$

and hence $h(\rho)=\eta_K(\rho)/(4\pi\rho)$ is decreasing.

When $n = 2$, the equatorial symmetral of K coincides with the semicircular symmetral introduced by Bonnesen [2, p. 67]. This process is called annular symmetrization by Bonnesen and Fenchel [3, p. 77]. Bonnesen proved that if the center of the minimal annulus containing ∂K is at the origin, then the perimeter of E_K does not exceed that of K. The semicircular symmetral is used by Campi [5] to obtain stability estimates connected to the isoperimetric inequality.

Another form of symmetral, the spherical symmetral, is the natural generalization of the circular symmetral introduced by Pólya [17] and discussed at length by Pólya and Szegö [18, p. 193]. The spherical symmetral of K is the body produced by replacing $K \cap \rho S^{n-1}$ by a cap in ρS^{n-1} of the same $\mathcal{H}^{n-1}$-measure, centered at the point on the positive x_1-axis a distance ρ from the origin. The spherical symmetral of an o-symmetric convex body does not generally enjoy the corresponding quasiconvexity property, however; this is shown by straightforward computation when $K = \{-r_K \leq x_n \leq r_K\} \cap RB$ where $R > r_K$.

There seems to be no obvious direct relationship between Conjecture 4.6 and Theorem 7.1. Indeed, when $\eta_K'(\rho)$ exists, Theorem 7.1 states that

$$\eta_K'(\rho) \leq \frac{\eta_K(\rho)}{\rho}. \tag{26}$$

Therefore when η_K is differentiable, Conjecture 4.6 implies Theorem 7.1 if $\alpha_K \geq \rho\eta_K/2$ and Theorem 7.1 implies Conjecture 4.6 if $\alpha_K \leq \rho\eta_K/2$, while we only know in general that $\alpha_K \geq \rho\eta_K/3$.

8. Related functions

There is another pair of functions that are related to those considered above and seem worthy of study. Let K and L be convex bodies in $\mathbb{R}^n$. Let

$$m(K, L) = \max_{x \in \mathbb{R}^n} V\left(K \cap (L + x)\right)$$

and let

$$s(K, L) = \max_{x \in \mathbb{R}^n} \mathcal{H}^{n-1}\left(K \cap (\partial L + x)\right).$$

We shall restrict our attention to the former function and define its kernel by

$$\overline{K}_L = \{x \in \mathbb{R}^n : V\left(K \cap (L + x)\right) = m(K, L)\}.$$

Note that when $\rho > 0$ is such that $\rho B + x \subset K$ for some $x \in \mathbb{R}^n$, $\overline{K}_{\rho B}$ is just an inner parallel body of K (see [20, p. 134]).

For $r(K, L) \leq \rho \leq R(K, L)$, let

$$\overline{\alpha}_K(L, \rho) = m(K, \rho L).$$

Note that when K and L are o-symmetric, $\overline{\alpha}_K(L, \rho) = \alpha_K(L, \rho)$.

It is worth remarking that the function $m(K,L)$ appears in a fascinating conjecture of Dar [6], that if K and L are convex bodies in $\mathbb{R}^n$, then

$$V(K+L)^{1/n} \geq m(K,L)^{1/n} + \left(\frac{V(K)V(L)}{m(K,L)}\right)^{1/n}. \tag{27}$$

Dar shows that (27) implies the Brunn-Minkowski inequality for convex bodies. He proves (27) when K and L are ellipsoids and for some other special cases, but his conjecture seems to be open even for planar o-symmetric bodies.

Theorem 8.1. *If K and L are convex bodies in $\mathbb{R}^n$, then $\overline{\alpha}_K(L,\rho)^{1/n}$ is concave.*

Proof. This is similar to the proof of Theorem 3.1. Let $0 < \rho_1 < \rho_2$ and let $z_i \in \overline{K}_{\rho_i L}$, $i = 1, 2$. Let $H = K \times [\rho_1, \rho_2] \subset \mathbb{R}^{n+1}$, for $i = 1, 2$ let

$$D_i = \{(x_1, \ldots, x_{n+1}) : (x_1, \ldots, x_n) \in \rho_i L + z_i,\ x_{n+1} = \rho_i\}$$

and let $C = \operatorname{conv}\{D_1, D_2\}$. Finally, let $M = H \cap C$.

Let $\rho_1 < \rho < \rho_2$, and choose $0 < t < 1$ such that $\rho = (1-t)\rho_1 + t\rho_2$. Let $z = (1-t)z_1 + tz_2$. Note that

$$C \cap \{x_{n+1} = \rho\} = \{(x_1, \ldots, x_{n+1}) : (x_1, \ldots, x_n) \in \rho L + z,\ x_{n+1} = \rho\},$$

so identifying the plane $\{x_{n+1} = \rho\}$ with $\mathbb{R}^n$, we have

$$M \cap \{x_{n+1} = \rho\} = K \cap (\rho L + z).$$

Similarly, identifying the plane $\{x_{n+1} = \rho_i\}$ with $\mathbb{R}^n$, we have

$$M \cap \{x_{n+1} = \rho_i\} = H \cap D_i = K \cap (\rho_i L + z_i),$$

for $i = 1, 2$. By the Brunn-Minkowski inequality, we obtain

$$\begin{aligned} \overline{\alpha}_K(L,\rho)^{1/n} &\geq V\left(K \cap (\rho L + z)\right)^{1/n} \\ &\geq (1-t)V\left(K \cap (\rho_1 L + z_1)\right)^{1/n} + tV\left(K \cap (\rho_2 L + z_2)\right)^{1/n} \\ &= (1-t)\overline{\alpha}_K(L,\rho_1)^{1/n} + t\overline{\alpha}_K(L,\rho_2)^{1/n}, \end{aligned}$$

as required. □

In view of the fact that $\overline{\alpha}_K(L,\rho) = \alpha_K(L,\rho)$ and $\overline{\eta}_K(L,\rho) = \eta_K(L,\rho)$ when K and L are o-symmetric, several of our earlier results apply to the new functions under this restriction. Natural questions arise when K or L is not o-symmetric, but we leave these for a future study.

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